

Fast Root-finding of Nonlinear Equations in Geometric Computation

Changchun Geng, Zhong Li, Tianhe Zhou, Bin Yang
 Department of Mathematical Sciences
 Zhejiang Sci-Tech University
 Hangzhou, 310018, China
 e-mail: lizhong@zstu.edu.cn

Abstract—Computing the roots of polynomials is an important issue in various geometric problems. In this paper, we introduce a new family of iterative methods with sixth and seventh order convergence for nonlinear equations (or polynomials). The new method is obtained by combining a different fourth-order iterative method with Newton’s method and using the approximation based on the divided difference to replace the derivative. It can improve the order of convergence and reduce the required number of functional evaluations per step. Numerical comparisons demonstrate the performance of the presented methods.

Keywords - Newton’s method; Convergence order; Divided difference; Non-linear equation

I. INTRODUCTION

Efficient and robust algorithms for finding the solutions of (systems of) polynomial equations are frequently used in geometric computing, modeling and processing, such as curves and surface intersection, surface rendering by ray-tracing, collision detection, etc [1].

Recently, Barton and Juttler provided the quadratic clipping to compute all the roots of a univariate polynomial equation within an interval whose convergence order is 3 for single roots [2]. Liu et al generalized it to the cubic clipping to compute the roots of polynomial equations which can attain the fourth-order convergence for single roots [3]. Both methods are based on the degree reduction of the polynomial approximation and may suffer numerical instability. They actually can be converted into the problem for computing roots of non-linear equations. In this paper, we study the iterative method to find a simple root α to get $f(\alpha)=0$ and $f'(\alpha)\neq 0$ of a non-linear equation $f(x)=0$.

Classical Newton’s iteration is the well-known method which converges quadratically for a single non-linear equation. Some higher order iterative methods have recently been proposed and analyzed for solving nonlinear equations. In order to improve the order of convergence, a known technique consists of the combining two iterative methods of orders p and q , to obtain a method of order pq [4], respectively. However, the process increases the number of functional evaluations per step. We can use some approximations of the derivatives to reduce the number of functional evaluations, still maintaining the

convergence order as high as possible to improve the efficiency of the proposed method.

For example, Parhi and Gupta [5] composed Weerakoon and Fernando’s third order method [6] with Newton’s method and approximated the derivative by linear interpolation to obtain a sixth-order method. Sharma and Guha [7] derived a one-parameter family of sixth order methods based on Ostrowski fourth order multipoint method. Mir and Zaman in [8] composed Ostrowski’s method with Halley’s method, introducing a parameter that allowed them to obtain families of iterative methods with sixth, seventh and eighth orders of convergence. Cordero and Hueso [9] provided an iterative method with sixth and seventh order convergence helped by using Kou et al’s fourth-order method [10].

In this paper, we present two new three-step iterative methods for solving nonlinear equations with sixth and seventh convergence order. The new methods are obtained by composing known methods with Newton’s method and using an adequate approximation for the last derivative. The efficiency index of these methods are 1.565 and 1.627, which makes them competitive and the new methods need only three functions and one first derivative, respectively.

II. DESCRIPTION OF THE METHODS

In order to compose new methods, we recall some notions about divided differences. Let us consider the nonlinear equation $f(x)$ which is a real function defined on R , where R is the smallest interval containing $n+1$ distinct nodes $[x_0, x_1, \dots, x_n]$. The divided difference $f[x_0, x_1, \dots, x_n]$ with n th-order is defined as follows:

$$f[x_0] = f(x_0),$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0},$$

...

$$f[x_0, x_1, \dots, x_n] = \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_1 - x_0}.$$

When $f^n(x)$ is continuous in the smallest interval R which contains x_0, x_1, \dots, x_n , we have

$$f[x_0, x_1, \dots, x_n] = \int_0^1 ds_1 \int_0^{s_1} ds_2 \dots \int_0^{s_{n-1}} f^n((1-s_1)x_0 + (s_1-s_2)x_1 + \dots + (s_{n-1}-s_n)x_{n-1} + s_n x_n) ds_n.$$

In our previous work, we obtained the following iterative method

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - h(\mu_n) \cdot \frac{f(y_n)}{f[y_n, x_n]}, \end{cases} \quad (1)$$

where $\mu_n = \frac{f(y_n)}{f(x_n)}$ and $h(\mu_n)$ represents a real-valued function.

We have proved that when $h(0)=1, h'(0) \neq 1$ and $|h''(0)| < \infty$, the convergence order of the iterative method is three, when $h(0)=1, h'(0)=1$ and $|h'''(0)| < \infty$, the convergence order of the iterative method is four.

Here, we use the divided difference with Newton's method and (1) to get a new three-step family of iterative methods.

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - h(\mu_n) \frac{f(y_n)}{f[y_n, x_n]}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}. \end{cases} \quad (2)$$

We have following theorem for the proposed method (2).

Theorem1. Let $\alpha \in I$ be a simple zero of a sufficiently function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ in an open interval I . If x_0 is sufficiently close to α , then the method defined by (2) have convergence order at six for $h(0)=1, h'(0) \neq 1$ and $|h''(0)| < \infty$. Furthermore, when $h(0)=1, h'(0)=1$ and $|h'''(0)| < \infty$, the order is seven.

Proof: Let α be a single root of $f(x), e_n = x_n - \alpha$. By using the Taylor expansion around $x=\alpha$ and taking into account $f(\alpha)=0$, we have

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + o(e_n^8)],$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + 7c_7 e_n^6 + o(e_n^7)],$$

where $c_k = \frac{f^{(k)}(\alpha)}{k! f'(\alpha)}, k = 2, 3, \dots$.

From above two formulas, we know

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + (2c_2^2 - 2c_3) e_n^3 - (4c_2^3 - 7c_2 c_3 + 3c_4) e_n^4 + (8c_2^4 - 20c_2^2 c_3 + 6c_3^2 + 10c_2 c_4 - 4c_5) e_n^5 - (16c_2^5 - 52c_2^3 c_3 + 28c_2^2 c_4 - 17c_3 c_4 + 33c_3^2 c_2 - 13c_2 c_5 + 5c_6) e_n^6 + (22c_3 c_5 - 36c_2^2 c_5 - 6c_7 + 16c_2 c_6 - 92c_2 c_3 c_4 + 12c_4^2 + 72c_4 c_2^3 - 18c_3^3 + 126c_2^2 c_3^2 - 128c_2^4 c_3 + 32c_2^6) e_n^7 + o(e_n^8).$$

Then we can get the expansions of $d_n, f(y_n), \mu_n, h(\mu_n), f[y_n, x_n], z_n, f(z_n)$ and $f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)$.

After the careful computation, we obtain

$$e_{n+1} = c_2^3 (-1 + h(0))^2 e_n^4 - 2c_2^2 (h(0) - 1) (-3c_3 h(0) + 3h(0)c_2^2 - h'(0)c_2^2 + 2c_3 - 2c_2^2) e_n^5 + \frac{1}{h(0)} (c_2 (-14h(0)c_2^2 h'(0)c_3 + 16h(0)^2 c_2^2 h'(0)c_3 - 21c_2^2 h(0)c_3 + 6c_2 h(0)c_4 + 16h(0)c_2^4 h'(0) - h(0)h''(0)c_2^4 + 60h(0)^2 c_2^2 c_3 - 15h(0)^2 c_2 c_4 - 41h(0)^3 c_2^2 c_3 - 18h(0)^2 c_2^4 h'(0) + h(0)^2 c_2^4 h'''(0) + 9h(0)^3 c_2 c_4 + h(0)h'(0)^2 c_2^4 - c_3 h(0)^4 c_2^2 + 17h(0)^3 c_2^4 + 12h(0)^3 c_3^2 + 2h(0)^4 c_2^4 + 10h(0)c_2^4 + c_2^2 c_3 - 28h(0)^2 c_2^4 + 4h(0)c_3^2 - 16h(0)^2 c_3^2) e_n^6 + (32c_2^3 c_4 - 18h(0)^3 c_2^6 + 27h(0)^3 c_2^4 c_3 + \dots - 20c_2^6) e_n^7 + o(e_n^8). \quad (3)$$

We can easily find when $h(0)=1$ and $h'(0) \neq 1$, the coefficients of e_n^4, e_n^5 in (3) are zero, and the coefficients of e_n^6 are not zero. The convergence order of the iterative method (2) is six.

When $h(0)=1$ and $h'(0)=1$, the coefficients of e_n^4, e_n^5 and e_n^6 in (3) are zero, and the coefficients of e_n^7 are not zero. The convergence order of the iterative method (2) is seven.

Remark1. Per iteration of the present method (2) requires three evaluations of the function and one of its first derivative. The efficiency index is usually used to compare different methods in the literature. We recall that the index [11] is defined as $p^{1/d}$, where p is the order of convergence and d is the total number of function evaluations per iteration. If we assume that all the evaluations have the same cost as function one, we have that the efficiency

index of our methods are 1.565 and 1.627, which makes them competitive.

III. NUMERICAL RESULTS

In this section, we do the experiment to test the efficiency of our numerical method. All computations were done by using MATLAB. We present some numerical results for various convergent iterative schemes in Table I-III. We compare with Newton's method (NM), the M3 method provided in [4], the PM method proposed in [2], the MK7 Method defined [6] and our new iteration (GM). In our iteration, we set

$$h(\mu_n) = \frac{\mu_n}{1 + \mu_n}$$

We remark that chosen for comparison are only the methods which do not require the computation of second or higher derivatives of the function to carry out iterations. We use following test functions

(a) $f(x) = x \exp(x^2) - \sin(x)^2 + 3 \cos(x) + 5;$

$\alpha \approx -1.207647827130919.$

(b) $f(x) = \exp(x^2 + 7x - 30) - 3;$

$\alpha \approx 3.083966303731217.$

(c) $f(x) = 10x \exp(-x^2) - 1;$

$\alpha \approx 1.6796306104284499.$

(d) $f(x) = (x - 1)^3 - \sqrt{2};$

$\alpha \approx 2.122462048309373.$

(e) $f(x) = \sin(x)^2 - x^2 + 1;$

$\alpha \approx 1.404491648215341.$

We accept an approximate solution rather than the exact root, depending on the precision ϵ of the computer. We use the following stopping criteria: (1). $|x_{n+1} - x_n| < \epsilon$, (2). $|f(x_n)| < \epsilon$. When the stopping criterion is satisfied, x_{n+1} is taken as the exact root. For numerical illustrations in this section we use the fixed stopping criterion $\epsilon = 10^{-14}$.

In Table I-III, COC is the computed order of convergence, N denotes the number of iterations to approximate the root to 14 decimal places, NOFE denotes the number of function evaluations, NA means that the method can not get the root of the equation.

TABLE I. COMPARISON OF THE COMPUTED ORDER OF CONVERGENCE FOR DIFFERENT ITERATIVE METHODS

$f(x)$	x_0	COC				
		NM	M3	PM	MK7	GM
(a)	-1.5	2	6	6	7	7
(b)	5	2	6	6	7	7
(c)	1	2	6	6	7	7

(d)	5	2	6	6	7	7
(e)	3	2	6	6	7	7

TABLE II. COMPARISON OF THE ITERATION NUMBER TO APPROXIMATE THE ROOT TO 14 DECIMAL PLACES FOR DIFFERENT ITERATIVE METHOD

$f(x)$	x_0	N				
		NM	M3	PM	MK7	GM
(a)	-1.5	6	3	3	2	2
(b)	5	34	14	9	NA	4
(c)	1	5	3	2	2	2
(d)	5	8	4	3	3	3
(e)	3	6	3	2	NA	3

TABLE III. COMPARISON OF THE NUMBER OF FUNCTION EVALUATIONS FOR DIFFERENT ITERATIVE METHODS

$f(x)$	x_0	NOFE				
		NM	M3	PM	MK7	GM
(a)	-1.5	12	12	12	8	8
(b)	5	68	56	36	NA	16
(c)	1	10	12	8	8	8
(d)	5	16	16	12	12	12
(e)	3	12	12	8	NA	12

IV. CONCLUSION

In this paper, we compose the different fourth-order method, Newton's method and the divided difference to obtain new iterative methods with sixth and seventh order convergence. Numerical experiment shows that our methods play at least equal performance compared to other methods with the same convergence order.

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