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Seiberg-Witten-like Equations on 7-Manifolds with G_2-Structure

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Abstract

The Seiberg-Witten equations are of great importance in the study of topology of smooth four-dimensional manifolds. In this work, we propose similar equations for 7-dimensional compact manifolds with G_2-structure.

1 Introduction

The Seiberg-Witten monopole equations are formulated for four dimensional compact $Spin^c$-manifolds. There are some analogues of these equations in higher dimensions (see [1, 9, 12, 13]). All of the higher dimensional equations are stated for even dimensional manifolds. The Seiberg-Witten monopole equations consist of two equations. The first equation is the harmonicity condition on the spinors; this condition is linear and can be stated for $Spin^c$-manifolds in any dimension. The second equation couples the self-dual part of the curvature 2-form with a spinor field and is non-linear. There is no natural generalization of the second equation to higher dimensions because self-duality of 2-forms in the sense of Hodge is meaningless if dimension $\neq 4$. Still, there are various definitions of self-duality in dimensions > 4 (see [2, 5, 6]).

Recently 7-dimensional manifolds with G_2-structure have become popular due to the works of Bryant, Joyce and Cleyton-Ivanov (see [3, 4, 7, 10]). In this work we also deal with 7-dimensional manifolds with G_2-structure. If M has a G_2-structure, then M is a $Spin$-manifold (see [11]). As a $Spin$-manifold, M is then automatically a $Spin^c$-manifold. The existence of a G_2-structure on M leads to a decompositions of the space of k-forms on M. The decomposition of 2-forms is especially crucial because it enables us to define an analog of self-duality of 2-forms on M.

Using this approach to self-duality and the standard spinor machinery, we suggest an analog of the Seiberg-Witten-like equations on 7-manifolds with G_2-structure and we show that these Seiberg-Witten-like equations possess non-trivial solutions.

2 Preliminaries

Let us consider $\mathbb{R}^7$ with a basis $e_1, \ldots, e_7$. Endow $\mathbb{R}^7$ with a metric for which the basis $e_1, \ldots, e_7$ is orthonormal and choose the orientation given by $[e_1, \ldots, e_7]$. Set

$$\Phi = dx_{124} + dx_{235} + dx_{346} + dx_{457} + dx_{156} + dx_{267} + dx_{137}, \tag{2.1}$$

where $dx_{ijk} = dx_i \wedge dx_j \wedge dx_k$. The subgroup of $Gl(7, \mathbb{R})$ fixing Φ is the exceptional Lie group G_2; it is a compact, connected and simply-connected Lie subgroup of $SO(7)$ of dimension 14 (see [3, 4]).

A G_2-structure on a 7−manifold M is a reduction of the structure group $SO(7)$ to G_2.

Let M be a 7-manifold with a G_2-structure. The action of G_2 on the tangent bundle induces an action of G_2 on $\wedge^k(M)$. This action gives the following orthogonal decompositions of $\wedge^k(M)$:

$$\wedge^1(M) = \wedge^1_7, \qquad \wedge^2(M) = \wedge^2_7 \oplus \wedge^2_{14}, \qquad \wedge^3(M) = \wedge^3_1 \oplus \wedge^3_7 \oplus \wedge^3_{27}$$

where

$$\begin{aligned}
\wedge^2_7 &= \left\{\alpha \in \wedge^2(M) : *(\alpha \wedge \Phi) = -2\alpha\right\}, \\
\wedge^2_{14} &= \left\{\alpha \in \wedge^2(M) : *(\alpha \wedge \Phi) = \alpha\right\}, \\
\wedge^3_1 &= \{t\Phi : t \in \mathbb{R}\}, \\
\wedge^3_7 &= \left\{*(\beta \wedge \Phi) : \beta \in \wedge^1(M)\right\}, \\
\wedge^3_{27} &= \left\{\gamma \in \wedge^3(M) : \gamma \wedge \Phi = 0, \gamma \wedge *\Phi = 0\right\}
\end{aligned}$$

and $\wedge^k_l$ denotes an l-dimensional G_2-irreducible subspace of $\wedge^k(M)$.

Recall that $Spin(7)$ is the double cover of $SO(7)$ and a 7-dimensional manifold is called a *Spin* one if the structure group $SO(7)$ of M (in the sense of G-structures) can be lifted to $Spin(7)$ (see [8, 11]).

Recall also that $Spin^c(7) = Spin(7) \times S^1/\mathbb{Z}_2$ with projection

$$\begin{aligned}
Spin^c(7) &\rightarrow SO(7) \\
[g, z] &\longmapsto \lambda g
\end{aligned}$$

where $\lambda : Spin(7) \rightarrow SO(7)$ is the covering map. A 7-dimensional manifold is called a $Spin^c(7)$ one if the structure group $SO(7)$ of M can be lifted to $Spin^c(7)$ (see [8, 11]).

Let M be a $Spin^c$-manifold of dimension n. By a complex spinor bundle for M we mean a complex vector bundle S associated to a representation of $Spin^c(n)$ by Clifford multiplication, i.e.,

$$S = P_{Spin^c(n)} \times_\kappa \Delta_n$$

where $\Delta_n \cong \mathbb{C}^{2^n}$ and $\kappa : Spin^c(n) \longrightarrow End(\Delta_n)$ is given by restriction of the $\mathbb{C}l_n$-representation to $Spin^c(n) \subset \mathbb{C}l_n$. These spinor bundles are bundles of complex modules over the Clifford algebra bundle $\mathbb{C}l(M)$. When n is even the spinor bundle S splits into a direct sum

$$S = S^+ \oplus S^-$$

where $S^{\pm} = (1 \pm \omega_{\mathbb{C}})\, S$ and where $\omega_{\mathbb{C}} = i^{n/2} e_1 e_2 \cdots e_n$ is the volume form (see [8, 11]).

It is known that a connection A in the principal $U(1)$-bundle P_1 and Levi-Civita connection on M determine a covariant derivative

$$\nabla^A : \Gamma(S) \longrightarrow \Gamma(T^*M \otimes S)$$

on the spinor bundle S. And it can be define a first-order differential operator $D_A : \Gamma(S) \longrightarrow \Gamma(S)$ called the Dirac operator of S by setting

$$D_A \psi = \sum_{j=1}^{n} e_j \cdot \nabla^A_{e_j} \psi$$

where $e_1, e_2, \cdots, e_n$ is an orthonormal basis of $T_m M$, at $m \in M$, where ∇^A denotes the covariant derivative on S, " $\cdot$ " denotes the complex Clifford module multiplication. If the dimension n is even, the Dirac operator decomposes into the sum of two operators, $D_A^{\pm} : \Gamma(S^{\pm}) \longrightarrow \Gamma(S^{\mp})$, since Clifford multiplication by vectors interchanges these summands (see [8, 11]).

3 Seiberg-Witten-like Equations in dimension 7

3.1 The Seiberg-Witten Equations in dimension 4

Let M be an oriented, compact 4-dimensional Riemannian manifold. It is known that every compact, orientable 4-dimensional manifold M is a *Spin*^{c}-manifold (see [8]). Fix a *Spin*^{c}−structure and a connection A in the principal $U(1)$-bundle P_1 associated to the *Spin*^{c}-structure. The Seiberg-Witten monopole equations on M are

$$D_A \psi = 0, \qquad \Omega_A^{+} = \sigma(\psi)$$

for $\psi \in \Gamma(S^+)$, where S^+ is the positive spinor bundle (see [8, 11]), and

$$\begin{array}{rccl} \sigma: & \Gamma(S^+) & \rightarrow & \wedge^{2,+} M \\ & \psi & \mapsto & -\frac{1}{4} \sum\limits_{i<j} \langle e_i e_j \psi, \psi \rangle \, e_i \wedge e_j \end{array}$$

and Ω_A^+ is the self dual part of the curvature 2-form Ω_A. The self-dual part Ω_A^+ of Ω_A can be expressed in terms of the Hodge star operator as $\Omega_A^+ = \frac{1}{2}(\Omega_A + *\Omega_A)$. It is also possible to write Ω_A^+ in terms of the basis elements $f_1, f_2, f_3 \in \wedge^2(M)$ as

$$\Omega_A^+ = \sum_{i=1}^{3} \langle f_i, \Omega_A \rangle \, f_i,$$

where

$$f_1 = e_1 \wedge e_2 + e_3 \wedge e_4, \quad f_2 = e_1 \wedge e_3 - e_2 \wedge e_4, \quad f_3 = e_1 \wedge e_4 + e_2 \wedge e_3.$$

Note that σ can also be given by the formula

$$\sigma(\psi) = -\frac{1}{4} \sum_{i=1}^{3} \langle f_i \cdot \psi, \psi \rangle \, f_i.$$

3.2 Seiberg-Witten-like Equations in dimension 7

Note that the first of Seiberg-Witten equations, $D_A\psi = 0$, is linear and meaningful for any $Spin^c$-manifolds whereas the second equation is non-linear and there is no natural generalization to higher dimensions, because self-duality of 2-forms in the Hodge sense is meaningful only for 4-manifolds. Our aim is to write similar equations on 7-dimensional manifolds with G_2-structure. Since such manifolds are *Spin* and thus $Spin^c$ ones, this enables us to construct the spinor bundle and Dirac operator on it. The G_2-structure also provides us with a decomposition $\wedge^2_7 \oplus \wedge^2_{14}$ of the space of 2−forms $\wedge^2(M)$.

Let M be a 7-dimensional manifold with a G_2-structure and A the 1-form of a connection in the principal $U(1)$-bundle P_1 associated to the $Spin^c$ structure on M; let S be the spinor bundle. Then we can define the Dirac operator $D_A : \Gamma(S) \longrightarrow \Gamma(S)$.

Note that, in this case, S does not split into positive and negative parts unlike the even-dimensional case. The 7-dimensional version of the first of Seiberg-Witten equations is

$$D_A\psi = 0 \text{ for } \psi \in \Gamma(S).$$

For the second equation we need a kind of self-duality of 2-forms — the decomposition $\wedge^2(M) = \wedge^2_7 \oplus \wedge^2_{14}$. Let $\pi_7 : \wedge^2(M) \to \wedge^2_7$ be the orthogonal projection. For $\eta \in \wedge^2(M)$, the *self-dual part* of η is, by definition, $\pi_7(\eta)$. If $\pi_7(\eta) = \eta$, then η is called *self-dual.* First, we define a quadratic map $\sigma : \Gamma(S) \longrightarrow \wedge^2_7$ by setting

$$\sigma(\psi) = \sum_{i=1}^{7} \frac{\langle f_i\psi, \psi\rangle}{\langle f_i, f_i\rangle} f_i,$$

where $f_1, \ldots, f_7$ is a basis of $\wedge^2_7$ associated to an orthonormal frame $e_1, \ldots, e_7$ of T_mM at any point $m \in M$. Then the 7-dimensional version of the second Seiberg-Witten equation is

$$\pi_7(\Omega_A) = \sigma(\psi).$$

Hence the Seiberg-Witten-like equations in 7-dimensions are

$$D_A\psi = 0, \qquad \pi_7(\Omega_A) = \sigma(\psi) \tag{3.1}$$

where A is the 1-form of the $i\mathbb{R}$-valued connection on P_1 and $\psi \in \Gamma(S)$.

These equations admit non-trivial solutions. Consider, for example, the flat case $M = \mathbb{R}^7$ with the G_2-structure given by (2.1). Then the spinor bundle is $\mathbb{R}^7 \times \mathbb{C}^8$. We use the spin representation emerging from the isomorphism $\mathbb{C}l_7 \cong End(\mathbb{C}^8) \oplus End(\mathbb{C}^8)$, where $\mathbb{C}l_7$ is the complex Clifford algebra with 7 generators and $End(\mathbb{C}^8)$ denotes the space of 8×8 complex matrices. A direct verification shows that $\psi = (\psi_1, i\psi_1, 0, 0, i\psi_1, \psi_1, 0, 0)$ with $\psi_1(x_1, x_2, \ldots, x_7) = e^{-\frac{i}{4}x_1^2 x_2}$ and $A(x_1, x_2, \ldots, x_7) = (ix_1x_2)dx_1 + (\frac{i}{2}x_1^2)dx_2$ satisfy our Seiberg-Witten-like equations (3.1).

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