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Box-Ball System with Reflecting End

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Abstract

A soliton cellular automaton on a one dimensional semi-infinite lattice with a reflecting end is presented. It extends a box-ball system on an infinite lattice associated with the crystal base of $U_q(\widehat{\mathfrak{sl}}_n)$. A commuting family of time evolutions are obtained by adapting the K matrices and the double row construction of transfer matrices in solvable lattice models to a crystal setting. Factorized scattering among the left and the right moving solitons and the boundary reflection rule are determined.

1 Introduction

The box-ball system [26, 24] is a soliton cellular automaton on a one dimensional lattice. It has been studied and generalized from a variety of aspects; ultradiscretization of soliton equations [28], solvable vertex models [1] at $q = 0$, connection to crystal base theory [5, 4, 8], description as particle and anti-particle systems [9, 20], factorized scattering [7, 27], Robinson-Schensted-Knuth correspondence [3], geometric crystal and tropical R [18, 19], inverse scattering method [17, 25], quantization [6, 10] and so forth.

The automaton is defined on an infinite lattice and these studies are concerned with the behavior without a boundary effect. Solitons undergo factorized scattering during their travel from the left to the right asymptotic regions where they acquire individualities being well separated from each other.

For instance, the standard box-ball system with $n-1$ kinds of balls is a dynamical system on an infinite lattice whose state is an element $p \in \cdots \otimes B_1 \otimes B_1 \otimes \cdots$, where

$B_1 = \{1, 2, \ldots, n\}$ and $1 \in B_1$ is interpreted as the empty box. Typical time evolution of this system goes as follows: (. denotes $1 \in B_1$.)

$$\begin{array}{ccccccccccccccccccc}
. & . & 3 & 3 & 2 & . & . & . & 4 & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & 3 & 3 & 2 & . & 4 & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & 3 & 3 & 4 & 2 & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 3 & . & 4 & 3 & 2 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 3 & . & . & . & 4 & 3 & 2 & .
\end{array}$$

This example shows two solitons of length 3 and 1, moving to the right. The basic ingredient in the crystal formulation of the model is the combinatorial R for $U_q(\widehat{\mathfrak{sl}}_n)$ which is a certain bijection satisfying the Yang-Baxter equation. This system has $\mathfrak{sl}_{n-1}$-invariance and the scattering of the solitons is most naturally described in terms of the combinatorial R for $U_q(\widehat{\mathfrak{sl}}_{n-1})$.

In this paper we formulate a soliton cellular automaton on a semi-infinite lattice having a reflecting end. A new feature here is the reflection at the boundary, which essentially doubles the soliton degrees of freedom into the right and the left moving ones. When there are only right moving solitons, their behavior far away from the boundary is the same as the original box-ball system [24] with $n-2$ kinds of balls. (The case $n = 2$ is trivial and we assume $n \geq 3$ throughout.)

In order to incorporate the left moving solitons we consider the states in the space $\cdots \otimes (B_1 \otimes B_1^\vee) \otimes (B_1 \otimes B_1^\vee)$, where $B_1^\vee = \{\overline{1}, \ldots, \overline{n}\}$ and $1 \otimes \overline{n} \in B_1 \otimes B_1^\vee$ plays the role of the empty box. Then the double row construction of transfer matrices in solvable lattice models [23, 2, 16] is adapted to the crystal setting to generate a commuting family of time evolutions. In the construction we need three kinds of combinatorial R: R, $R^\vee$, $R^{\vee\vee}$ and a combinatorial K: the map K that satisfies the reflection equation (2.13). We find three solutions $K = \mathrm{Rotateleft}, \mathrm{Switch}_{1n}, \mathrm{Switch}_{12}$ of the equation. See (2.10)–(2.12) and Remark 1. By using $R, R^\vee, R^{\vee\vee}$ and K, the commuting family of time evolutions $\{T_l\}$ as well as conserved quantities $\{N_l\}$ counting solitons are constructed.

Our system has asymptotic $\mathfrak{sl}_{n-2}$ symmetry and we identify special patterns $(i_1 \otimes \overline{n}) \otimes (i_2 \otimes \overline{n}) \otimes \cdots \otimes (i_l \otimes \overline{n})$ $(n-1 \geq i_1 \geq i_2 \geq \cdots \geq i_l \geq 2)$ and $(1 \otimes \overline{i_1}) \otimes (1 \otimes \overline{i_2}) \otimes \cdots \otimes (1 \otimes \overline{i_l})$ $(2 \leq i_1 \leq i_2 \leq \cdots \leq i_l \leq n-1)$ that behave as right/left-moving solitons respectively. (For $K = \mathrm{Switch}_{12}$ slightly different patterns are identified. See Sec. 4.1 for details.) Their reflection at the boundary and the scattering involving the left and the right moving ones are identified essentially with the combinatorial K and the combinatorial R of $U_q(\widehat{\mathfrak{sl}}_{n-2})$.

Here is a summary of some characteristic features of the automata associated with $U_q(\widehat{\mathfrak{sl}}_n)$ on the infinite and the semi-infinite lattices.

	infinite system	semi-infinite system
local state	B_1	$B_1 \otimes B_1^\vee$
symmetry	$\mathfrak{sl}_{n-1}$-invariance	asymptotic $\mathfrak{sl}_{n-2}$-invariance
soliton	$\mathrm{Aff}(B_l)$ of $U_q(\widehat{\mathfrak{sl}}_{n-1})$	$\mathrm{Aff}(B_l)$, $\mathrm{Aff}(B_l^\vee)$ of $U_q(\widehat{\mathfrak{sl}}_{n-2})$
reflection rule		K of $U_q(\widehat{\mathfrak{sl}}_{n-2})$
two body scattering rule	R of $U_q(\widehat{\mathfrak{sl}}_{n-1})$	$R, R^\vee, R^{\vee\vee}$ of $U_q(\widehat{\mathfrak{sl}}_{n-2})$

For symmetry, see Sec. 3.1 in [7] and Sec. 3.4 in this paper. The reflection and the scattering rules in the semi-infinite system are obtained (and more precisely stated) in Theorem 1 and Theorem 2, respectively.

The paper is arranged as follows. In Sec. 2, we recall basic facts on crystals, combinatorial R and the Yang-Baxter equation. A combinatorial version of the reflection equation is formulated and the combinatorial K is given. In Sec. 3, we present the automaton with a reflecting end. Conserved quantities $\{E_l\}$ are constructed and the asymptotic $\mathfrak{sl}_{n-2}$-invariance is explained. In Sec. 4, we identify solitons and determine their reflection and scattering rules. We omit most of the proofs but provide several examples. In Sec. 5, we formulate an automaton on a finite lattice surrounded by two reflecting ends and having a commuting family of time evolutions. We illustrate reflecting solitons with a few examples leaving a thorough study as a future problem. Appendix A contains a proof of the reflection equation in a tropical setting.

2 Yang-Baxter and reflection equations

2.1 Crystals $B_l, B_l^\vee$

Here we fix notations concerning the $U_q(\widehat{\mathfrak{sl}}_n)$ crystals used in this paper. We omit standard facts on crystals such as weight decomposition and tensor product rule, etc., for which we refer to [11, 12, 13, 14].

Let B_1 and $B_1^\vee$ be the crystals of the vector and the $n-1$ fold antisymmetric tensor representations of $U_q(\widehat{\mathfrak{sl}}_n)$, and let B_l and $B_l^\vee$ be the ones corresponding to their l fold symmetric tensor product representations. As a set B_l and $B_l^\vee$ are both presented as

$$\{x = (x_1, \ldots, x_n) \in (\mathbb{Z}_{\geq 0})^n \mid x_1 + \cdots + x_n = l\}.$$

This parameterization originates in the basis labeled with the semi-standard tableaux. For B_l, x_i corresponds to the number of the letter i in the length l row shape tableaux. For $B_l^\vee$, x_i corresponds to the number of columns without the letter i in the $n-1$ by l rectangular shape tableaux. A given array $x = (x_1, \ldots, x_n)$ can specify two crystal elements, one in B_l and the other one in $B_l^\vee$. When necessary we distinguish them by writing, for instance, $(2, 0, 1) \in B_3$ as 113 and $(0, 3, 1, 2) \in B_6^\vee$ as $\overline{222344}$, etc.

Action of Kashiwara operators $\tilde{f}_i, \tilde{e}_i (0 \leq i \leq n-1) : B_l \to B_l \sqcup \{0\}$ or $B_l^\vee \to B_l^\vee \sqcup \{0\}$ is defined by

$$\begin{aligned}
&(\tilde{f}_i x)_j = x_j - \delta_{j,i} + \delta_{j,i+1}, \quad (\tilde{e}_i x)_j = x_j + \delta_{j,i} - \delta_{j,i+1} \quad \text{for } x \in B_l, \\
&(\tilde{f}_i x)_j = x_j + \delta_{j,i} - \delta_{j,i+1}, \quad (\tilde{e}_i x)_j = x_j - \delta_{j,i} + \delta_{j,i+1} \quad \text{for } x \in B_l^\vee.
\end{aligned}$$

Here and in the remainder of the paper all the indices are to be understood in $\mathbb{Z}_n$. The right hand side is to be interpreted as 0 if it does not belong to B_l or $B_l^\vee$. We see that the crystal graphs of B_l and $B_l^\vee$ are identical by reversing all the arrows. $\varphi_i(x) = \max\{k \geq 0 \mid \tilde{f}_i^k x \neq 0\}$ and $\varepsilon_i(x) = \max\{k \geq 0 \mid \tilde{e}_i^k x \neq 0\}$ are given by

$$\begin{aligned}
&\varphi(x) = x_i, \quad \varepsilon_i(x) = x_{i+1} \quad \text{for } x \in B_l, \\
&\varphi(x) = x_{i+1}, \quad \varepsilon_i(x) = x_i \quad \text{for } x \in B_l^\vee.
\end{aligned}$$

For crystals B_l and $B_l^\vee$, we define their affinization $\mathrm{Aff}(B_l) = \{z^d x \mid x \in B_l, d \in \mathbb{Z}\}$ and $\mathrm{Aff}(B_l^\vee) = \{z^d x \mid x \in B_l^\vee, d \in \mathbb{Z}\}$. z is called the *spectral parameter*. They also admit the crystal structure by $\tilde{e}_i \cdot z^d x = z^{d+\delta_{i0}}(\tilde{e}_i x)$ and $\tilde{f}_i \cdot z^d x = z^{d-\delta_{i0}}(\tilde{f}_i x)$.

2.2 Combinatorial R and Yang-Baxter equation

The isomorphism of crystals $\mathrm{Aff}(B) \otimes \mathrm{Aff}(B') \overset{\sim}{\to} \mathrm{Aff}(B') \otimes \mathrm{Aff}(B)$ is called the combinatorial R. It has the following form:

$$\begin{aligned} R\ :\ \mathrm{Aff}(B) \otimes \mathrm{Aff}(B') \quad &\longrightarrow \quad \mathrm{Aff}(B') \otimes \mathrm{Aff}(B) \\ z^d x \otimes z^e y \quad &\longmapsto \quad z^{e+H(x\otimes y)}\tilde{y} \otimes z^{d-H(x\otimes y)}\tilde{x}, \end{aligned}$$

where $x \otimes y \mapsto \tilde{y} \otimes \tilde{x}$ under the isomorphism (classical combinatorial R) $B \otimes B' \overset{\sim}{\to} B' \otimes B$. $H(x \otimes y)$ is called the energy function and determined up to a global additive constant by

$$H(\tilde{e}_i(x \otimes y)) = \begin{cases} H(x \otimes y) + 1 & \text{if } i = 0,\ \varphi_0(x) \geq \varepsilon_0(y),\ \varphi_0(\tilde{y}) \geq \varepsilon_0(\tilde{x}), \\ H(x \otimes y) - 1 & \text{if } i = 0,\ \varphi_0(x) < \varepsilon_0(y),\ \varphi_0(\tilde{y}) < \varepsilon_0(\tilde{x}), \\ H(x \otimes y) & \text{otherwise.} \end{cases}$$

In this paper we are concerned with the following combinatorial R:

$$\begin{aligned} R:\ \mathrm{Aff}(B_l) \otimes \mathrm{Aff}(B_m) \quad &\longrightarrow \quad \mathrm{Aff}(B_m) \otimes \mathrm{Aff}(B_l) \\ z^d x \otimes z^e y \quad &\longmapsto \quad z^{e-Q_0(x,y)}\tilde{y} \otimes z^{d+Q_0(x,y)}\tilde{x}, \\ \tilde{x}_i = x_i + Q_i(x,y) - Q_{i-1}(x,y), &\quad \tilde{y}_i = y_i + Q_{i-1}(x,y) - Q_i(x,y), \\ Q_i(x,y) = \min\{\sum_{j=1}^{k-1} x_{i+j} &+ \sum_{j=k+1}^{n} y_{i+j} \mid 1 \leq k \leq n\}. \end{aligned} \tag{2.1}$$

$$\begin{aligned} R^\vee\ :\ \mathrm{Aff}(B_l) \otimes \mathrm{Aff}(B_m^\vee) \quad &\longrightarrow \quad \mathrm{Aff}(B_m^\vee) \otimes \mathrm{Aff}(B_l) \\ z^d x \otimes z^e y \quad &\longmapsto \quad z^{e-P_0(x,y)}\tilde{y} \otimes z^{d+P_0(x,y)}\tilde{x}, \\ \tilde{x}_i = x_i + P_i(x,y) - P_{i-1}(x,y), &\quad \tilde{y}_i = y_i + P_i(x,y) - P_{i-1}(x,y), \\ P_i(x,y) = P_i(y,x) &= \min(x_{i+1}, y_{i+1}). \end{aligned} \tag{2.2}$$

$$\begin{aligned} {}^\vee R:\ \mathrm{Aff}(B_l^\vee) \otimes \mathrm{Aff}(B_m) \quad &\longrightarrow \quad \mathrm{Aff}(B_m) \otimes \mathrm{Aff}(B_l^\vee) \\ z^d x \otimes z^e y \quad &\longmapsto \quad z^{e-P_{-1}(x,y)}\tilde{y} \otimes z^{d+P_{-1}(x,y)}\tilde{x}, \\ \tilde{x}_i = x_i + P_{i-2}(x,y) - P_{i-1}(x,y), &\quad \tilde{y}_i = y_i + P_{i-2}(x,y) - P_{i-1}(x,y). \end{aligned} \tag{2.3}$$

$$\begin{aligned} R^{\vee\vee}\ :\ \mathrm{Aff}(B_l^\vee) \otimes \mathrm{Aff}(B_m^\vee) \quad &\longrightarrow \quad \mathrm{Aff}(B_m^\vee) \otimes \mathrm{Aff}(B_l^\vee) \\ z^d x \otimes z^e y \quad &\longmapsto \quad z^{e-Q_0(y,x)}\tilde{y} \otimes z^{d+Q_0(y,x)}\tilde{x}, \\ \tilde{x}_i = x_i + Q_{i-1}(y,x) - Q_i(y,x), &\quad \tilde{y}_i = y_i + Q_i(y,x) - Q_{i-1}(y,x). \end{aligned} \tag{2.4}$$

Except (2.6) below, ${}^\vee R$ will not be used in the rest of the paper. We have normalized the energy as

$$H(x \otimes y) = \begin{cases} -Q_0(x,y) & \text{for } R, \\ -P_0(x,y) & \text{for } R^\vee, \\ -Q_0(y,x) & \text{for } R^{\vee\vee} \end{cases} \tag{2.5}$$

so as to simplify the definition of the conserved quantity E_l in Sec. 3.3.

These combinatorial R commute with the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$. They satisfy the inversion relations:

$$\begin{aligned} RR &= \mathrm{id}, \\ R^\vee({}^\vee R) = {}^\vee R R^\vee &= \mathrm{id}, \\ R^{\vee\vee} R^{\vee\vee} &= \mathrm{id}, \end{aligned} \tag{2.6}$$

where we have suppressed the l, m-dependence. For instance, the first line actually means that $R_{B_m,B_l} R_{B_l,B_m}$ is the identity map on $\mathrm{Aff}(B_l) \otimes \mathrm{Aff}(B_m)$. They also satisfy the Yang-Baxter equations:

$$\begin{aligned} &(1 \otimes R)(R \otimes 1)(1 \otimes R) = (R \otimes 1)(1 \otimes R)(R \otimes 1), \\ &(1 \otimes R)(R^\vee \otimes 1)(1 \otimes R^\vee) = (R^\vee \otimes 1)(1 \otimes R^\vee)(R \otimes 1), \\ &(1 \otimes R^\vee)(R^\vee \otimes 1)(1 \otimes R^{\vee\vee}) = (R^{\vee\vee} \otimes 1)(1 \otimes R^\vee)(R^\vee \otimes 1), \\ &(1 \otimes R^{\vee\vee})(R^{\vee\vee} \otimes 1)(1 \otimes R^{\vee\vee}) = (R^{\vee\vee} \otimes 1)(1 \otimes R^{\vee\vee})(R^{\vee\vee} \otimes 1), \end{aligned} \tag{2.7}$$

which correspond to the isomorphisms that reverse the order of tensor products $\mathrm{Aff}(B_k) \otimes \mathrm{Aff}(B_l) \otimes \mathrm{Aff}(B_m)$, $\mathrm{Aff}(B_k) \otimes \mathrm{Aff}(B_l) \otimes \mathrm{Aff}(B_m^\vee)$, $\mathrm{Aff}(B_k) \otimes \mathrm{Aff}(B_l^\vee) \otimes \mathrm{Aff}(B_m^\vee)$ and $\mathrm{Aff}(B_k^\vee) \otimes \mathrm{Aff}(B_l^\vee) \otimes \mathrm{Aff}(B_m^\vee)$.

We attach the elements in B_m and $B_m^\vee$ with solid and dotted lines, respectively. The combinatorial R are depicted as in Fig. 1.

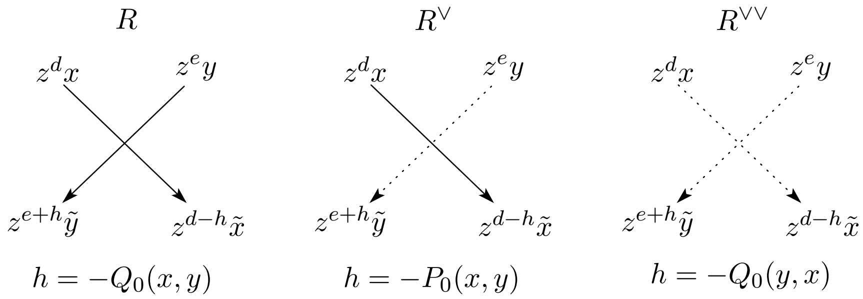


Figure 1. Diagrams for $R, R^\vee$ and $R^{\vee\vee}$

For convenience the classical combinatorial R (combinatorial R up to z^d part) will be denoted by $\overline{R} : B_l \otimes B_m \to B_m \otimes B_l$, $\overline{R}^\vee : B_l \otimes B_m^\vee \to B_m^\vee \otimes B_l$ and $\overline{R}^{\vee\vee} : B_l^\vee \otimes B_m^\vee \to B_m^\vee \otimes B_l^\vee$. The relations like $R(z^d x \otimes z^e y) = z^{\tilde{e}} \tilde{y} \otimes z^{\tilde{d}} \tilde{x}$ and $\overline{R}(x \otimes y) = \tilde{y} \otimes \tilde{x}$ will also be denoted by $z^d x \otimes z^e y \simeq z^{\tilde{e}} \tilde{y} \otimes z^{\tilde{d}} \tilde{x}$ and $x \otimes y \simeq \tilde{y} \otimes \tilde{x}$.

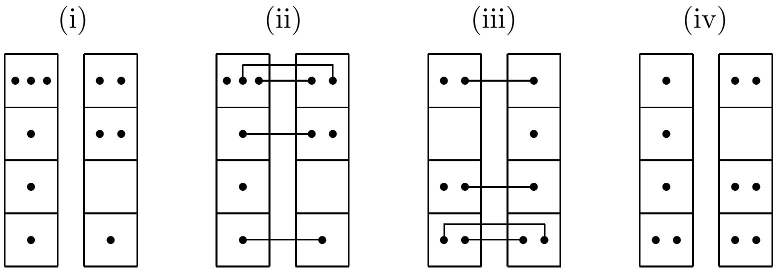
There is a simple rule to compute R graphically [21]. In particular, Q_0 is known as the 'unwinding number'. Although R and $R^{\vee\vee}$ act on different kind of crystals, their formulas are intertwined by the transposition of the components of the tensor product. Thus $R^{\vee\vee}$ is also calculated graphically. Here we illustrate a graphical procedure to compute $R^\vee$ along the example $R^\vee : z^d(3,1,1,1) \otimes z^e(2,2,0,1) \mapsto z^{e-2}(1,1,1,2) \otimes z^{d+2}(2,0,2,2)$, which reads $z^d 111234 \otimes z^e \overline{11224} \simeq z^{e-2} \overline{12344} \otimes z^{d+2} 113344$ in the tableau notation.

(i) Draw pictures corresponding to $(3,1,1,1) \otimes (2,2,0,1)$.

(ii) Connect dots horizontally to make as many pairs as possible.

(iii) Shift the pairs upward by one cyclically leaving the unpaired dots.
(iv) Exchange the left and the right components.

The result yields the image $(1,1,1,2)\otimes(2,0,2,2)$. The minus energy $P_0 = 2$ is the number of pairs sent from the top to the bottom in step (iii).



Example 1. For $\widehat{\mathfrak{sl}}_3$, one has

$$R(z^d 123 \otimes z^e 12) = z^{e+H} 13 \otimes z^{d-H} 122, \quad H = -1,$$
$$R^\vee(z^d 112 \otimes z^e\, \overline{1123}) = z^{e+H}\, \overline{1333} \otimes z^{d-H} 133, \quad H = -2,$$
$$R^{\vee\vee}(z^d\, \overline{12} \otimes z^e\, \overline{2233}) = z^{e+H}\, \overline{1222} \otimes z^{d-H}\, \overline{33}, \quad H = 0.$$

2.3 Combinatorial K and Reflection equation

We introduce the maps

$$\begin{aligned} K: \quad & \mathrm{Aff}(B_l) \longrightarrow \mathrm{Aff}(B_l^\vee) \\ & z^d x \mapsto z^{-d+I(x)} \kappa(x), \end{aligned} \tag{2.8}$$

$$\begin{aligned} K^\vee: \quad & \mathrm{Aff}(B_l^\vee) \longrightarrow \mathrm{Aff}(B_l) \\ & z^d x \mapsto z^{-d-I(x)} \kappa(x). \end{aligned} \tag{2.9}$$

Here we have three choices for the pair (κ, I). For $x = (x_1, x_2, \ldots, x_n)$, they are given by

$$\begin{cases} \kappa(x) = \mathrm{Rotateleft}(x) = (x_2, \ldots, x_n, x_1) \\ I(x) = -x_1 \end{cases} \quad \text{(any } n\text{)}, \tag{2.10}$$

$$\begin{cases} \kappa(x) = \mathrm{Switch}_{1n}(x) = (x_n, x_3, x_2, x_5, x_4, \ldots, x_{n-1}, x_{n-2}, x_1) \\ I(x) = x_n - x_1 \end{cases} \quad \text{(even } n\text{)}, \tag{2.11}$$

$$\begin{cases} \kappa(x) = \mathrm{Switch}_{12}(x) = (x_2, x_1, x_4, x_3, \ldots, x_{n-1}, x_n) \\ I(x) = 0 \end{cases} \quad \text{(even } n\text{)}. \tag{2.12}$$

Rotateleft is a cyclic permutation, whereas Switch_{1n} and Switch_{12} are transposition of adjacent coordinates. We call K and $K^\vee$ the combinatorial K. To the relations (2.8) and (2.9) we assign the diagrams in Fig. 2.

In Fig. 2, the vertical line in K and $K^\vee$ stands for the right and left reflecting end, respectively. The diagonal line segments switch from the one corresponding to $\mathrm{Aff}(B_l)$ and $\mathrm{Aff}(B_l^\vee)$ upon the contact with the reflecting ends. Actually $K^\vee$ is irrelevant to our automaton until Sec. 5, where its classical part is denoted by κ_{left}.

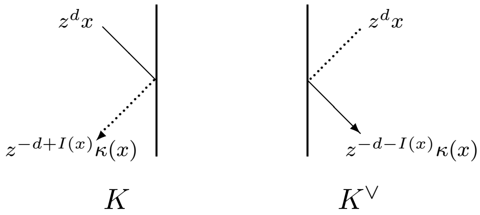


Figure 2. Diagrams for K and $K^\vee$

We let K_2 and $K_1^\vee$ denote the operators acting as

$$K_2(z^d x \otimes z^e y) = z^d x \otimes K(z^e y), \quad K_1^\vee(z^d x \otimes z^e y) = K^\vee(z^d x) \otimes z^e y,$$

where the untouched tensor component can be either B_m or $B_m^\vee$.

Proposition 1 (Combinatorial reflection equation).

$$K_2 R^\vee K_2 R = R^{\vee\vee} K_2 R^\vee K_2, \tag{2.13}$$
$$K_1^\vee R^\vee K_1^\vee R^{\vee\vee} = R K_1^\vee R^\vee K_1^\vee. \tag{2.14}$$

This is proved in a tropical setting in Appendix A. (2.13) is an identity as the maps $\mathrm{Aff}(B_l) \otimes \mathrm{Aff}(B_m) \to \mathrm{Aff}(B_l^\vee) \otimes \mathrm{Aff}(B_m^\vee)$, and (2.14) is the one for $\mathrm{Aff}(B_l^\vee) \otimes \mathrm{Aff}(B_m^\vee) \to \mathrm{Aff}(B_l) \otimes \mathrm{Aff}(B_m)$. They are depicted in Fig. 3. ('Aff' omitted)

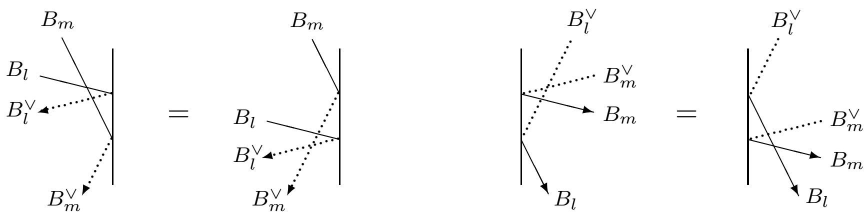


Figure 3. Diagrams for (2.13) and (2.14)

Remark 1. By a computer experiment we have checked for small n that solutions of the combinatorial reflection equation are exhausted by those in (2.10)–(2.12) provided that they are given as a permutation of coordinates $x_1, \dots, x_n$.

3 Automaton on semi-infinite lattice

3.1 States

Consider the one dimensional semi-infinite lattice which extends towards left from an end. To the end we assign the coordinate 1 and successively $2, 3, \dots,$ to the sites located in the left. To each site we assign an element of

$$B = B_1 \otimes B_1^\vee,$$

which we call a local state. Thus there are n^2 local states. We impose the boundary condition that all the local states sufficiently distant from the end are vac $\in B$ defined by

$$\mathrm{vac} = 1 \otimes \kappa(1) = \begin{cases} 1 \otimes \overline{n} & \text{for } \kappa = \text{Rotateleft, Switch}_{1n}, \\ 1 \otimes \overline{2} & \text{for } \kappa = \text{Switch}_{12}. \end{cases} \tag{3.1}$$

See (2.10), (2.11) and (2.12). An assignment of local states to the semi-infinite lattice satisfying the boundary condition is called a state. See Fig. 4.

$$\begin{array}{ccccccccc} & B & & B & & B & B & B \\ \cdots\cdots & \bullet & \text{———} & \bullet & \cdots\cdots\cdots & \bullet & \bullet & \bullet \\ & \mathrm{vac} & & \mathrm{vac} & & p_3 & p_2 & p_1 \end{array}$$

Figure 4. States

Let $\mathcal{P}$ denote the set of states:

$$\mathcal{P} = \{\cdots \otimes p_2 \otimes p_1 \in \cdots \otimes B \otimes B \mid p_k = \mathrm{vac} \ \text{ for } k \gg 1\}. \tag{3.2}$$

Our automaton is a dynamical system on $\mathcal{P}$.

3.2 Time evolution

The dynamics is given by a commuting family of time evolution operators $T_l : \mathcal{P} \to \mathcal{P}$ with $l \in \mathbb{Z}_{\geq 1}$. We set

$$u_l = (l, 0, \ldots, 0) = \overbrace{\boxed{11\ldots1}}^{l} \in B_l,$$
$$u_l^\vee = \kappa(u_l) = \begin{cases} \boxed{\overline{nn\ldots n}} \in B_l^\vee & \text{for } \kappa = \text{Rotateleft, Switch}_{1n}, \\ \boxed{\overline{22\ldots 2}} \in B_l^\vee & \text{for } \kappa = \text{Switch}_{12}. \end{cases}$$

For any l and m, the relations

$$z^d u_l \otimes z^e u_m \simeq z^e u_m \otimes z^d u_l,\ z^d u_l \otimes z^e u_m^\vee \simeq z^e u_m^\vee \otimes z^d u_l,\ z^d u_l^\vee \otimes z^e u_m^\vee \simeq z^e u_m^\vee \otimes z^d u_l^\vee \tag{3.3}$$

are valid. It is easy to verify

Lemma 1. *By iterating $B_l \otimes B \simeq B \otimes B_l$ and $B_l^\vee \otimes B \simeq B \otimes B_l^\vee$, we consider maps*

$$\begin{array}{llcl} (i) & B_l \otimes B \otimes \cdots \otimes B & \stackrel{\sim}{\longrightarrow} & B \otimes \cdots \otimes B \otimes B_l \\ & x \otimes b_1 \otimes \cdots \otimes b_N & \mapsto & \tilde{b}_1 \otimes \cdots \otimes \tilde{b}_N \otimes \tilde{x}, \\ (ii) & B \otimes \cdots \otimes B \otimes B_l^\vee & \stackrel{\sim}{\longrightarrow} & B_l^\vee \otimes B \otimes \cdots \otimes B \\ & c_N \otimes \cdots \otimes c_1 \otimes y & \mapsto & \tilde{y} \otimes \tilde{c}_N \otimes \cdots \otimes \tilde{c}_1, \end{array}$$

where we assume (i) $b_j = vac$ ((ii) $c_j = vac$) for $N - N_0 \leq j \leq N$ with sufficiently large N_0. Then we have (i) $\tilde{x} = u_l$ ((ii) $\tilde{y} = u_l^\vee$).

Pick any state $p \in \mathcal{P}$ and $l \in \mathbb{Z}_{\geq 1}$. By repeated applications of combinatorial R, one can determine another state $T_l(p) \in \mathcal{P}$ along with the subsidiary data $v \in B_l$ and $p^\dagger \in \mathcal{P}$ as follows:

$$\begin{aligned} & B_l \otimes (\cdots \otimes B \otimes B) \simeq (\cdots \otimes B \otimes B) \otimes B_l \\ & u_l \otimes p \simeq p^\dagger \otimes v, \end{aligned} \tag{3.4}$$

$$\begin{aligned} & (\cdots \otimes B \otimes B) \otimes B_l^\vee \simeq B_l^\vee \otimes (\cdots \otimes B \otimes B) \\ & p^\dagger \otimes \kappa(v) \simeq u_l^\vee \otimes T_l(p). \end{aligned} \tag{3.5}$$

In (3.4), the tensor product $u_l \otimes p$ makes sense due to the boundary condition on $\mathcal{P}$ and the property $u_l \otimes \mathrm{vac} \simeq \mathrm{vac} \otimes u_l$ which follows from (3.3). Similarly in (3.5), the right hand side takes the specified form due to the boundary condition, Lemma 1 (ii) and $u_l^\vee \otimes \mathrm{vac} \simeq \mathrm{vac} \otimes u_l^\vee$. The definition of $T_l(p)$ via (3.4) and (3.5) is a crystal theory analogue of the well known double row construction of transfer matrices in solvable lattice models with a boundary. Schematically it is shown in Fig. 5, where the vertical solid lines except the rightmost reflecting wall represent elements of B instead of B_1. (The same convention is used in Fig. 8 and Fig. 10.)

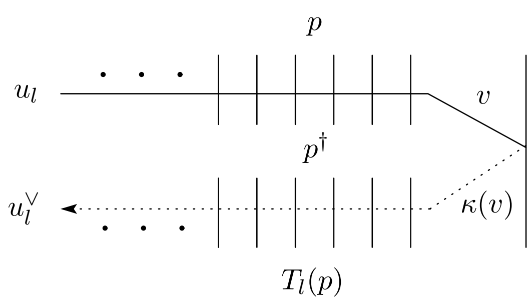


Figure 5. Diagram for $T_l(p)$

The family $\{T_l\}$ possesses a saturation property.

Proposition 2. *For any fixed element $p \in \mathcal{P}$, there exists an integer l_0 such that $T_l(p) = T_{l_0}(p)$ for $l \geq l_0$.*

Proof. For any $1 \leq i \leq n$, let x^+ be the array $x = (x_1, \ldots, x_n)$ with only x_i increased by 1. Suppose $x \otimes y \simeq \tilde{y} \otimes \tilde{x}$ under $\overline{R}, \overline{R}^\vee$ or $\overline{R}^{\vee\vee}$. Then from the concrete form of the combinatorial R, it follows that there exist an integer N such that $x^+ \otimes y \simeq \tilde{y} \otimes (\tilde{x})^+$ for $x_i \geq N$. Similarly there exists an integer N' such that $x \otimes y^+ \simeq (\tilde{y})^+ \otimes \tilde{x}$ for $y_i \geq N'$. The assertion follows from $i = 1, 2, n$ cases of this and the analogous property of the map κ. ■

Remark 2. In the infinite box-ball system without a boundary [24], T_1 is the global translation. This is not the case in the present automaton where the translational invariance is absent. See Example 4.

Example 2. $\widehat{\mathfrak{sl}}_5$, $\kappa =$Rotateleft. Time evolution T_3^t of the state in the first line is presented for $0 \leq t \leq 4$. The symbol $\otimes$ is omitted and .. stands for $\mathrm{vac} = 1 \otimes \overline{5}$ and $4\overline{5} = 4 \otimes \overline{5} \in B$, etc. There supposed to be an infinite tail consisting of .. only in the left on each line.

$$
\begin{array}{ll}
0: & ..\ 4\overline{5}\ 2\overline{5}\ 2\overline{5}\ ..\ ..\ ..\ .. \\
1: & ..\ ..\ ..\ ..\ 4\overline{5}\ 2\overline{5}\ 2\overline{5}\ .. \\
2: & ..\ ..\ ..\ ..\ ..\ ..\ 1\overline{1}\ 4\overline{1} \\
3: & ..\ ..\ ..\ ..\ ..\ 1\overline{3}\ 1\overline{4}\ 1\overline{4} \\
4: & ..\ ..\ 1\overline{3}\ 1\overline{4}\ 1\overline{4}\ ..\ ..\ ..
\end{array}
$$

In this case T_l with any $l \geq 3$ yields the same evolution pattern. The action of T_3 on the $t = 1$ state is calculated according to the rule (3.4) and (3.5) as in Fig. 6.

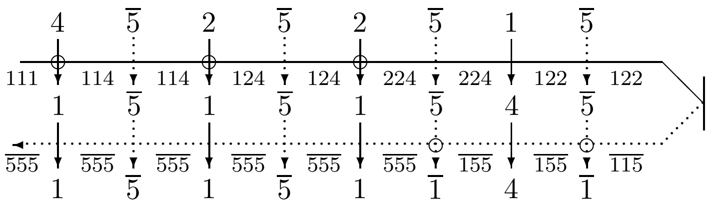


Figure 6. Calculation of T_3 on the $t = 1$ state

In Fig. 6, the combinatorial $\overline{R}^{\vee}$ is acting just as a transposition of the components everywhere. The symbol $\circ$ is put for convenience in Example 5.

Example 3. $\widehat{\mathfrak{sl}}_4$, $\kappa =$Switch$_{14}$. Time evolution T_2^t of the state in the first line is presented for $0 \leq t \leq 7$. The symbol .. stands for vac $= 1 \otimes \overline{4}$ and $3\overline{4} = 3 \otimes \overline{4} \in B$, etc.

$$
\begin{array}{ll}
0: & ..\ ..\ ..\ ..\ 3\overline{4}\ 3\overline{4}\ 2\overline{4}\ ..\ ..\ ..\ ..\ 1\overline{3} \\
1: & ..\ ..\ ..\ ..\ ..\ ..\ 3\overline{4}\ 3\overline{4}\ 2\overline{4}\ ..\ 1\overline{3}\ .. \\
2: & ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 3\overline{4}\ 3\overline{3}\ 2\overline{4}\ .. \\
3: & ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\overline{2}\ ..\ 3\overline{4}\ 2\overline{3} \\
4: & ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\overline{2}\ ..\ 1\overline{2}\ 1\overline{3}\ 1\overline{3} \\
5: & ..\ ..\ ..\ ..\ ..\ 1\overline{2}\ 1\overline{2}\ ..\ 1\overline{3}\ 1\overline{3}\ ..\ .. \\
6: & ..\ ..\ ..\ 1\overline{2}\ 1\overline{2}\ 1\overline{3}\ ..\ 1\overline{3}\ ..\ ..\ ..\ .. \\
7: & ..\ 1\overline{2}\ 1\overline{2}\ 1\overline{3}\ ..\ ..\ 1\overline{3}\ ..\ ..\ ..\ ..\ ..
\end{array}
$$

These states have the same evolution pattern under T_l with any $l \geq 3$.

The action of T_2 on the $t = 2$ state is calculated as in Fig. 7.

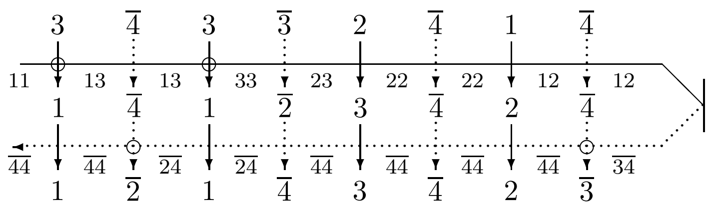


Figure 7. Calculation of T_2 on the $t = 2$ state

Again the symbol $\circ$ is put for convenience in Example 5.

Example 4. $\widehat{\mathfrak{sl}}_4$, κ =Switch$_{12}$. Time evolution T_1^t of the state in the first line is presented for $0 \le t \le 4$. The symbol .. stands for vac $= 1 \otimes \overline{2}$ and $2\overline{3} = 2 \otimes \overline{3} \in B$, etc.

$$\begin{array}{ll}
0: & ..\,..\,..\,..\,..\,..\,..\,..\,..\,2\overline{3}\,..\,..\,..\,..\,4\overline{1}\,..\,..\,..\,..\,..\,..\,..\,..\,..\,2\overline{1}\,..\,..\,.. \\
1: & ..\,..\,..\,..\,..\,..\,..\,..\,1\overline{3}1\overline{1}\,..\,..\,..\,1\overline{4}4\overline{2}4\overline{2}\,..\,..\,..\,..\,..\,..\,..\,1\overline{4}4\overline{1}\,..\,..\,.. \\
2: & ..\,..\,..\,..\,..\,..\,..\,1\overline{3}1\overline{4}\,..\,4\overline{2}\,..\,1\overline{4}\,..\,..\,4\overline{2}4\overline{2}\,..\,..\,..\,..\,..\,1\overline{4}1\overline{4}4\overline{2}4\overline{2}\,..\,.. \\
3: & ..\,..\,..\,..\,..\,..\,1\overline{3}1\overline{4}\,..\,..\,..\,4\overline{4}\,..\,..\,..\,..\,4\overline{2}4\overline{2}\,..\,..\,..\,1\overline{4}1\overline{4}\,..\,..\,4\overline{2}4\overline{2}\,.. \\
4: & ..\,..\,..\,..\,..\,1\overline{3}1\overline{4}\,..\,..\,..\,1\overline{3}\,..\,3\overline{2}\,..\,..\,..\,..\,4\overline{2}4\overline{2}\,..\,1\overline{4}1\overline{4}\,..\,..\,..\,..\,4\overline{2}4\overline{2}
\end{array}$$

On the other hand, time evolution of the initial state under T_l^t with any $l \ge 2$ are the same, and given as follows:

$$\begin{array}{ll}
0: & ..\,..\,..\,..\,..\,..\,..\,..\,..\,2\overline{3}\,..\,..\,..\,..\,4\overline{1}\,..\,..\,..\,..\,..\,..\,..\,..\,..\,2\overline{1}\,..\,..\,.. \\
1: & ..\,..\,..\,..\,..\,..\,..\,1\overline{3}1\overline{4}4\overline{2}\,..\,..\,..\,1\overline{4}\,..\,4\overline{2}4\overline{2}\,..\,..\,..\,..\,..\,1\overline{4}1\overline{4}4\overline{2}4\overline{2}\,..\,.. \\
2: & ..\,..\,..\,..\,..\,1\overline{3}1\overline{4}\,..\,..\,..\,4\overline{2}\,..\,1\overline{4}\,..\,..\,..\,..\,4\overline{2}4\overline{2}\,..\,1\overline{4}1\overline{4}\,..\,..\,..\,..\,4\overline{2}4\overline{2} \\
3: & ..\,..\,..\,1\overline{3}1\overline{4}\,..\,..\,..\,..\,..\,..\,4\overline{4}\,..\,..\,..\,..\,..\,..\,1\overline{3}4\overline{4}3\overline{2}\,..\,..\,..\,..\,..\,1\overline{3}1\overline{3} \\
4: & ..\,1\overline{3}1\overline{4}\,..\,..\,..\,..\,..\,..\,..\,1\overline{3}\,..\,3\overline{2}\,..\,..\,..\,1\overline{3}\;1\overline{3}\,..\,..\,..\,3\overline{2}3\overline{2}\,..\,1\overline{3}1\overline{3}\,..\,..
\end{array}$$

Proposition 3. *$\{T_l\}$ forms a commuting family, i.e., $T_l T_m = T_m T_l$.*

Proof. This is shown by a standard argument based on the successive transformations in Fig. 8. In this figure, each line segment is assigned with an element of a crystal and each cross represents a relation $x \otimes y \simeq \tilde{y} \otimes \tilde{x}$ under $\overline{R}, \overline{R}^\vee$ or $\overline{R}^{\vee\vee}$. A touch with the end stands for an application of κ. The first and the last equalities are due to (3.3). The second and the fourth ones are results of repeated applications of the Yang-Baxter equation. The third one is the reflection equation. ■

3.3 Conserved quantity E_l

To each time evolution T_l, a conserved quantity $E_l : \mathcal{P} \to \mathbb{Z}_{\ge 0}$ is associated. It is a consequence of the affinization of the construction (3.4), (3.5) and Proposition 3. To see this we set $p_i = s_i \otimes t_i$, $p_i^\dagger = s_i^\dagger \otimes t_i^\dagger$ and $T_l(p)_i = s_i' \otimes t_i' \in B = B_1 \otimes B_1^\vee$ in (3.4) and (3.5). Then Fig. 5 is detailed as Fig. 9.

In Fig. 9, we have introduced $v_i, v_i^\sharp \in B_l$ such that $u_l \otimes (\cdots \otimes p_{i+1}) \simeq (\cdots \otimes p_{i+1}^\dagger) \otimes v_i$ and $v_i \otimes s_i \simeq s_i^\dagger \otimes v_i^\sharp$. Similarly, $w_i, w_i^\sharp \in B_l^\vee$ are determined by $(p_{i-1}^\dagger \otimes \cdots \otimes p_1^\dagger) \otimes \kappa(v) \simeq w_i \otimes (T_l(p)_{i-1} \otimes \cdots \otimes T_l(p)_1)$ and $t_i^\dagger \otimes w_i \simeq w_i^\sharp \otimes t_i'$.

Now consider the affinization of Fig. 9. We replace $u_l \in B_l$ by attaching the spectral parameter as $z^0 u_l \in \mathrm{Aff}(B_l)$ and keep track of it along the arrow, namely the affinization of $\ldots, v_i, v_{i-1}, \ldots, v_0, w_1, w_2, \ldots w_i, \ldots$. Due to the boundary condition and the property (3.3), as the classical part of w_i converges to $u_l^\vee$ for i large, the spectral parameter also tends to a finite value. We define $E_l(p) \in \mathbb{Z}$ by saying that $z^0 u_l$ comes back as $z^{-E_l(p)+I(u_l)} u_l^\vee \in \mathrm{Aff}(B_l^\vee)$ as in Fig. 10.

Figure 9 is useful to write down a formula for $E_l(p)$. From (2.1) and (2.2) we have $d = \sum_{i \ge 1}(Q_0(v_i, s_i) + P_0(v_i^\sharp, t_i))$ in Fig. 10. Combining this with a similar calculation in the bottom horizontal arrow we obtain

$$E_l(p) = \sum_{i \ge 1}(Q_0(v_i, s_i) + P_0(v_i^\sharp, t_i)) + \sum_{i \ge 1}(Q_0(w_i, t_i^\dagger) + P_0(w_i^\sharp, s_i^\dagger)) - I(v) + I(u_l). \quad (3.6)$$

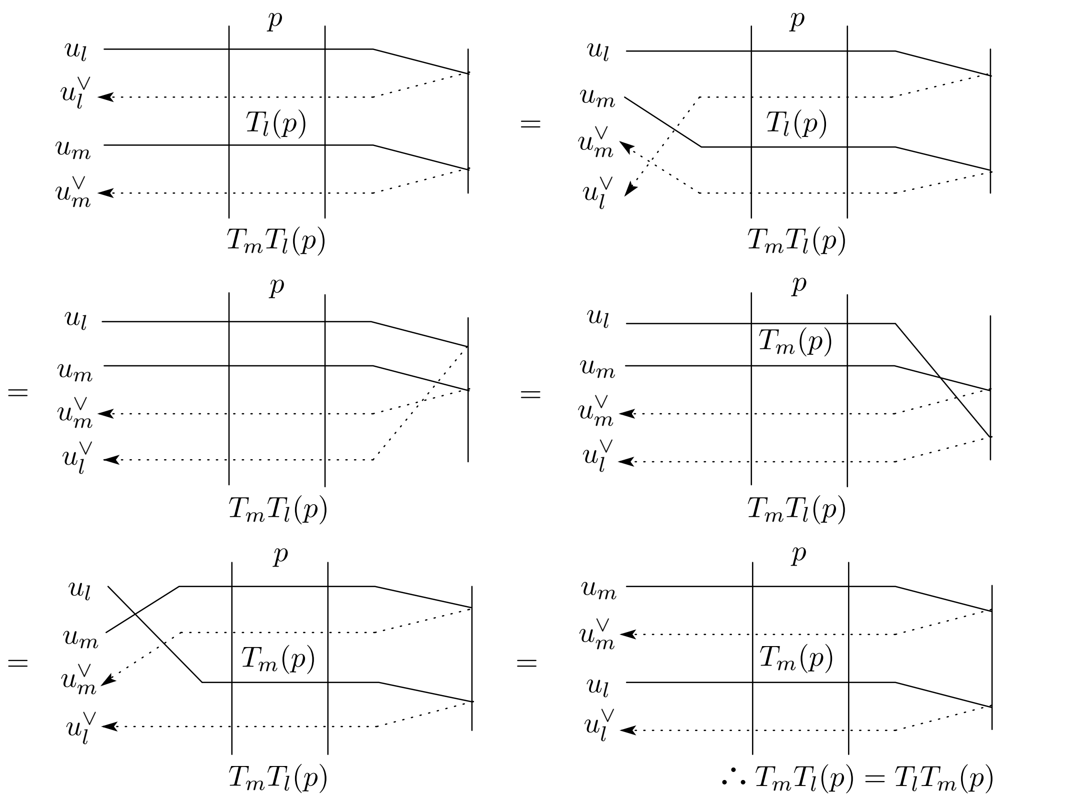


Figure 8. Explanation of $T_mT_l(p) = T_lT_m(p)$

Here $I(u_l) = -l$ (Rotateleft, Switch_{1n}), $= 0$ (Switch_{12}) is a normalization constant for a later convenience.

Proposition 4. *E_l is invariant under time evolutions, i.e., $E_l(T_m(p)) = E_l(p)$ for any $l, m \in \mathbb{Z}_{\geq 1}$ and $p \in \mathcal{P}$.*

Proof. We consider the affinization of the series of identities depicted in Fig. 8, which makes sense since the Yang-Baxter and the reflection equations are both valid in the affine setting. In the first diagram, let the affinization of u_l and u_m be $z^0 u_l$ and $z^0 u_m$, respectively. Then by definition $u_l^\vee$ and $u_m^\vee$ are replaced by $z^{-E_l(p)+I(u_l)}u_l^\vee$ and $z^{-E_m(T_l(p))+I(u_m)}u_m^\vee$, respectively. In the diagrams in Fig. 8, these four elements on the left change their positions only because of (3.3). Therefore in the last diagram they are aligned as $z^0 u_m$, $z^{-E_m(T_l(p))+I(u_m)}u_m^\vee$, $z^0 u_l$ and $z^{-E_l(p)+I(u_l)}u_l^\vee$ from the top to the bottom. On the other hand, the last diagram itself tells that if the first and the third ones are chosen as $z^0 u_m$ and $z^0 u_l$, the second and the fourth ones should be $z^{-E_m(p)+I(u_m)}u_m^\vee$ and $z^{-E_l(T_m(p))+I(u_l)}u_l^\vee$, respectively. Therefore we conclude $E_l(T_m(p)) = E_l(p)$ (and $E_m(T_l(p)) = E_m(p)$ as well). ■

Remark 3. Given a state p, $E_l(p) = E_{l_0}(p)$ holds for any $l \geq l_0$ with the same l_0 as in Proposition 2.

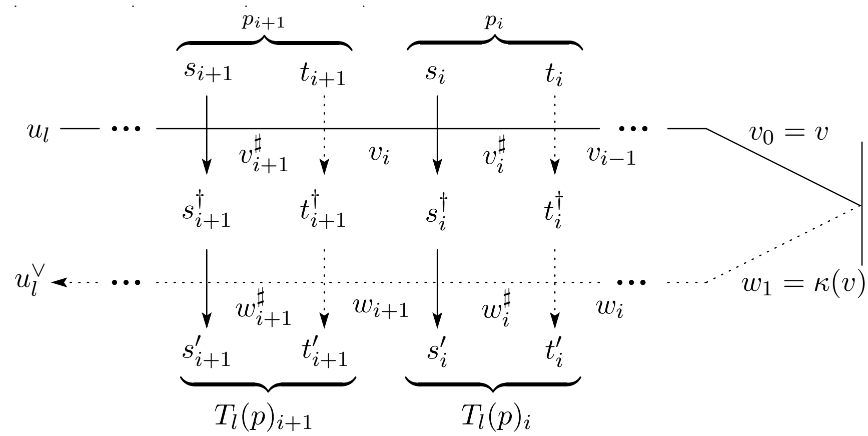


Figure 9. Another diagram for $T_l(p)$

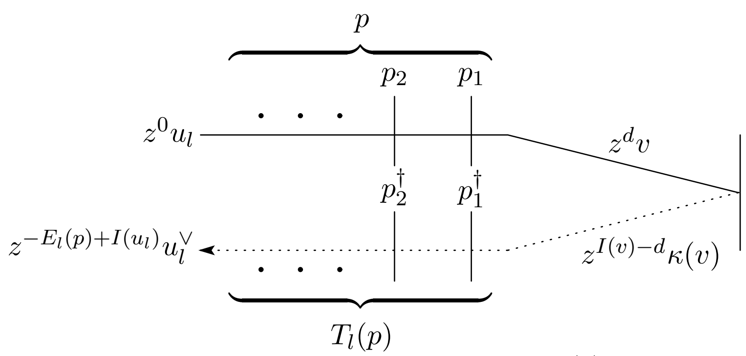


Figure 10. Definition of $E_l(p)$

Example 5. E_l in the previous examples reads

	E_1	E_2	E_3	E_4	$\dots$
Example 2 :	1	2	3	3	$\dots$
Example 3 :	2	3	4	4	$\dots$
Example 4 :	6	10	10	10	$\dots$

In Example 2, $E_3 = 3$ can be read off Fig. 6. Only those vertices marked with $\circ$ contribute to the $i-$sums in (3.6) each by 1, adding up to 5. Together with $I(v) = I(122) = -1$ and $I(u_3) = -3$, one finds $E_3 = 5 + 1 - 3 = 3$. Similarly in Example 3, the $i-$sums for E_2 (3.6) equal to 4 from the marked vertices in Fig. 7. Therefore $E_2 = 4 - I(12) + I(u_2) = 4 - (-1) + (-2) = 3$.

3.4 Asymptotic $\mathfrak{sl}_{n-2}$-invariance

Suppose a Kashiwara operator $\tilde{f}_i$ act on the defining relations (3.4) and (3.5) of T_l as

$$\begin{array}{ccccccc} \tilde{f}_i(u_l \otimes p) & \simeq & \tilde{f}_i(p^\dagger \otimes v), & \qquad & \tilde{f}_i(p^\dagger \otimes \kappa(v)) & \simeq & \tilde{f}_i(u_l^\vee \otimes T_l(p)), \\ || & & || & & || & & || \\ u_l \otimes \tilde{f}_i p & \simeq & \tilde{f}_i p^\dagger \otimes v, & & \tilde{f}_i p^\dagger \otimes \kappa(v) & \simeq & u_l^\vee \otimes \tilde{f}_i T_l(p). \end{array} \tag{3.7}$$

Then by definition the bottom line implies the commutativity $T_l(\tilde{f}_i p) = \tilde{f}_i T_l(p)$. The same argument holds also for $\tilde{e}_i$. The maximal set of Kashiwara operators $\{\tilde{e}_i, \tilde{f}_i\}$ commuting with the time evolutions is the fundamental data to characterize the quantum group symmetry of the system. In the scheme (3.7), the leftmost and the rightmost equalities (for $\tilde{e}_i$ as well) are automatically satisfied if

$$\begin{aligned} &i \in \{2, 3, \ldots, n-2\} \quad &&\text{for } \kappa = \text{Rotateleft},\ \text{Switch}_{1n}, \\ &i \in \{3, 4, \ldots, n-1\} \quad &&\text{for } \kappa = \text{Switch}_{12}. \end{aligned} \tag{3.8}$$

On the other hand the middle two equalities in (3.7) are essentially dependent on $p^\dagger$ and v, and are not guaranteed by the simple condition (3.8). To remedy this, we give up coping with the full set of states $\mathcal{P}$ but confine ourselves to the subset

$$\mathcal{P}_{\text{asy}} = \{p \in \mathcal{P} \mid u_l \otimes p \simeq p^\dagger \otimes u_l \text{ for any } l \text{ with some } (l\text{-dependent})\ p^\dagger \in \mathcal{P}\}. \tag{3.9}$$

We call an element of $\mathcal{P}_{\text{asy}}$ an *asymptotic* state. In view of Lemma 1 (i), the states whose deviation from $\cdots \otimes \text{vac} \otimes \text{vac}$ is sufficiently far from the right end of the lattice are asymptotic states. In (3.9) there are actually finitely many conditions on p, because for sufficiently large l they become equivalent owing to the saturation property explained in the proof of Proposition 2. The set $\mathcal{P}_{\text{asy}}$ is not stable under the time evolutions $\{T_l\}$. For $p \in \mathcal{P}_{\text{asy}}$, one has $v = u_l$ and $\kappa(v) = u_l^\vee$ in (3.7), hence all the equalities therein are valid under the choice (3.8). To summarize we have shown

Proposition 5 (asymptotic $\mathfrak{sl}_{n-2}$-invariance). *For any asymptotic state $p \in \mathcal{P}_{\text{asy}}$ and i specified in (3.8), the commutativity $T_l(\tilde{e}_i p) = \tilde{e}_i T_l(p)$, $T_l(\tilde{f}_i p) = \tilde{f}_i T_l(p)$ is valid.*

Note that the choice of the $\widehat{\mathfrak{sl}}_{n-2}$ subalgebra is dependent on κ. For a comparison, the box-ball system without a boundary associated with $U_q(\widehat{\mathfrak{sl}}_n)$ [24] possesses $\mathfrak{sl}_{n-1}$-invariance. See [5, 4, 7].

Remark 4. The identity $\varepsilon_i(\kappa(v)) = \varepsilon_i(v)(= v_{i+1})$ is valid for $\kappa =$Rotateleft (any i), Switch_{1n} (even i) and Switch_{12} (odd i). Therefore for these i further satisfying (3.8), the commutativity $T_l(\tilde{e}_i p) = \tilde{e}_i T_l(p)$ (resp. $T_l(\tilde{f}_i p) = \tilde{f}_i T_l(p)$) holds under the condition $\varphi_i(p^\dagger) \geq v_{i+1}$ (resp. $\varphi_i(p^\dagger) > v_{i+1}$) in (3.7). There are such p's not belonging to $\mathcal{P}_{\text{asy}}$.

Example 6. A commutative diagram exemplifying the asymptotic $\mathfrak{sl}_2$-invariance in the $\widehat{\mathfrak{sl}}_4$ automaton with $\kappa =\text{Switch}_{14}$ $(.. = 1\overline{4})$.

$$\begin{array}{ccc} ..\ ..\ 1\overline{4}\ 2\overline{1}\ 3\overline{2}\ 1\overline{4}\ 1\overline{4} & \overset{\tilde{e}_2}{\longmapsto} & ..\ ..\ 1\overline{4}\ 2\overline{1}\ 3\overline{3}\ 1\overline{4}\ 1\overline{4} \\ T_2 \downarrow & & \downarrow T_2 \\ ..\ ..\ 1\overline{4}\ 4\overline{3}\ 1\overline{4}\ 3\overline{4}\ 3\overline{4} & \overset{\tilde{e}_2}{\longmapsto} & ..\ ..\ 1\overline{4}\ 4\overline{3}\ 1\overline{4}\ 3\overline{4}\ 2\overline{4} \end{array}$$

The states on the top line are asymptotic states. They exhibit the same evolution pattern under T_l with any $l \geq 2$.

4 Solitons

By a soliton we mean some localized pattern of local states. Under time evolutions it remains stable as long as it is surrounded by sufficiently many vac states. Here we formulate solitons in our automaton and study their scattering and reflection rule. We demonstrate the results mainly with examples and include only sketchy proofs, for they are straightforward or quite similar to those for the infinite systems [8, 4, 7].

In this section we let B'_l and $B'^{\vee}_l$ denote the crystals B_l and $B^{\vee}_l$ for $U_q(\widehat{\mathfrak{sl}}_{n-2})$.

4.1 Basic feature of solitons

Example 2 illustrates a reflection of a soliton at the end. There are two kinds of solitons, one moving to the right and the other one moving to the left. In order to describe them we introduce an injection

$$\imath_l : B'_l \sqcup B'^{\vee}_l \to B^{\otimes l}$$

for each $l \in \mathbb{Z}_{\geq 1}$. In terms of the tableau notation $x = i_1 i_2 \dots i_l \in B'_l$ or $x = \overline{i_1}\,\overline{i_2}\dots\overline{i_l} \in B'^{\vee}_l$ with $1 \leq i_1 \leq \cdots \leq i_l \leq n-2$, it is defined by

$$\kappa = \text{Rotateleft, Switch}_{1n}:$$
$$\imath_l(x) = \begin{cases} \big((i_l+1)\otimes\overline{n}\big)\otimes\cdots\otimes\big((i_2+1)\otimes\overline{n}\big)\otimes\big((i_1+1)\otimes\overline{n}\big) & \text{for } x \in B'_l, \\ \big(1\otimes\overline{i_1+1}\big)\otimes\big(1\otimes\overline{i_2+1}\big)\otimes\cdots\otimes\big(1\otimes\overline{i_l+1}\big) & \text{for } x \in B'^{\vee}_l. \end{cases}$$

$$\kappa = \text{Switch}_{12}:$$
$$\imath_l(x) = \begin{cases} \big((i_l+2)\otimes\overline{2}\big)\otimes\cdots\otimes\big((i_2+2)\otimes\overline{2}\big)\otimes\big((i_1+2)\otimes\overline{2}\big) & \text{for } x \in B'_l, \\ \big(1\otimes\overline{i_1+2}\big)\otimes\big(1\otimes\overline{i_2+2}\big)\otimes\cdots\otimes\big(1\otimes\overline{i_l+2}\big) & \text{for } x \in B'^{\vee}_l. \end{cases}$$

A state of the form

$$\text{......}[l_1]\text{........}[l_2]\text{.....}\cdots\text{.....}[l_m]\text{.......} \tag{4.1}$$

is an example of m-soliton states of length $l_1, l_2, \dots, l_m$ defined generally in the sequel. Here $..[l]..$ denotes a local configuration such as

$$\cdots\otimes\text{vac}\otimes\text{vac}\otimes\imath_l(x)\otimes\text{vac}\otimes\text{vac}\otimes\cdots \qquad \text{for some } x \in B'_l \sqcup B'^{\vee}_l \tag{4.2}$$

surrounded by sufficiently many vac's. It is called a right (resp. left) soliton if $x \in B'_l$ (resp. $x \in B'^{\vee}_l$). We call x the *label* of a soliton. (For (4.1) to be an m-soliton state, it is sufficient (but not necessary in general) to assume there are at least $\text{vac}^{\otimes l_i}$ between $[l_i]$ and $[l_{i+1}]$ if they are both right solitons, $\text{vac}^{\otimes l_{i+1}}$ if the both are left solitons, and $\text{vac}^{\otimes l_i+l_{i+1}}$ if $[l_i]$ is a right and $[l_{i+1}]$ is a left soliton.)

According to these terminologies, Example 2 is the reflection of a length 3 soliton with label 113 into $\overline{233}$.

By a direct calculation one can check

Lemma 2. *Let p be the length l one-soliton state (4.2).*

(1) The k-th conserved quantity of p is given by $E_k(p) = \min(k, l)$.

(2) *Under the evolution $p \to T_k(p)$, the right (resp. left) soliton moves to the right (resp. left) by* $\min(k,l)$ *lattice steps. (For a right soliton we assume there are at least $vac^{\otimes \min(k,l)}$ in its right until the end.)*

For any state p, define the numbers $N_l = N_l(p)$ $(l = 1, 2, \dots)$ by

$$E_k(p) = \sum_{l \geq 1} \min(k,l) N_l \tag{4.3}$$

By definition $\{N_l\}$ is a set of conserved quantities such that $N_l = 0$ for sufficiently large l. See also Remark 3.

Motivated by the property of individual solitons in Lemma 2 and apparent independence of sufficiently separated solitons, we define the number of length l solitons in p to be N_l for $l = 1, 2, \dots$. The total number of solitons is $N_1 + N_2 + \cdots = E_1$. The general definition of m-soliton states is those p satisfying $E_1(p) = m$, and the state (4.1) is such an example. In fact it is not difficult to show

Lemma 3. *Suppose a state of the form $p = \cdots \otimes vac \otimes vac \otimes b \otimes \overbrace{vac \otimes \cdots \otimes vac}^{L}$ with $b \in B^{\otimes l}$ and $L \gg l$ satisfies the one-soliton condition $E_1(p) = 1$. Then $b = \imath_l(x)$ for some $x \in B'_l \sqcup B'^{\vee}_l$.*

Example 7. Example 5 is translated into $\{N_l\}$ as

	N_1	N_2	N_3	N_4	$\dots$	label of solitons
Example 2 :	0	0	1	0	$\dots$	$\overline{233}$ $(t{=}4)$
Example 3 :	1	0	1	0	$\dots$	$\overline{112}$, $\overline{2}$ $(t{=}7)$
Example 4 :	2	4	0	0	$\dots$	$\overline{12}$, $\overline{1}$, 1, 22, $\overline{22}$, 22 (T_1^4 case)

Labels of solitons are not conserved quantities. It is our subject in the following subsections to determine the transformation rule of labels combined with another important data, phase, under collisions and reflection.

4.2 Scattering data

Let us introduce the *position* and the *phase* of a soliton. These are the data defined only for those states where solitons are enough separated as in (4.1). Consider a length l label x soliton

$$\cdots \otimes \mathrm{vac} \otimes \mathrm{vac} \otimes \overbrace{p_{i+l} \otimes \cdots \otimes p_{i+2} \otimes p_{i+1}}^{\imath_l(x)} \otimes \mathrm{vac} \otimes \mathrm{vac} \otimes \cdots$$

appearing as a portion of the state $p = \cdots \otimes p_2 \otimes p_1$. See Fig. 4. We define its position to be $i+1$ if it is a right soliton and $1-i-l$ for a left soliton. By the definition, the position of the length l soliton, either it is a right or left one, always decreases by $\min(k,l)$ under T_k from Lemma 2 (2) as long as it remains isolated. Therefore within such a time interval, the position $\gamma(t)$ of the soliton at time t should be written as $\gamma(t) = -\min(k,l)t + d$ for some d. This constant d is called the phase of the soliton during the time interval in question. The arbitrariness of d due to the freedom $t \to t + \mathrm{const}$ does not matter since we will actually be concerned only with the phase shift from one time interval to another.

To the one-soliton state $...[l]...$ containing a length l soliton with label $x \in B'_l \sqcup B'^{\vee}_l$ and phase d, we assign the *scattering data*:

$$\begin{aligned} z^d x &\in \mathrm{Aff}(B'_l) && \text{for a right soliton,} \\ z^{-d} x &\in \mathrm{Aff}(B'^{\vee}_l) && \text{for a left soliton.} \end{aligned}$$

Similarly the scattering data for the m-soliton state (4.1) is defined by

$$z^{\pm d_1} x_1 \otimes z^{\pm d_2} x_2 \otimes \cdots \otimes z^{\pm d_m} x_m \in \mathrm{Aff}(B'^{\pm}_{l_1}) \otimes \mathrm{Aff}(B'^{\pm}_{l_2}) \otimes \cdots \otimes \mathrm{Aff}(B'^{\pm}_{l_m}), \tag{4.4}$$

where $B'^{+}_l = B'_l, B'^{-}_l = B'^{\vee}_l$ and the $\pm$ symbols are to be taken $+(-)$ for right (left) solitons.

Example 8. $\widehat{\mathfrak{sl}}_5$, κ =Rotateleft. The symbol .. stands for $1\overline{5}$. Time evolution $T_2^t(p)$ for $t = 0, 1$ is given. ($p =$ the middle line.) The top line shows the coordinates of the lattice.

	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
$t=0:$	..	$4\overline{5}$	$3\overline{5}$	$2\overline{5}$	..	..	$4\overline{5}$	$2\overline{5}$	..	..	..	..	$1\overline{3}$	..	..	$1\overline{2}$	$1\overline{4}$
$t=1:$	..	..	..	$4\overline{5}$	$3\overline{5}$	$2\overline{5}$	..	..	$4\overline{5}$	$2\overline{5}$	..	$1\overline{3}$	..	$1\overline{2}$	$1\overline{4}$	..	..

The number of solitons are $N_1 = 1$, $N_2 = 2$ and $N_3 = 1$. Under T_2, phases d are determined by fitting the positions to the formula $-t + d$ for the length 1 soliton and $-2t + d$ for the other solitons. The resulting scattering data is

$$z^{14} 123 \otimes z^{10} 13 \otimes z^4\, \overline{2} \otimes z^1\, \overline{13}$$

for the both $t = 0$ and $t = 1$ states.

4.3 Reflection rule

Let d_1 and d_2 be the phases of a soliton before and after a reflection, respectively. We define the phase shift of the reflection to be $d_2 - d_1$.

Example 9. In Example 2, the position $-3t + d$ and the phase d of the length 3 soliton are given as follows:

t	position	phase
0	5	5
1	2	5
2		
3	-2	7
4	-5	7

The phase shift of this reflection is $7 - 5 = 2$. At $t = 2$, neither data are defined nor needed to determine the phase shift.

Suppose a length l soliton with label $x \in B'_l$ and phase d is transformed into the one with label $y \in B'^{\vee}_l$ and phase $d + \delta$ by the reflection. We introduce the following map on the scattering data that integrates the reflection rule:

$$\begin{aligned} L: \quad \mathrm{Aff}(B'_l) &\longrightarrow \mathrm{Aff}(B'^{\vee}_l) \\ z^d x &\longmapsto z^{-\delta-d} y. \end{aligned} \tag{4.5}$$

We call L the *reflection operator*. (The letter L is chosen as the next one to K.)

Example 10. In view of Example 9, the reflection in Example 2 ($\widehat{\mathfrak{sl}}_5$, $\kappa =$ Rotateleft) is expressed as $L(z^5 113) = z^{-7}\,\overline{233}$.

Theorem 1. *Let $K'_{\text{Rotateleft}}$ and $K'_{\text{Switch}_{12}}$ denote the K defined by (2.10) and (2.12) for $\widehat{\mathfrak{sl}}_{n-2}$, i.e., those with n replaced by n–2, respectively. Then the reflection operator is given by*

$$L = \begin{cases} K'_{\text{Rotateleft}} & \textit{for } \kappa = \textit{Rotateleft}, \\ K'_{\text{Switch}_{12}} & \textit{for } \kappa = \textit{Switch}_{1n},\ \textit{Switch}_{12}. \end{cases}$$

In particular, the phase shift in (4.5) is given by $\delta = -I(x)$.

Proof. We give the proof only for $\kappa =$Rotateleft. By Proposition 3 and Lemma 2 (2) one can assume that at $t = 0$ our right soliton is given by

$$p = \mathrm{vac}^{\otimes L} \otimes i_l\overline{n} \otimes \cdots \otimes i_2\overline{n} \otimes i_1\overline{n},$$

where $1 < i_1 \le i_2 \le \cdots \le i_l < n$ and L is sufficiently large. Set $a = \sharp\{s \mid i_s = 2\}$. A direct calculation shows

$$T_l(p) = \begin{cases} \mathrm{vac}^{\otimes(L+a)} \otimes 1\overline{i_{a+1}-1} \otimes \cdots \otimes 1\overline{i_{l-a}-1} \otimes n\overline{i_{l-a+1}-1} \otimes \cdots \otimes n\overline{i_l - 1} \\ \qquad \text{if } l-2a \ge 0, \\ \mathrm{vac}^{\otimes(L+\frac{l}{2})} \otimes (n\overline{1})^{\otimes\frac{2a-l}{2}} \otimes n\overline{i_{a+1}-1} \otimes \cdots \otimes n\overline{i_l-1} \\ \qquad \text{if } l-2a < 0,\ l \text{ is even}, \\ \mathrm{vac}^{\otimes(L+\frac{l-1}{2})} \otimes 1\overline{1} \otimes (n\overline{1})^{\otimes\frac{2a-l-1}{2}} \otimes n\overline{i_{a+1}-1} \otimes \cdots \otimes n\overline{i_l-1} \\ \qquad \text{if } l-2a < 0,\ l \text{ is odd}, \end{cases}$$

$$T_l^2(p) = \mathrm{vac}^{\otimes(L-l+a)} \otimes 1\overline{i_{a+1}-1} \otimes \cdots \otimes 1\overline{i_l-1} \otimes (1\overline{n-1})^{\otimes a} \otimes \mathrm{vac}^{\otimes(l-a)}.$$

Under our notation this reflection is written as

$$L: \quad z^1 \cdot 1^a (i_{a+1}-1)\cdots(i_l-1) \longmapsto z^{-(1+a)} \cdot (i_{a+1}-2)\cdots(i_l-2)(n-2)^a.$$

This agrees with $K'_{\text{Rotateleft}}$ as desired. ■

In Example 10, $L = K'_{\text{Rotateleft}}$ indeed holds as $I(113) = -2$. Since $I = 0$ in (2.12), no phase shift takes place for $\kappa =$ Switch$_{1n}$ and Switch$_{12}$. The following two are such examples.

Example 11. $\widehat{\mathfrak{sl}}_6$, $\kappa =$Switch$_{16}$. Time evolution under T_3^t for $0 \le t \le 4$. The symbol .. stands for $1\overline{6}$.

0 : .. .. $5\overline{6}$ $4\overline{6}$ $4\overline{6}$ $2\overline{6}$ $2\overline{6}$ .. .. ..
1 : .. .. $5\overline{6}$ $4\overline{6}$ $4\overline{6}$ $2\overline{6}$ $2\overline{6}$
2 : .. .. $1\overline{3}$ $5\overline{3}$ $4\overline{5}$
3 : .. .. $1\overline{3}$ $1\overline{3}$ $1\overline{4}$ $1\overline{5}$ $1\overline{5}$ ..
4 : .. $1\overline{3}$ $1\overline{3}$ $1\overline{4}$ $1\overline{5}$ $1\overline{5}$ .. ..

Fitting the positions of the length 5 soliton to $-3t + d$, we find the phase remains $d = 4$ before and after the reflection. Hence $L(z^4 11334) = z^{-4}\,\overline{22344}$, which agrees with $K'_{\text{Switch}_{12}}(z^4 11334)$.

Example 12. $\widehat{\mathfrak{sl}}_6$, κ =Switch$_{12}$. Time evolution under T_4^t for $0 \le t \le 4$. The symbol .. stands for $1\overline{2}$.

$$\begin{array}{ll}
0: & ..\ 6\overline{2}\ 5\overline{2}\ 5\overline{2}\ 3\overline{2}\ 3\overline{2}\ 3\overline{2}\ ..\ ..\ ..\ ..\ ..\ .. \\
1: & ..\ ..\ ..\ ..\ ..\ 6\overline{2}\ 5\overline{2}\ 5\overline{2}\ 3\overline{2}\ 3\overline{2}\ 3\overline{2}\ ..\ .. \\
2: & ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 6\overline{2}\ 5\overline{2}\ 5\overline{4}\ 3\overline{4} \\
3: & ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\overline{4}\ 1\overline{4}\ 1\overline{4}\ 1\overline{5}\ 1\overline{6}\ 1\overline{6} \\
4: & ..\ ..\ ..\ 1\overline{4}\ 1\overline{4}\ 1\overline{4}\ 1\overline{5}\ 1\overline{6}\ 1\overline{6}\ ..\ ..\ ..\ ..
\end{array}$$

Fitting the positions of the length 6 soliton to $-4t + d$, we find the phase remains $d = 7$ before and after the reflection. Hence $L(z^7 111334) = z^{-7}\,\overline{222344}$, which agrees with $K'_{\mathrm{Switch}_{12}}(z^7 111334)$.

4.4 Scattering rule

Here we consider the scattering among solitons which takes place far from the end without a boundary effect. The result is the same for all the choices κ =Rotateleft, Switch$_{1n}$ and Switch$_{12}$ when expressed in terms of the scattering data (4.4). (All the examples in this subsection are taken from the first two.) In particular the scattering involving only right solitons are essentially the same as the infinite system studied in [5, 4, 27].

Let us observe scattering of two solitons, which is the most fundamental case.

Example 13. $\widehat{\mathfrak{sl}}_5$. Scattering of length 3 and 2 right solitons under T_l with any $l \ge 3$.

$$\begin{array}{l}
\cdots\ ..\ 4\overline{5}\ 3\overline{5}\ 2\overline{5}\ ..\ ..\ 3\overline{5}\ 2\overline{5}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ \cdots \\
\cdots\ ..\ ..\ ..\ ..\ 4\overline{5}\ 3\overline{5}\ 2\overline{5}\ ..\ 3\overline{5}\ 2\overline{5}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ \cdots \\
\cdots\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 4\overline{5}\ 2\overline{5}\ ..\ 3\overline{5}\ 3\overline{5}\ 2\overline{5}\ ..\ ..\ ..\ ..\ \cdots \\
\cdots\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 4\overline{5}\ 2\overline{5}\ ..\ ..\ 3\overline{5}\ 3\overline{5}\ 2\overline{5}\ ..\ \cdots
\end{array}$$

The phases, say d_1 and d_2, of the larger and smaller solitons before the collision have been changed into $d_1 - 3$ and $d_2 + 3$, respectively. In terms of the scattering data the event is expressed as

$$z^{d_1} 123 \otimes z^{d_2} 12 \ \mapsto z^{d_2+3} 13 \otimes z^{d_1-3} 122.$$

Example 14. $\widehat{\mathfrak{sl}}_5$. Scattering of a length 3 right and a length 4 left soliton under T_l with any $l \ge 4$.

$$\begin{array}{l}
\cdots\ ..\ ..\ ..\ ..\ 3\overline{5}\ 2\overline{5}\ 2\overline{5}\ ..\ ..\ ..\ ..\ 1\overline{2}\ 1\overline{2}\ 1\overline{3}\ 1\overline{4}\ ..\ \cdots \\
\cdots\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 3\overline{1}\ 2\overline{2}\ 1\overline{3}\ 1\overline{4}\ ..\ ..\ ..\ ..\ ..\ \cdots \\
\cdots\ ..\ ..\ ..\ ..\ ..\ 1\overline{2}\ 1\overline{4}\ 1\overline{4}\ 5\overline{5}\ 4\overline{5}\ 2\overline{5}\ ..\ ..\ ..\ ..\ ..\ \cdots \\
\cdots\ ..\ 1\overline{2}\ 1\overline{4}\ 1\overline{4}\ 1\overline{4}\ ..\ ..\ ..\ ..\ ..\ ..\ 4\overline{5}\ 4\overline{5}\ 2\overline{5}\ ..\ ..\ \cdots
\end{array}$$

The phases, say d_1 and d_2, of the right and left solitons before the collision have been changed into $d_1 + 2$ and $d_2 + 2$, respectively. In terms of the scattering data the event is expressed as

$$z^{d_1} 112 \otimes z^{-d_2}\, \overline{1123} \ \mapsto z^{-d_2-2}\, \overline{1333} \otimes z^{d_1+2} 133.$$

The same result is valid under T_l with any $l \ge 1$.

Example 15. $\widehat{\mathfrak{sl}}_5$. Scattering of a length 2 and 4 left solitons under T_l with any $l \ge 4$.

$$
\begin{array}{l}
\cdots\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\overline{2}\,1\overline{3}\ ..\ ..\ ..\ ..\ ..\ 1\overline{3}\,1\overline{3}\,1\overline{4}\,1\overline{4}\ ..\ \cdots \\
\cdots\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\overline{2}\,1\overline{3}\ ..\ ..\ ..\ 1\overline{3}\,1\overline{3}\,1\overline{4}\,1\overline{4}\ ..\ ..\ ..\ ..\ ..\ \cdots \\
\cdots\ ..\ ..\ ..\ ..\ ..\ 1\overline{2}\,1\overline{3}\,1\overline{3}\ ..\ ..\ 1\overline{3}\,1\overline{4}\,1\overline{4}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ \cdots \\
\cdots\ ..\ 1\overline{2}\,1\overline{3}\,1\overline{3}\,1\overline{3}\ ..\ ..\ ..\ 1\overline{4}\,1\overline{4}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ \cdots
\end{array}
$$

The phases, say d_1 and d_2, of the smaller and larger solitons before the collision have been changed into d_1+4 and d_2-4, respectively. In terms of the scattering data the event is expressed as

$$z^{-d_1}\,\overline{12} \otimes z^{-d_2}\,\overline{2233} \ \mapsto z^{-d_2+4}\,\overline{1222} \otimes z^{-d_1-4}\,\overline{33}.$$

The same result is valid also under T_3.

In general suppose that the initially enough separated two solitons with length l, m, labels x, y and phases d_1, d_2 are scattered into those with labels $\tilde{x}, \tilde{y}$ and phases $\tilde{d}_1, \tilde{d}_2$, respectively.

$$\ldots\ldots\, \imath_l(x)_{d_1} \ldots\ldots.\, \imath_m(y)_{d_2} \ldots\ldots\ldots \quad \longrightarrow \quad \ldots\ldots\, \imath_m(\tilde{y})_{\tilde{d}_2} \ldots\ldots\ldots\, \imath_l(\tilde{x})_{\tilde{d}_1} \ldots\ldots.,$$

where the phases are attached as further indices and . denotes vac. We define the three kinds of *scattering operators* by

$$
\begin{aligned}
&S: \quad \mathrm{Aff}(B'_l) \otimes \mathrm{Aff}(B'_m) \longrightarrow \mathrm{Aff}(B'_m) \otimes \mathrm{Aff}(B'_l) \quad (l \neq m) && (4.6)\\
&z^{d_1} x \otimes z^{d_2} y \longmapsto z^{\tilde{d}_2} \tilde{y} \otimes z^{\tilde{d}_1} \tilde{x}, \\
&S^{\vee}: \quad \mathrm{Aff}(B'_l) \otimes \mathrm{Aff}({B'}^{\vee}_m) \longrightarrow \mathrm{Aff}({B'}^{\vee}_m) \otimes \mathrm{Aff}(B'_l) && (4.7)\\
&z^{d_1} x \otimes z^{-d_2} y \longmapsto z^{-\tilde{d}_2} \tilde{y} \otimes z^{\tilde{d}_1} \tilde{x}, \\
&S^{\vee\vee}: \quad \mathrm{Aff}({B'}^{\vee}_l) \otimes \mathrm{Aff}({B'}^{\vee}_m) \longrightarrow \mathrm{Aff}({B'}^{\vee}_m) \otimes \mathrm{Aff}({B'}^{\vee}_l) \quad (l \neq m) && (4.8)\\
&z^{-d_1} x \otimes z^{-d_2} y \longmapsto z^{-\tilde{d}_2} \tilde{y} \otimes z^{-\tilde{d}_1} \tilde{x}.
\end{aligned}
$$

These operators are the fundamental objects that integrates the two body scattering rule, which is common under T_k with $k > \min(l, m)$ for S and $S^{\vee\vee}$, and any $k \ge 1$ for $S^{\vee}$. The action of $S, S^{\vee}$ and $S^{\vee\vee}$ is illustrated in Examples 13, 14 and 15, respectively. The operators S and $S^{\vee\vee}$ with $l = m$ are not determined by (4.6) and (4.8) because solitons of equal length and direction do not collide. Here we formally define them by declaring that the following theorem can be extrapolated to the $l = m$ case.

Theorem 2. *Let $R', {R'}^{\vee}$ and ${R'}^{\vee\vee}$ be the combinatorial R defined by (2.1), (2.2) and (2.4) for $\widehat{\mathfrak{sl}}_{n-2}$ with the energy $H(x \otimes y)$ (2.5) replaced with*

$$
\Delta(x \otimes y) := \begin{cases} 2\min(l, m) - Q_0(x, y) & \text{for } R', \\ -P_0(x, y) & \text{for } {R'}^{\vee}, \\ 2\min(l, m) - Q_0(y, x) & \text{for } {R'}^{\vee\vee}. \end{cases} \tag{4.9}
$$

Then scattering operators are given by

$$S = R', \quad S^{\vee} = {R'}^{\vee}, \quad S^{\vee\vee} = {R'}^{\vee\vee}.$$

Proof. Let

$$p = \mathrm{vac}^{\otimes L} \otimes \imath_l(x) \otimes \mathrm{vac}^{\otimes d} \otimes \imath_m(y) \otimes \mathrm{vac}^{\otimes M},$$

be a truncated state in the semi-infinite system. One may observe the scattering after applying the following isomorphism of crystals.

$$\begin{array}{cccc} \Phi : & (B_1 \otimes B_1^\vee)^{\otimes \tilde{L}} & \longrightarrow & B_1^{\otimes \tilde{L}} \otimes (B_1^\vee)^{\otimes \tilde{L}} \\ & p & \longmapsto & p^{(1)} \otimes p^{(2)}, \end{array}$$

since one can show Φ commutes with T_l by the inversion relation (2.6) and the Yang-Baxter equation (2.2). Here $\tilde{L} = L + l + d + m + M$ and we assume L, M are sufficiently large.

First suppose both solitons are right ones. Then one finds $p^{(2)} = \overline{n}^{\otimes \tilde{L}}$ and notices that the transition $p^{(1)} \mapsto$ (the left component of $\Phi(T_l(p))$) is nothing but the time evolution of the original box-ball system. Therefore it is natural to see our scattering rule agrees with that in [5, 4]. If both solitons are left ones, the proof reduces to the above by taking the dual.

The proof is left when $\imath_l(x)$ is a right soliton and $\imath_m(y)$ is a left one. By the asymptotic $\mathfrak{sl}_{n-2}$-invariance (Proposition 5) it suffices to show when p is an $\mathfrak{sl}_{n-2}$ highest weight element, i.e.,

$$p = \mathrm{vac}^{\otimes L} \otimes (2\overline{n})^{\otimes l} \otimes \mathrm{vac}^{\otimes d} \otimes (1\overline{2})^{\otimes a} \otimes (1\overline{n-1})^{\otimes (m-a)} \otimes \mathrm{vac}^{\otimes M}.$$

Here $a \le l$ and one can assume $l + m > d$. Then one has

$$\Phi(p) = 1^L \otimes 2^l \otimes 1^{d+m+M} \otimes \overline{n}^{L+l+d} \otimes \overline{2}^a \otimes \overline{n-1}^{m-a} \otimes \overline{n}^M.$$

(We omitted $\otimes$ in the superscript.) A direct calculation shows

$$T_l^j(\Phi(p)) = 1^{L+jl} \otimes 2^l \otimes 1^{d+m+M-jl} \otimes \overline{n}^{L+l+d-jm} \otimes \overline{2}^a \otimes \overline{n-1}^{m-a} \otimes \overline{n}^{jm+M}.$$

for $j \ge 1$. Applying Φ^{-1} when $j = 3$ (scattering is finished), we get

$$\begin{aligned} T_l^3(p) = {} & \mathrm{vac}^{\otimes (L+l+d-3m+a)} \otimes (1\overline{n-1})^{\otimes m} \otimes \mathrm{vac}^{\otimes (2l+2m-d-2a)} \\ & \otimes (n-1\overline{n})^{\otimes a} \otimes (2\overline{n})^{\otimes (l-a)} \otimes \mathrm{vac}^{\otimes (d+m+M-3l+a)}. \end{aligned}$$

In our notation, this scattering reads as

$$\begin{aligned} & z^{M+m+d+1} \cdot 1^l \otimes z^{-(1-M-m)} \cdot \overline{1}^a \overline{n-2}^{m-a} \\ & \qquad \longmapsto z^{-(1-M-m)-a} \cdot \overline{n-2}^m \otimes z^{M+m+d+1+a} \cdot 1^{l-a}(n-2)^a. \end{aligned}$$

This shows $S^\vee = R'^\vee$. ■

Example 16. The transformations of the scattering data in Examples 13, 14 and 15 should coincide with the $\widehat{\mathfrak{sl}}_3$ combinatorial R with the modified energy (4.9). In fact, they agree with the three formulas in Example 1 with H replaced by $\Delta = 3$ (for R), -2 (for $R^\vee$) and 4 (for $R^{\vee\vee}$).

Remark 5. The phase (d_1, d_2) of the colliding two solitons are changed into $(d_1 - \Delta, d_2 + \Delta)$ under S, $(d_1 - \Delta, d_2 - \Delta)$ under $S^\vee$ and $(d_1 + \Delta, d_2 - \Delta)$ under $S^{\vee\vee}$. Thus the modified energy Δ plays the role of the phase shift. For S and $S^{\vee\vee}$, $\min(l, m) \le \Delta \le 2\min(l, m)$, whereas $-\min(l, m) \le \Delta \le 0$ for $S^\vee$.

4.5 Factorized scattering and reflection

Here we briefly discuss the scattering and reflection in multi-soliton state. Theorem 2 implies that the scattering operators $S, S^\vee$ and $S^{\vee\vee}$ satisfy the Yang-Baxter equations (2.2). Moreover from Theorem 1 and Proposition 1, the reflection operator L and the scattering operators satisfy the reflection equation:

$$L_2 S^\vee L_2 S = S^{\vee\vee} L_2 S^\vee L_2. \tag{4.10}$$

These properties lead to the factorized scattering and reflection. Namely, multi-soliton scattering and reflection is expressed as a composition of two body scattering and single body reflection whose order does not alter the final state. This statement is made most transparent in terms of the scattering data. Let p be an N-soliton state with length $l_1 \geq l_2 \geq \cdots \geq l_N$ and consider the time evolution $T_r^t(p)$ with $r \geq l_1$. The scattering data of $T_r^t(p)$ is convergent in the limit $t \to -\infty$ (resp. $t \to +\infty$), where there are only right (resp. left) solitons. Thus the time evolution specifies a map

$$\mathrm{Aff}(B'_{l_1}) \otimes \cdots \otimes \mathrm{Aff}(B'_{l_N}) \to \mathrm{Aff}({B'}^\vee_{l_1}) \otimes \cdots \otimes \mathrm{Aff}({B'}^\vee_{l_N}) \tag{4.11}$$

between the scattering data. By factorized scattering and reflection it is meant that this map coincides with any product of

$$\begin{array}{cc} 1 \otimes \cdots \otimes 1 \otimes S \otimes 1 \otimes \cdots \otimes 1, & 1 \otimes \cdots \otimes 1 \otimes S^\vee \otimes 1 \otimes \cdots \otimes 1, \\ 1 \otimes \cdots \otimes 1 \otimes S^{\vee\vee} \otimes 1 \otimes \cdots \otimes 1, & 1 \otimes 1 \otimes \cdots \otimes 1 \otimes L \end{array}$$

that achieves the rearrangement (4.11).

Example 17. $\widehat{\mathfrak{sl}}_6, \kappa =$Switch$_{16}$. Time evolution under T_5.

$$\begin{array}{l}
..\ 5\bar{6}\ 4\bar{6}\ 3\bar{6}\ 3\bar{6}\ 2\bar{6}\ ..\ ..\ 5\bar{6}\ 4\bar{6}\ 2\bar{6}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
..\ ..\ ..\ ..\ ..\ ..\ 5\bar{6}\ 4\bar{6}\ 3\bar{6}\ 3\bar{6}\ ..\ 5\bar{6}\ 4\bar{6}\ 2\bar{6}\ 2\bar{6}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 5\bar{6}\ 4\bar{6}\ 3\bar{6}\ ..\ ..\ 5\bar{6}\ 4\bar{6}\ 3\bar{6}\ 2\bar{6}\ 2\bar{6}\ ..\ ..\ .. \\
..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 5\bar{6}\ 4\bar{6}\ 3\bar{6}\ ..\ ..\ ..\ ..\ 5\bar{6}\ 4\bar{2}\ 2\bar{3} \\
..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 5\bar{2}\ 4\bar{2}\ 2\bar{3}\ 1\bar{4}\ 1\bar{5}\ ..\ .. \\
..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ 1\bar{3}\ 1\bar{4}\ ..\ ..\ ..\ 4\bar{6}\ 3\bar{6}\ 2\bar{6}\ .. \\
..\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ 1\bar{3}\ 1\bar{4}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 4\bar{3} \\
..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ 1\bar{3}\ 1\bar{4}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{3}\ 1\bar{5}\ ..\ ..
\end{array}$$

Here scattering and reflection are occurring along the order in the left hand side of the reflection equation (4.10).

$$\begin{array}{l}
..\ 5\bar{6}\ 4\bar{6}\ 3\bar{6}\ 3\bar{6}\ 2\bar{6}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 5\bar{6}\ 4\bar{6}\ 2\bar{6}\ ..\ ..\ .. \\
..\ ..\ ..\ ..\ ..\ ..\ 5\bar{6}\ 4\bar{6}\ 3\bar{6}\ 3\bar{6}\ 2\bar{6}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 5\bar{6}\ 4\bar{6}\ 2\bar{6} \\
..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 5\bar{6}\ 4\bar{6}\ 3\bar{6}\ 3\bar{6}\ 2\bar{6}\ ..\ ..\ ..\ ..\ 1\bar{3}\ 1\bar{4}\ 1\bar{5} \\
..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 5\bar{6}\ 4\bar{2}\ 3\bar{4}\ 2\bar{5}\ 2\bar{6}\ ..\ .. \\
..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{3}\ 1\bar{4}\ ..\ ..\ ..\ 1\bar{2}\ 4\bar{2}\ 2\bar{3} \\
..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{3}\ 1\bar{4}\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ 1\bar{3}\ 1\bar{5}\ ..\ ..\ .. \\
..\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ 1\bar{3}\ 1\bar{4}\ ..\ 1\bar{2}\ 1\bar{3}\ 1\bar{5}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ 1\bar{3}\ 1\bar{4}\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{3}\ 1\bar{5}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..
\end{array}$$

Here scattering and reflection are occurring along the order in the right hand side of the reflection equation (4.10). In terms of the scattering data the both cases are expressed as the same transformation (4.11):

$$z^{d_1}12234 \otimes z^{d_2}134 \mapsto z^{-d_1+4}\,\overline{11223} \otimes z^{-d_2-4}\,\overline{124}$$

for some d_1, d_2 and d_3.

For a scattering without a boundary effect, the factorization is due to the Yang-Baxter equation of the scattering operators only. We finish with such an example involving both right and left solitons.

Example 18. $\widehat{\mathfrak{sl}}_5$. Scattering of two right and one left solitons under T_3. The two patterns correspond to the two sides of the second Yang-Baxter equation in (2.2):

$$(1 \otimes S)(S^\vee \otimes 1)(1 \otimes S^\vee) = (S^\vee \otimes 1)(1 \otimes S^\vee)(S \otimes 1).$$

$$\begin{array}{l}
..\ 4\bar{5}\ 3\bar{5}\ 2\bar{5}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 2\bar{5}\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ ..\ .. \\
..\ ..\ ..\ ..\ 4\bar{5}\ 3\bar{5}\ 2\bar{5}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 2\bar{5}\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ .. \\
..\ ..\ ..\ ..\ ..\ ..\ ..\ 4\bar{5}\ 3\bar{5}\ 2\bar{5}\ ..\ ..\ ..\ ..\ ..\ ..\ 5\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 4\bar{5}\ 3\bar{5}\ 2\bar{5}\ 1\bar{2}\ 1\bar{3}\ 1\bar{4}\ ..\ 4\bar{5}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 5\bar{3}\ 3\bar{5}\ 2\bar{5}\ ..\ ..\ ..\ 4\bar{5}\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{3}\ 1\bar{4}\ ..\ ..\ ..\ ..\ 4\bar{5}\ 3\bar{5}\ 2\bar{5}\ ..\ 4\bar{5}\ ..\ ..\ ..\ ..\ ..\ .. \\
..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{3}\ 1\bar{4}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 4\bar{5}\ 3\bar{5}\ 4\bar{5}\ 2\bar{5}\ ..\ ..\ ..\ .. \\
..\ ..\ 1\bar{2}\ 1\bar{3}\ 1\bar{4}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 3\bar{5}\ ..\ 4\bar{5}\ 4\bar{5}\ 2\bar{5}\ ..
\end{array}$$

$$\begin{array}{l}
..\ 4\bar{5}\ 3\bar{5}\ 2\bar{5}\ ..\ ..\ ..\ 2\bar{5}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ ..\ .. \\
..\ ..\ ..\ ..\ 4\bar{5}\ 3\bar{5}\ 2\bar{5}\ ..\ 2\bar{5}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ .. \\
..\ ..\ ..\ ..\ ..\ ..\ ..\ 4\bar{5}\ ..\ 3\bar{5}\ 2\bar{5}\ 2\bar{5}\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 4\bar{5}\ ..\ ..\ ..\ 3\bar{1}\ 2\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 4\bar{5}\ ..\ 1\bar{2}\ 1\bar{4}\ 5\bar{5}\ 4\bar{5}\ 2\bar{5}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{3}\ 3\bar{4}\ ..\ ..\ ..\ ..\ ..\ 4\bar{5}\ 4\bar{5}\ 2\bar{5}\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{3}\ 1\bar{4}\ ..\ ..\ ..\ 3\bar{5}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 4\bar{5}\ 4\bar{5}\ 2\bar{5}\ ..\ ..\ ..\ .. \\
..\ ..\ 1\bar{2}\ 1\bar{3}\ 1\bar{4}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 3\bar{5}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 4\bar{5}\ 4\bar{5}\ 2\bar{5}\ ..
\end{array}$$

In terms of the scattering data the both patterns are expressed as

$$z^{d_1}123 \otimes z^{d_2}1 \otimes z^{-d_3}\,\overline{112} \mapsto z^{-d_3-2}\,\overline{123} \otimes z^{d_2+2}2 \otimes z^{d_1}133$$

for some d_1, d_2 and d_3.

5 Automaton with two reflecting ends

Here we formulate an automaton on a finite lattice surrounded by two reflecting ends. We will only display its soliton behavior leaving a thorough study as a future problem.

5.1 General case

First we consider a rather general setting where the set of states of the automaton is taken as

$$\mathcal{P} = B^{\epsilon_1}_{m_1} \otimes B^{\epsilon_2}_{m_2} \otimes \cdots \otimes B^{\epsilon_L}_{m_L}, \tag{5.1}$$

where $\epsilon_i = \pm 1$ and $B^+_m = B_m$, $B^-_m = B^\vee_m$. m_i and L are arbitrary positive integers. This is an inhomogeneous system in that the local states belong to different crystals according to the data $\{(\epsilon_i, m_i)\}$. We denote by $\mathcal{P}^{(i)}$ the $L-1$ fold tensor product without the i th component $B^{\epsilon_i}_{m_i}$ in (5.1). Let $\kappa_{\mathrm{right}} : B_l \to B^\vee_l$ and $\kappa_{\mathrm{left}} : B^\vee_l \to B_l$ be any one of the maps (2.10), (2.11) and (2.12). Given $b_1 \otimes \cdots \otimes b_L \in \mathcal{P}$, its time evolution $T^{(i)}(b_1 \otimes \cdots \otimes b_L) = \tilde{b}_1 \otimes \cdots \otimes \tilde{b}_L \in \mathcal{P}$ $(1 \le i \le L)$ is defined as follows:

$$\begin{aligned}
\text{If } b_i \in B_{m_i}, \quad & b_1 \otimes \cdots \otimes b_L \simeq \mathbf{p} \otimes b'_i \in \mathcal{P}^{(i)} \otimes B_{m_i}, \\
& \mathbf{p} \otimes \kappa_{\mathrm{right}}(b'_i) \simeq b''_i \otimes \mathbf{p}' \in B^\vee_{m_i} \otimes \mathcal{P}^{(i)}, \\
& \kappa_{\mathrm{left}}(b''_i) \otimes \mathbf{p}' \simeq \tilde{b}_1 \otimes \cdots \otimes \tilde{b}_L.
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
\text{If } b_i \in B^\vee_{m_i}, \quad & b_1 \otimes \cdots \otimes b_L \simeq b'_i \otimes \mathbf{p} \in B^\vee_{m_i} \otimes \mathcal{P}^{(i)}, \\
& \kappa_{\mathrm{left}}(b'_i) \otimes \mathbf{p} \simeq \mathbf{p}' \otimes b''_i \in \mathcal{P}^{(i)} \otimes B_{m_i}, \\
& \mathbf{p}' \otimes \kappa_{\mathrm{right}}(b''_i) \simeq \tilde{b}_1 \otimes \cdots \otimes \tilde{b}_L.
\end{aligned} \tag{5.3}$$

Here $\simeq$ is obtained by repeated applications of the combinatorial R. The definitions (5.2) and (5.3) are depicted in Fig. 11, where we have assumed $\epsilon_1 = \epsilon_{i-1} = \epsilon_{i+1} = \epsilon_L = 1$. (Otherwise, the corresponding vertical lines therein should be dotted ones.) Unlike the semi-infinite system, we impose no boundary condition as in (3.2).

By using the Yang-Baxter equation (2.2) and the reflection equation (2.13) and (2.14), one can show

Proposition 6. $\{T^{(1)}, \ldots, T^{(L)}\}$ *forms a commuting family, i.e.,* $T^{(i)}T^{(j)} = T^{(j)}T^{(i)}$.

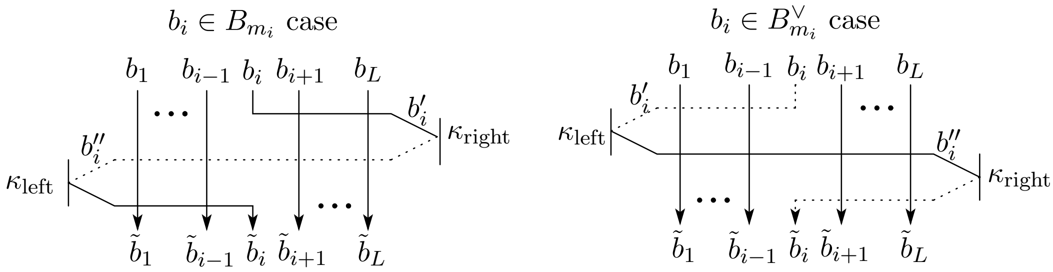


Figure 11. Diagram for $T^{(i)}$

5.2 Soliton behavior

Next we report the soliton behavior in the automaton corresponding to the almost homogeneous specialization of (5.1):

$$\mathcal{P} = B_l \otimes B^{\otimes L}, \tag{5.4}$$

where $B = B_1 \otimes B_1^\vee$ as before and l and L are arbitrary positive integers. According to Proposition 6, there are commuting operators $T^{(1)}, T^{(2)}, \dots, T^{(2L+1)}$. However they are not independent under the choice (5.4). In fact, $T^{(2)} = T^{(4)} = \cdots = T^{(2L)}$ and $T^{(3)} = T^{(5)} = \cdots = T^{(2L+1)}$ can be proved by means of the inversion relation (2.6). In what follows we shall concentrate on $T := T^{(1)}$, which is an analogue of T_l in the semi-infinite system. ($T^{(2)}$ is an analogue of T_1.) For $u \in B_l$ and $p \in B^{\otimes L}$, the time evolution $T(u \otimes p) \in \mathcal{P}$ defined in Sec. 5.1 is rephrased as follows:

$$\begin{aligned} &u \otimes p \simeq p^\dagger \otimes v \in B^{\otimes L} \otimes B_l, \\ &p^\dagger \otimes \kappa_{\text{right}}(v) \simeq w \otimes p' \in B_l^\vee \otimes B^{\otimes L}, \\ &T(u \otimes p) = \kappa_{\text{left}}(w) \otimes p'. \end{aligned} \tag{5.5}$$

In place of a boundary condition, we postulate that there exist an element $a \in B_1$ such that

$$\kappa_{\text{left}}(\kappa_{\text{right}}(a)) = a. \tag{5.6}$$

Under this condition one can let the local state $a \otimes \kappa_{\text{right}}(a) \in B$ play a role analogous to vac (3.1). The condition (5.6) is satisfied only for even n and the following choices:

κ_{left}	κ_{right}	a
Switch_{1n}	Rotateleft	odd
Switch_{12}	Rotateleft	even
Rotateleft	Switch_{1n}	even
Switch_{1n}	Switch_{1n}	arbitrary
Rotateleft	Switch_{12}	odd
Switch_{12}	Switch_{12}	arbitrary

Here the freedom of a can be absorbed into a relabeling and is not essential. By regarding the local state $a \otimes \kappa_{\text{right}}(a)$ as the vacuum one, the present automaton contains the semi-infinite case (with time evolution T_l) as a natural limit $L \to \infty$. By computer experiments we have observed that in each case in the list, the system behaves as a soliton cellular automaton on finite lattice with two reflecting ends. (Unless (5.6) is satisfied, we observed a chaotic behavior.) We include two examples with $a = 1$.

Example 19. $\widehat{\mathfrak{sl}}_4$, $\kappa_{\text{right}} = \text{Rotateleft}$, $\kappa_{\text{left}} = \text{Switch}_{14}$. $\mathcal{P} = B_3 \otimes B^{\otimes 13}$. The symbol .. stands for $1\overline{4}$. The leftmost components $111, 113$ etc. are the tableau representation of the elements in B_3. Successive collisions and reflections of solitons of length 2 and 3.

$$
\begin{array}{l}
111\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ .. \\
111\ ..\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ .. \\
113\ 1\bar{3}\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
113\ 2\bar{2}\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
133\ 4\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
113\ 2\bar{4}\ ..\ 3\bar{4}\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
111\ ..\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ 3\bar{4}\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ ..\ .. \\
111\ ..\ ..\ ..\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ 3\bar{4}\ 3\bar{4}\ 2\bar{4}\ ..\ .. \\
111\ ..\ ..\ ..\ ..\ ..\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ ..\ 3\bar{4}\ 3\bar{1} \\
111\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 3\bar{4}\ 2\bar{4}\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3} \\
111\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 4\bar{2}\ 3\bar{3}\ ..\ ..\ .. \\
111\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ 3\bar{4}\ 2\bar{4}\ .. \\
111\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 3\bar{1} \\
113\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{3} \\
133\ 2\bar{4}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{3}\ ..\ .. \\
111\ ..\ 3\bar{4}\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ .. \\
111\ ..\ ..\ ..\ ..\ 3\bar{4}\ 3\bar{1}\ 1\bar{3}\ ..\ ..\ ..\ ..\ ..\ .. \\
111\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{3}\ 3\bar{4}\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ .. \\
111\ ..\ ..\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ ..\ 3\bar{4}\ 3\bar{4}\ 2\bar{4}\ .. \\
111\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\bar{1}\ 3\bar{2} \\
123\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ .. \\
111\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ .. \\
111\ ..\ ..\ 3\bar{4}\ 2\bar{2}\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
111\ ..\ 1\bar{2}\ 1\bar{2}\ 4\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
133\ 1\bar{3}\ ..\ ..\ ..\ ..\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ ..\ ..\ .. \\
113\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ ..\ ..\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ .. \\
111\ ..\ ..\ 3\bar{4}\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ ..\ 3\bar{4}\ 2\bar{4}\ ..\ ..
\end{array}
$$

Example 20. $\widehat{\mathfrak{sl}}_6$, $\kappa_{\text{right}} = \kappa_{\text{left}} = \text{Switch}_{12}$. $\mathcal{P} = B_4 \otimes B^{\otimes 21}$. The symbol .. stands for

$1\bar{2}$. Successive collisions and reflections of solitons of length 1, 1, 2 and 3.

$$
\begin{array}{l}
1111\; ..\; ..\; ..\; 1\bar{6}\; 1\bar{6}\; 1\bar{6}\; ..\; ..\; ..\; ..\; ..\; 1\bar{3}\; 1\bar{4}\; ..\; ..\; ..\; 1\bar{3}\; ..\; ..\; ..\; 1\bar{4} \\
1111\; 1\bar{6}\; 1\bar{6}\; 1\bar{6}\; ..\; ..\; ..\; ..\; ..\; ..\; 1\bar{3}\; 1\bar{4}\; ..\; ..\; ..\; ..\; 1\bar{3}\; ..\; ..\; ..\; 1\bar{4}\; .. \\
1555\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; 1\bar{3}\; 1\bar{4}\; ..\; ..\; ..\; ..\; ..\; 1\bar{3}\; ..\; ..\; ..\; 1\bar{4}\; ..\; .. \\
1111\; 5\bar{2}\; 5\bar{2}\; 5\bar{2}\; ..\; ..\; 1\bar{3}\; 1\bar{4}\; ..\; ..\; ..\; ..\; ..\; ..\; 1\bar{3}\; ..\; ..\; ..\; 1\bar{4}\; ..\; ..\; .. \\
1111\; ..\; ..\; ..\; 5\bar{3}\; 5\bar{4}\; 5\bar{2}\; ..\; ..\; ..\; ..\; ..\; ..\; 1\bar{3}\; ..\; ..\; ..\; 1\bar{4}\; ..\; ..\; ..\; ..\; .. \\
1111\; ..\; 1\bar{3}\; 1\bar{4}\; ..\; ..\; ..\; 5\bar{2}\; 5\bar{2}\; 5\bar{2}\; ..\; ..\; 1\bar{3}\; ..\; ..\; ..\; 1\bar{4}\; ..\; ..\; ..\; ..\; .. \\
1114\; 1\bar{4}\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; 5\bar{2}\; 5\bar{3}\; 5\bar{2}\; ..\; ..\; 1\bar{4}\; ..\; ..\; ..\; ..\; ..\; .. \\
1114\; 3\bar{2}\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; 1\bar{3}\; ..\; ..\; 5\bar{2}\; 5\bar{4}\; 5\bar{2}\; ..\; ..\; ..\; ..\; ..\; .. \\
1111\; ..\; 4\bar{2}\; 3\bar{2}\; ..\; ..\; ..\; ..\; ..\; 1\bar{3}\; ..\; ..\; ..\; 1\bar{4}\; ..\; ..\; 5\bar{2}\; 5\bar{2}\; 5\bar{2}\; ..\; ..\; .. \\
1111\; ..\; ..\; ..\; 4\bar{2}\; 3\bar{2}\; ..\; ..\; 1\bar{3}\; ..\; ..\; ..\; 1\bar{4}\; ..\; ..\; ..\; ..\; ..\; ..\; 5\bar{2}\; 5\bar{2}\; 5\bar{2} \\
1111\; ..\; ..\; ..\; ..\; ..\; 4\bar{2}\; 3\bar{3}\; ..\; ..\; ..\; 1\bar{4}\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; 1\bar{6}\; 1\bar{6}\; 1\bar{6} \\
1111\; ..\; ..\; ..\; ..\; ..\; ..\; 1\bar{1}\; 4\bar{2}\; ..\; 1\bar{4}\; ..\; ..\; ..\; ..\; ..\; 1\bar{6}\; 1\bar{6}\; 1\bar{6}\; ..\; ..\; .. \\
1111\; ..\; ..\; ..\; ..\; ..\; 1\bar{6}\; ..\; ..\; 6\bar{3}\; 3\bar{2}\; ..\; ..\; 1\bar{6}\; 1\bar{6}\; 1\bar{6}\; ..\; ..\; ..\; ..\; ..\; .. \\
1111\; ..\; ..\; ..\; ..\; 1\bar{6}\; ..\; ..\; 1\bar{3}\; ..\; 1\bar{5}\; 5\bar{6}\; 3\bar{6}\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; .. \\
1111\; ..\; ..\; ..\; 1\bar{6}\; 1\bar{3}\; 1\bar{5}\; 1\bar{6}\; ..\; 1\bar{6}\; ..\; ..\; ..\; 5\bar{2}\; 3\bar{2}\; ..\; ..\; ..\; ..\; ..\; ..\; .. \\
1111\; 1\bar{3}\; 1\bar{6}\; 1\bar{6}\; 1\bar{5}\; ..\; ..\; ..\; 1\bar{6}\; ..\; ..\; ..\; ..\; ..\; ..\; 5\bar{2}\; 3\bar{2}\; ..\; ..\; ..\; ..\; .. \\
1455\; ..\; ..\; 1\bar{5}\; ..\; ..\; ..\; 1\bar{6}\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; 5\bar{2}\; 3\bar{2}\; ..\; ..\; .. \\
1111\; 5\bar{2}\; 5\bar{5}\; 4\bar{2}\; ..\; ..\; 1\bar{6}\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; 5\bar{2}\; 3\bar{2}\; .. \\
1111\; 1\bar{4}\; ..\; ..\; 5\bar{2}\; 4\bar{6}\; 4\bar{2}\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; 5\bar{4} \\
1113\; ..\; ..\; ..\; 1\bar{6}\; ..\; ..\; 5\bar{2}\; 4\bar{2}\; 4\bar{2}\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; 1\bar{4}\; 1\bar{6}\; .. \\
1111\; 3\bar{2}\; ..\; 1\bar{6}\; ..\; ..\; ..\; ..\; ..\; ..\; 5\bar{2}\; 4\bar{2}\; 4\bar{2}\; ..\; ..\; ..\; ..\; 1\bar{4}\; 1\bar{6}\; ..\; ..\; .. \\
1111\; ..\; 3\bar{6}\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; 5\bar{2}\; 4\bar{2}\; 4\bar{4}\; 1\bar{6}\; ..\; ..\; ..\; ..\; .. \\
1111\; 1\bar{6}\; ..\; 3\bar{2}\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; 1\bar{3}\; 1\bar{6}\; ..\; 5\bar{2}\; 4\bar{2}\; 3\bar{2}\; ..\; ..\; .. \\
1115\; ..\; ..\; ..\; 3\bar{2}\; ..\; ..\; ..\; ..\; ..\; ..\; 1\bar{3}\; 1\bar{6}\; ..\; ..\; ..\; ..\; ..\; ..\; 5\bar{2}\; 4\bar{2}\; 3\bar{2} \\
1111\; 5\bar{2}\; ..\; ..\; ..\; 3\bar{2}\; ..\; ..\; ..\; 1\bar{3}\; 1\bar{6}\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; 1\bar{3}\; 1\bar{4}\; 1\bar{6} \\
1111\; ..\; 5\bar{2}\; ..\; ..\; ..\; 3\bar{2}\; 1\bar{3}\; 1\bar{6}\; ..\; ..\; ..\; ..\; ..\; ..\; ..\; 1\bar{3}\; 1\bar{4}\; 1\bar{6}\; ..\; ..\; ..
\end{array}
$$

A Proof of Proposition 1

A.1 Equivalence of (2.14) and (2.13)

Let $P(z^d x \otimes z^e y) = z^e y \otimes z^d x$ be the transposition and let ι be the map $\mathrm{Aff}(B_l) \to \mathrm{Aff}(B_l^\vee)$ defined by $\iota(z^d x) = z^{-d} x$.

Lemma 4.

$$(\iota^{-1} \otimes \iota^{-1}) R^{\vee\vee} (\iota \otimes \iota) = PRP, \tag{A.1}$$

$$(\iota \otimes \iota) K_1^\vee R^\vee K_1^\vee (\iota \otimes \iota) = P K_2 R^\vee K_2 P. \tag{A.2}$$

(A.1) is also written as $(\iota\otimes\iota)R(\iota^{-1}\otimes\iota^{-1}) = PR^{\vee\vee}P$. Thus one has $(\iota\otimes\iota)(2.14)(\iota\otimes\iota) = P(2.13)P$, hence the two relations are equivalent.

Proof. We apply the formulas (2.1), (2.2) and (2.4). Pick any $z^d x \otimes z^e y \in \mathrm{Aff}(B_l) \otimes \mathrm{Aff}(B_m)$ and set $R(z^e y \otimes z^d x) = z^{d-Q_0(y,x)} \tilde{x} \otimes z^{e+Q_0(y,x)} \tilde{y}$. Then the both sides of (A.1) applied to $z^d x \otimes z^e y$ become $z^{e+Q_0(y,x)} \tilde{y} \otimes z^{d-Q_0(y,x)} \tilde{x}$. Similarly, the both sides of (A.2) send $z^d x \otimes z^e y$ to $z^{-e-p+I(\tilde{y})} \kappa(\tilde{y}) \otimes z^{-d+I(x)-p} \tilde{w}$, where $w = \kappa(x)$, $\tilde{y} \otimes \tilde{w} = \overline{R}^\vee (w \otimes y)$ and $p = P_0(w,y) = P_0(y,w)$. ∎

A.2 Classical and affine parts of the reflection equation

In the rest of the appendix, we concentrate on (2.13). We let its two sides act on $z^d x \otimes z^e y \in \mathrm{Aff}(B_l) \otimes \mathrm{Aff}(B_m)$ and name the generated elements as in Fig. 12.

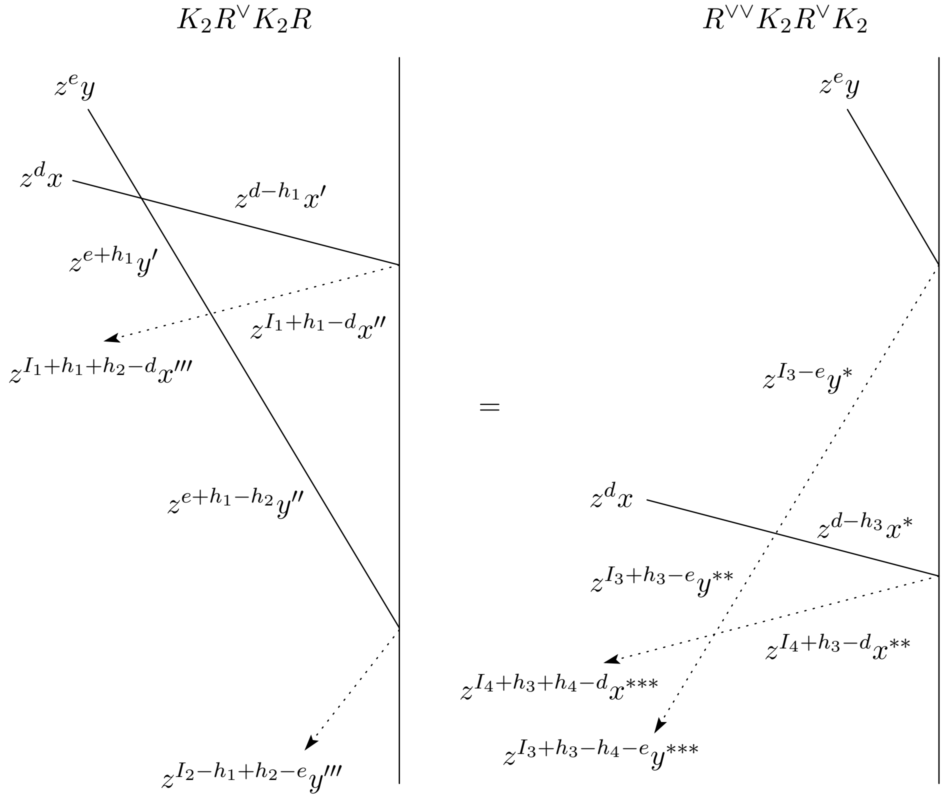


Figure 12. Diagram for (2.13)

Here the energy $h_1, \ldots, h_4$ and $I_1, \ldots, I_4$ are given by

$$
\begin{aligned}
h_1 &= -Q_0(x,y), & I_1 &= I(x'),\\
h_2 &= -P_0(x'',y'), & I_2 &= I(y''),\\
h_3 &= -P_0(x,y^*), & I_3 &= I(y),\\
h_4 &= -Q_0(x^{**},y^{**}), & I_4 &= I(x^*).
\end{aligned}
$$

Comparing the final outputs of the two diagrams in Fig. 12, we find that the reflection equation (2.13) consists of

classical part:

$$x''' = x^{***}, \tag{A.3}$$

$$y''' = y^{***}, \tag{A.4}$$

affine part:

$$Q_0(x, y) + P_0(x'', y') - I(x') = Q_0(x^{**}, y^{**}) + P_0(x, y^*) - I(x^*), \tag{A.5}$$

$$Q_0(x, y) - P_0(x'', y') + I(y'') = Q_0(x^{**}, y^{**}) - P_0(x, y^*) + I(y). \tag{A.6}$$

A.3 Switch to tropical version

There are three cases (2.10), (2.11) and (2.12) to treat, to which not only I in the affine part but also the classical part is dependent. In any case, (A.3) and (A.5) are piecewise linear relations among the $2n$ coordinates $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ involving min through Q_i and P_i functions. This allows us to employ the tropical analysis [15, 22]. Namely, we are going to show the totally positive rational relations obtained by replacing all the $+$, $-$ and min in the original piecewise linear ones by $\times$, $/$ and $+$, respectively. This is justified, at a calculative level, by the simple identities $\lim_{\epsilon\to-0} \epsilon \log(X_1 \times X_2) = x_1 + x_2$, $\lim_{\epsilon\to-0} \epsilon \log(X_1/X_2) = x_1 - x_2$ and $\lim_{\epsilon\to-0} \epsilon \log(X_1 + X_2) = \min(x_1, x_2)$ under the correspondence $X_i = \exp(x_i/\epsilon)$ with $x_i \in \mathbb{R}$ [28]. To save the notation, we let the same letter x_i to denote the tropical coordinates X_i. Now the piecewise linear formulas (2.1), (2.2) and (2.4) are replaced by rational maps $R(x, y)$ called the tropical R [29, 18, 19].

$$R(x, y) = \left(\left(y_i \frac{Q_{i-1}(x, y)}{Q_i(x, y)} \right)_{i=1}^n, \left(x_i \frac{Q_i(x, y)}{Q_{i-1}(x, y)} \right)_{i=1}^n \right), \tag{A.7}$$

$$R^\vee(x, y) = \left(\left(y_i \frac{P_i(x, y)}{P_{i-1}(x, y)} \right)_{i=1}^n, \left(x_i \frac{P_i(x, y)}{P_{i-1}(x, y)} \right)_{i=1}^n \right), \tag{A.8}$$

$$R^{\vee\vee}(x, y) = \left(\left(y_i \frac{Q_i(y, x)}{Q_{i-1}(y, x)} \right)_{i=1}^n, \left(x_i \frac{Q_{i-1}(y, x)}{Q_i(y, x)} \right)_{i=1}^n \right), \tag{A.9}$$

$$Q_i(x, y) = \sum_{1\le k\le n} \prod_{j=1}^{k-1} x_{i+j} \prod_{j=k+1}^{n} y_{i+j}, \quad P_i(x, y) = x_{i+1} + y_{i+1}. \tag{A.10}$$

We write $P_i(x, y)$ and $Q_i(x, y)$ without any change of the entry as $P_i(x, y^*), Q_i(y, x)$, etc. simply as P_i and Q_i. Obviously, Q_i changes into Q_{i+1} under the transformation $(x, y) \to (\mathrm{Rotateleft}(x), \mathrm{Rotateleft}(y))$. One has [29]

$$x_{i+1}Q_{i+1} + y_i Q_{i-1} = (x_i + y_{i+1})Q_i, \tag{A.11}$$

$$x_i + y_{i+1} = x'_{i+1} + y'_i. \tag{A.12}$$

In fact, $x_i Q_i - y_i Q_{i-1} = x_1 \cdots x_n - y_1 \cdots y_n$ is independent of i, showing (A.11). In view of $(y', x') = R(x, y)$ (see Fig. 12), (A.12) is equivalent to (A.11) upon substitution of (A.7).

A.4 $\kappa =$ Rotateleft case (2.10)

First we show the classical part $(x''', y''') = (x^{***}, y^{***})$. From Fig. 12, we know $(x''', y'') = R^{\vee}(y', x'')$, $x''_i = x'_{i+1}$ and $y'''_i = y''_{i+1}$. Thus x''' and y''' are calculated as

$$x'''_i \overset{(A.8)}{=} x''_i \frac{P_i(y', x'')}{P_{i-1}(y', x'')} \overset{(A.10)}{=} x'_{i+1} \frac{x'_{i+2} + y'_{i+1}}{x'_{i+1} + y'_i} \overset{(A.12),(A.7)}{=} \frac{x_{i+1}(x_{i+1} + y_{i+2})Q_{i+1}}{(x_i + y_{i+1})Q_i},$$
$$y'''_i \overset{(A.8)}{=} y'_{i+1} \frac{P_{i+1}(y', x'')}{P_i(y', x'')} \overset{(A.10)}{=} y'_{i+1} \frac{x'_{i+3} + y'_{i+2}}{x'_{i+2} + y'_{i+1}} \overset{(A.12),(A.7)}{=} \frac{y_{i+1}(x_{i+2} + y_{i+3})Q_i}{(x_{i+1} + y_{i+2})Q_{i+1}}.$$

Similarly, Fig. 12 tells $(x^{***}, y^{***}) = R^{\vee\vee}(y^{**}, x^{**})$, hence

$$x^{***}_i = x^{**}_i \frac{Q_i(x^{**}, y^{**})}{Q_{i-1}(x^{**}, y^{**})}, \quad y^{***}_i = y^{**}_i \frac{Q_{i-1}(x^{**}, y^{**})}{Q_i(x^{**}, y^{**})}. \tag{A.13}$$

By using (2.10) and (A.8) we find

$$x^{**}_i = x_{i+1} \frac{x_{i+2} + y_{i+3}}{x_{i+1} + y_{i+2}}, \quad y^{**}_i = y_{i+1} \frac{x_{i+1} + y_{i+2}}{x_i + y_{i+1}}. \tag{A.14}$$

From (A.14) it is easy to show

$$Q_i(x^{**}, y^{**}) = \frac{x_{i+1} + y_{i+2}}{x_{i+2} + y_{i+3}} Q_{i+1}. \tag{A.15}$$

Upon substitution of (A.14) and (A.15), x^{***}_i and y^{***}_i in (A.13) coincide with x'''_i and y'''_i obtained in the above.

Next we proceed to the affine part (A.5) and (A.6), which read, in the tropical setting as

$$(x''_1 + y'_1)x'_1 Q_0 = (x_1 + y^*_1)x^*_1 Q_0(x^{**}, y^{**}),$$
$$\frac{Q_0}{(x''_1 + y'_1)y''_1} = \frac{Q_0(x^{**}, y^{**})}{(x_1 + y^*_1)y_1}.$$

By applying $x''_1 = x'_2$, $x'_1 = x_1 Q_1/Q_0$, $x^*_1 = x_1(x_2 + y_3)/(x_1 + y_2)$ to the former, and $y''_1 = y'_1(x''_2 + y'_2)/(x''_1 + y'_1) = y_1 Q_0(x'_3 + y'_2)/((x''_1 + y'_1)Q_1)$ to the latter along with $y^*_1 = y_2$, they are simplified into

$$(x'_2 + y'_1)x_1 Q_1 = x_1(x_2 + y_3)Q_0(x^{**}, y^{**}),$$
$$\frac{Q_1}{x'_3 + y'_2} = \frac{Q_0(x^{**}, y^{**})}{x_1 + y_2},$$

which are obvious from (A.12) and (A.15).

A.5 $\kappa =$ Switch$_{1n}$, Switch$_{12}$ cases (2.11), (2.12)

Here we assume n is even. Set $\mathrm{Switch}_{12}(x) = (x_{\underline{1}}, x_{\underline{2}}, \ldots, x_{\underline{n}})$, namely, $\underline{i} = i - 1\,(i + 1)$ for i even (odd). A direct calculation leads to

$$x^{**}_i = x_{\underline{i}} \frac{\alpha_{\underline{i}+1}}{\alpha_{\underline{i}}}, \quad y^{**}_i = y_{\underline{i}} \frac{\alpha_{i+1}}{\alpha_i}, \quad \alpha_i = x_i + y_{\underline{i}}. \tag{A.16}$$

Lemma 5. *For $\kappa = Switch_{12}$,*

$$Q_i(x^{**}, y^{**}) = \begin{cases} Q_i & i \ even, \\ \frac{(x_{i+1}+y_i)(x_iQ_{i+1}+y_{i+1}Q_{i-1})}{(x_i+y_{i+1})(x_{i+2}+y_{i+3})} & i \ odd. \end{cases}$$

For $\kappa = Switch_{1n}$, the same relation with the opposite alternative with respect to the parity of i holds.

Proof. It suffices to show Switch_{12} case only. Suppose $Q_i(x^{**}, y^{**}) = Q_i$ for any even i. Then in (A.11) with odd i, replacement of (x, y) by (x^{**}, y^{**}) and application of (A.16) lead to the sought formula for $Q_i(x^{**}, y^{**})$ with odd i. Thus we are left with i even case, which is done by checking $Q_0(x^{**}, y^{**}) = Q_0$ only. We use the expression $Q_0 = \sum_{k=1}^{n/2} x_1x_2 \cdots x_{2k-2}\alpha_{2k-1}y_{2k+1}y_{2k+2} \cdots y_n$. Under the replacement (x, y) by (x^{**}, y^{**}), α_{2k-1} changes into α_{2k+1} while $x_1 \cdots x_{2k-2}$ and $y_{2k+1}y_{2k+2} \cdots y_n$ acquire the extra factors α_{2k-1}/α_1 and α_1/α_{2k+1}, respectively. Hence the summand for each k remains invariant. ∎

First we prove the classical part $x''' = x^{***}$ and $y''' = y^{***}$ in Switch_{12} case. By using (A.12), we have

$$x_i''' = \frac{x_{i+1}(x_iQ_{i+1} + y_{i+1}Q_{i-1})}{Q_{i-1}(x_i + y_{i+1})} \ (i \text{ odd}), \quad \frac{Q_ix_{i-1}(x_{i+1} + y_{i+2})}{x_{i-1}Q_i + y_iQ_{i-2}} \ (i \text{ even}),$$
$$y_i''' = \frac{Q_{i-1}y_{i+1}(x_{i+2} + y_{i+3})}{x_iQ_{i+1} + y_{i+1}Q_{i-1}} \ (i \text{ odd}), \quad \frac{y_{i-1}(x_{i-1}Q_i + y_iQ_{i-2})}{Q_i(x_{i-1} + y_i)} \ (i \text{ even}).$$

On the other hand,

$$x_i^{***} = x_i^{**}\frac{Q_i(x^{**}, y^{**})}{Q_{i-1}(x^{**}, y^{**})}, \quad y_i^{***} = y_i^{**}\frac{Q_{i-1}(x^{**}, y^{**})}{Q_i(x^{**}, y^{**})}.$$

Applying Lemma 5 and (A.16), one finds $x_i''' = x_i^{***}$ and $y_i''' = y_i^{***}$.

Second we show $x''' = x^{***}$ and $y''' = y^{***}$ for Switch_{1n}. This is a corollary of Switch_{12} case. To see this, note that x_i''' in the two cases have the same expressions except the opposite alternative concerning the parity of i, and the same holds for x_i^{***}, y_i''' and y_i^{***} as well.

Third we prove the affine part (A.5) and (A.6) in Switch_{12} case. Since I (2.12) is trivial, we are to show

$$Q_0 = Q_0(x^{**}, y^{**}), \quad P_0(x'', y') = P_0(x, y^*).$$

The former is due to Lemma 5, and it is easy to check the both sides of the latter agree with $x_1 + y_2$ by using (A.12).

Finally we prove (A.5) and (A.6) in Switch_{1n} case. Their tropical version read

$$Q_0P_0(x'', y')\frac{x_1'}{x_n'} = Q_0(x^{**}, y^{**})P_0(x, y^*)\frac{x_1^*}{x_n^*},$$
$$\frac{Q_0y_n''}{P_0(x'', y')y_1''} = \frac{Q_0(x^{**}, y^{**})y_n}{P_0(x, y^*)y_1}.$$

With the help of Lemma 5 and (A.12), these relations can be directly checked. This completes the proof of Proposition 1.

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