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On A Group Of Automorphisms Of The Noncommutative Burgers Hierarchy

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Abstract

Bäcklund transformations are constructed for the noncommutative Burgers hierarchy, generalizing the commutative ones of Weiss, Tabor, Carnevale, and Pickering. These transformations are shown to be invertible and form a group.

1 The Burgers Hierarchy and Its Basic Properties

The original Burgers equation on a function $u(x, t)$ has the form:

$$u_t + uu_x = \nu u_{xx}, \quad \nu = \text{const}; \tag{1.1}$$

the subscripts t and x, here and everywhere, denote the corresponding partial derivatives, with respect to the time coordinate t and space coordinate x, respectively.

Rescalings of u, x, t allow one to bring the coefficients entering the Burgers equation (1.1) into any desirable form; from now on, we shall be dealing with the following one:

$$u_t = 2uu_x + u_{xx} = \tag{1.2a}$$
$$= (u^2 + u_x)_x. \tag{1.2b}$$

Over the years, various Bäcklund transformations were found for the Burgers equation. Thus, Fokas [2; 8, p. 523] found that if u is a solution of the Burgers equation (1.2) then so is

$$\bar{u} = u + (lnu)_x. \tag{1.3}$$

More generally, Weiss, Tabor and Carnevale [9] showed that if φ satisfies

$$\varphi_t = 2u\varphi_x + \varphi_{xx} \tag{1.4}$$

and u is a solution of the Burgers equation, then

$$\bar{u} = u + (ln\varphi)_x \tag{1.5}$$

is again a solution of the Burgers equation. When $\varphi = u$, formula (1.5) yields formula (1.3).

Finally, Pickering [7] generalized formulae (1.4,5) to the whole Burgers *hierarchy*. The latter was defined by the Choodnovsky brothers [1] as follows. Let

$$v_t = v^{(n)}, \qquad n \in \mathbf{Z}_{>0}, \tag{1.6}$$

$$(\cdot)^{(n)} = \partial^n(\cdot), \tag{1.7}$$

$$\partial = \partial/\partial x, \tag{1.8}$$

be the hierarchy of "higher heat equations."

Then

$$u_t = \partial(L_n(u)), \quad n \in \mathbf{Z}_{>0} \tag{1.9}$$

for the variable

$$u = (lnv)_x \tag{1.10}$$

is the Burgers hierarchy. Pickering's formula is this: if u satisfies the n^{th} Burgers equation (1.9) and φ satisfies

$$\varphi_t = \frac{DL_n}{Du}\,(\varphi_x) \tag{1.11}$$

then

$$\bar{u} = u + (ln\varphi)_x \tag{1.12}$$

again satisfies the n^{th} Burgers equation (1.9); here $\dfrac{DF}{Du}$ is the Fréchet derivative of F.

Two problems have remained open: How to find *all* solutions of the auxiliary equation (1.11) on φ? Is the Bäcklund transformation (1.12) invertible? Both of these problems are solved below, and in a more general context of the *noncommutative* Burgers hierarchy.

2 The Noncommutative Burgers Hierarchy

The variable u of the Burgers equation (1.2) and the Burgers hierarchy (1.9) is *scalar*. Over the years, this restriction has been weakened in various directions: first, to allow u be a matrix, by Levy, Ragnisco, and Bruschi [6]; and last, by allowing u to be an element of an arbitrary left-symmetric algebra by Svinolupov [8]. The left-symmetric algebras are, however, *nonassociative*; as a result, no Bäcklund transformations have been ever found for the Svinolupov-Burgers systems.

We shall deal below with the most general *universal associative* Burgers systems introduced in [3]; more details can be found in §2.5 of [4]. The set-up is as follows. Consider all the variables as noncommuting but associative. We start off the heat picture, with the n^{th} flow

$$X_n(v) = \frac{\partial v}{\partial t_n} = v^{(n)}, \quad n \in \mathbf{Z}_{>0}, \tag{2.1}$$

where X_n is the evolutionary derivation of the differential ring

$$C_v = \mathbf{C}[v, v^{-1}; v^{(1)}, v^{(2)}, ...] \tag{2.2}$$

Now set

$$v^{(1)} = vu, \tag{2.3}$$
$$u = v^{-1}v^{(1)}. \tag{2.4}$$

Then,

$$v^{(n)} = vQ_n(u), \tag{2.5}$$

where

$$Q_n(u) = (\partial + \hat{L}_u)^n(1), \tag{2.6}$$

where $\hat{L}_f$ and $\hat{R}_f$ are the operators of left and right multiplication by f, respectively:

$$\hat{L}_f(a) = fa, \quad \hat{R}_f(a) = af, \quad \forall a, f. \tag{2.7}$$

Formula (2.6) follows from the following calculation:

$$vQ_{n+1} = v^{(n+1)} = \partial(v^{(n)}) = \partial(vQ_n) = v^{(1)}Q_n + vQ_n^{(1)} =$$
$$= vuQ_n + vQ_n^{(1)} = v(Q_n^{(1)} + uQ_n) \Rightarrow \tag{2.8}$$
$$Q_{n+1} = (\partial + \hat{L}_u)(Q_n). \tag{2.9}$$

The derivations X_n's (2.1) obviously commute in the ring C_v (2.2). Therefore, they will still commute in any differential subring of C_v, such as C_u, and we have:

$$X_n(u) = X_n(v^{-1}v^{(1)}) = X_n(v^{-1})v^{(1)} + v^{-1}X_n(v^{(1)}) =$$
$$= -v^{-1}X_n(v)v^{-1}v^{(1)} + v^{-1}(X_n(v))^{(1)} = -v^{-1}v^{(n)}u + v^{-1}v^{(n+1)} =$$
$$= -Q_nu + Q_{n+1} \text{ [by (2.9)]} = Q_n^{(1)} + uQ_n - Q_nu : \tag{2.10}$$
$$X_n(u) = (\partial + ad_u)(Q_n), \quad n \in \mathbf{Z}_{>0}. \tag{2.11}$$

This is our noncommutative Burgers hierarchy. Since

$$Q_0 = 1, \quad Q_1 = u, \quad Q_2 = u^{(1)} + u^2, \tag{2.12}$$

for $n = 2$ we find from formula (2.11) that

$$X_2(u) = (\partial + ad_u)(Q_2) = u^{(2)} + u^{(1)}u + uu^{(1)} + uu^{(1)} + u^3 -$$
$$-u^{(1)}u - u^3 = u^{(2)} + 2uu^{(1)} : \tag{2.13}$$
$$u_t = u_{xx} + 2uu_x \tag{2.14}$$

is the noncommutative Burgers equation, with u and u_x no longer commuting. Had we started with u being defined not as $v^{-1}v^{(1)}$ but as

$$u = v^{(1)}v^{-1} \tag{2.15}$$

instead, equation (2.14) would have been

$$u_t = u_{xx} + 2u_x u, \tag{2.16}$$

etc: all formulae being mirror-inverted.

3 Powers Of The Operator $\partial + \hat{L}_u$

In order to write down the noncommutative version of the evolution equation on φ, (1.11), we need first to establish a few useful formulae.
Proposition 3.1

$$(\partial + \hat{L}_u)^n = \sum_{k=0}^{n} \binom{n}{k} \hat{L}_{Q_{n-k}} \partial^k, \quad n \in \mathbf{Z}_{\geq 0}. \tag{3.2}$$

Proof. Formula (3.2) is obviously true for $n = 0, 1$. Induction on n then finishes the job. ■

Proposition 3.3

$$X_n(Q_k) = ((\partial + \hat{L}_u)^k - \hat{R}_{Q_k})(Q_n). \tag{3.4}$$

Proof. For $k = 0$ formula (3.4) is obviously true, and for $k = 1$ it becomes equation (2.11). Inducting on k, we have:

$$\begin{aligned} &X_n(Q_{k+1}) = X_n(Q_k^{(1)} + uQ_k) = \partial(X_n(Q_k)) + uX_n(Q_k) + \\ &+ X_n(u)Q_k = (\partial + \hat{L}_u)(X_n(Q_k)) + \hat{R}_{Q_k}(\partial + \hat{L}_u - \hat{R}_u)(Q_n) = \\ &= \{(\partial + \hat{L}_u)((\partial + \hat{L}_u)^k - \hat{R}_{Q_k}) + \hat{R}_{Q_k}\partial + \hat{R}_{Q_k}\hat{L}_u - \hat{R}_{uQ_k}\}(Q_n), \end{aligned}$$

so that we need to verify that

$$-(\partial + \hat{L}_u)\hat{R}_{Q_k} + \hat{R}_{Q_k}\partial + \hat{R}_{Q_k}\hat{L}_u - \hat{R}_{uQ_k} = -\hat{R}_{Q_{k+1}}, \tag{3.5}$$

which amounts to

$$-Q_k^{(1)} - uQ_k = -Q_{k+1},$$

and this is equation (2.9). ■

Corollary 3.6.

$$X_n(Q_k) = ((\partial + \hat{L}_u)^n - \hat{L}_{Q_n})(Q_k). \tag{3.7}$$

Proof. By formulae (3.4) and (2.6),

$$X_n(Q_k) = ((\partial + \hat{L}_u)^k - \hat{R}_{Q_k})(\partial + \hat{L}_u)^n(1) =$$
$$= (\partial + \hat{L}_u)^n(\partial + \hat{L}_u)^k(1) - Q_nQ_k = ((\partial + \hat{L}_u)^n - \hat{L}_{Q_n})(Q_k).$$

■

4 Symmetries

If v satisfies the n^{th} heat equation

$$X_n(v) = v_t = v^{(n)} \tag{4.1}$$

then so does

$$\tilde{v} = v^{(k)}, \quad \forall k \in \mathbf{Z}_{\geq 0}. \tag{4.2}$$

Therefore,

$$\tilde{u} = \tilde{v}^{-1}\tilde{v}^{(1)} \tag{4.3}$$

satisfies the n^{th} Burgers flow (2.11):

$$X_n(u) = (\partial + ad_u)(Q_n(u)). \tag{4.4}$$

But

$$\tilde{u} = \tilde{v}^{-1}\tilde{v}^{(1)} = v^{(k)-1}v^{(k+1)} = (vQ_k(u))^{-1}vQ_{k+1}(u) =$$
$$= Q_k(u)^{-1}Q_{k+1}(u) = Q_k(u)^{-1}(Q_k(u)^{(1)} + uQ_k(u)) :$$
$$\tilde{u} = Q_k^{-1}Q_k^{(1)} + Q_k^{(-1)}uQ_k. \tag{4.5}$$

By formula (3.7),

$$X_n(Q_k) = ((\partial + \hat{L}_u)^n - \widehat{L}_{Q_n})(Q_k). \tag{4.6}$$

Now, k above is *arbitrary*. We therefore shall be not too reckless to assume

Theorem 4.7. If u satisfies the n^{th} Burger equation (4.4) and φ satisfies

$$X_n(\varphi) = ((\partial + \hat{L}_u)^n - \hat{L}_{Q_n(u)})(\varphi), \tag{4.8}$$

then

$$\bar{u} = \varphi^{-1}\varphi_x + \varphi^{-1}u\varphi \tag{4.9}$$

again satisfies the n^{th} Burgers flow (4.4).

Proof. The idea is this: we imagine that

$$\bar{u} = \bar{v}^{-1}\bar{v}^{(1)} \tag{4.10}$$

and then show that

$$X_n(\bar{v}) = \bar{v}^{(n)}. \tag{4.11}$$

Thus, $\bar{v}$ satisfies the n^{th} heat flow, and therefore $\bar{u}$ satisfies the n^{th} Burgers flow.

Now for the details. Given the differential ring C_u, we enlarge it by a new variable v:

$$C_{u,v} = \mathbf{C}[u, u^{(1)}, ...; v, v^{-1}]. \tag{4.12}$$

We make $C_{u,v}$ into a differential ring by setting

$$\partial(v) = vu, \quad \partial(v^{-1}) = -uv^{-1}. \tag{4.13}$$

We then extend the evolutionary (i.e., commuting with ∂) derivation X_n of C_u onto $C_{u,v}$, by setting

$$X_n(v) = \partial^n(v) = vQ_n(u). \tag{4.14}$$

The calculation (2.10) shows that this extension of X_n is self-consistent

We can do the same extensions starting with another variable $\bar{u}$, even though we don't know yet but suspect that $X_n(\bar{u})$ satisfies the n^{th} Burgers equation (4.4).

But if our suspicion *were* correct, then formula (4.9) could be rewritten as

$$\begin{aligned} &\bar{v}^{-1}\bar{v}_x = \bar{u} = \varphi^{-1}\varphi_x + \varphi^{-1}u\varphi = \varphi^{-1}(\varphi_x + v^{-1}v_x\varphi) = \\ &= \varphi^{-1}v^{-1}(v\varphi_x + v_x\varphi) = (v\varphi)^{-1}(v\varphi)_x. \end{aligned} \tag{4.15}$$

Thus,

$$\bar{v} = Cv\varphi, \tag{4.16}$$

where C is a "constant" : $\partial(C) = 0$; C, therefore, can be absorbed into v without affecting u. Hence, the relations

$$\bar{v} = v\varphi; \tag{4.17}$$

$$\varphi = v^{-1}\bar{v} \tag{4.18}$$

from the essence of the symmetry formula (4.9). To make this statement precise, we use formula (3.2) and calculate:

$$\begin{aligned} X_n(v\varphi) &= X_n(v)\varphi + vX_n(\varphi) = v^{(n)}\varphi + v((\partial + \hat{L}_u)^n - \hat{L}_{Q_n})(\varphi) = \\ &= v^{(n)}\varphi + v\sum_{k=1}^{n}\binom{n}{k}Q_{n-k}\varphi^{(k)} = v^{(n)}\varphi + \sum_{k=1}^{n}\binom{n}{k}v^{(n-k)}\varphi^{(k)} = \\ &= \sum_{k=0}^{n}\binom{n}{k}v^{(n-k)}\varphi^{(k)} = (v\varphi)^{(n)} = \bar{v}^{(n)}. \end{aligned} \tag{4.19}$$

Conversely, if

$$X_n(\bar{v}) = \bar{v}^{(n)}$$

then

$$\begin{aligned} X_n(\varphi) &= X_n(v^{-1}\bar{v}) = -v^{-1}v^{(n)}v^{-1}\bar{v} + v^{-1}\bar{v}^{(n)} = \\ &= -Q_n(u)\varphi + v^{-1}\partial^n(v\varphi) = -Q_n\varphi + (v^{-1}\partial v)^n(\varphi) = \\ &= ((\partial + \hat{L}_u)^n - \hat{L}_{Q_n})(\varphi), \end{aligned} \tag{4.20}$$

because

$$v^{-1}\partial v = \partial + v^{-1}v^{(1)} = \partial + u. \tag{4.21}$$

All our claims have been now verified. In addition, formula (4.18) shows that every solution of the φ equation (4.8) is the noncommutative "ratio" of two arbitrary solutions of the n^{th} heat equation. ■

Remark 4.22 The symmetry formula $\bar{u} = \varphi^{-1}\varphi_x + \varphi^{-1}u\varphi$ — like all known formulae about the Burgers equation outside of Svinolupov's work — is misleading in its simplicity. The true nature of the Burgers equation — that it is a natural part of various finite- and infinite-component *systems* — has yet to be recognized. I leave this task for another occasion, and restrict myself here to a simple illustration.

Let

$$u_t = u_{xx} + 2uu_x, \tag{4.23a}$$
$$a_t = a_{xx} + 2au_x, \tag{4.23b}$$

be a noncommuntative version of the dark Burgers extension $(10.47)|_{\rho=0}$ from [5]. Let φ and ψ satisfy

$$\varphi_t = \varphi_{xx} + 2u\varphi_x, \tag{4.24a}$$
$$\psi_t = \psi_{xx} + 2a\varphi_x. \tag{4.24b}$$

Then the pair

$$\bar{u} = \varphi^{-1}\varphi_x + \varphi^{-1}u\varphi, \tag{4.25a}$$
$$\bar{a} = a\varphi + \psi_x - \psi\varphi^{-1}(u\varphi + \varphi_x), \tag{4.25b}$$

again satisfies the two-component system (4.23).

5 The Bäcklund Transformation Is An Automorphism

The Bäcklund transformation (4.3):

$$\bar{u} = \varphi^{-1}\varphi_x + \varphi^{-1}u\varphi \tag{5.1}$$

is *invertible*: it can be rewritten as

$$u = \varphi(\bar{u} - \varphi^{-1}\varphi_x)\varphi^{-1} = -\varphi_x\varphi^{-1} + \varphi\bar{u}\varphi^{-1}, \tag{5.2}$$

and

$$-\varphi_x\varphi^{-1} = \varphi(\varphi^{-1})_x. \tag{5.3}$$

The same conclusion follows directly from formula (4.18): if $\varphi = v^{-1}\bar{v}$ then

$$\varphi^{-1} = \bar{v}^{-1}v. \tag{5.4}$$

The direct form of this fact is far from being obvious: if u satisfies the n^{th} Burgers equation (4.4) and φ satisfies equation (4.8), then φ^{-1} satisfies the equation

$$X_n(\varphi^{-1}) = ((\partial + \hat{L}_{\bar{u}})^n - \hat{L}_{Q_n(\bar{u})})(\varphi^{-1}), \tag{5.5}$$

where

$$\bar{u} = \varphi^{-1}\varphi_x + \varphi^{-1}u\varphi. \tag{5.6}$$

Moreover, formula (4.18) implies that the automorphisms (5.1) form a *group*. Indeed, let

$$w = \bar{v}^{-1} \quad \bar{\bar{v}}, \tag{5.7}$$

and

$$\bar{\bar{u}} = w^{-1}w_x + w^{-1}\bar{u}w. \tag{5.8}$$

Then

$$\begin{aligned}\bar{\bar{u}} &= w^{-1}w_x + w^{-1}(\varphi^{-1}\varphi_x + \varphi^{-1}u\varphi)w = \\ &= w^{-1}w_x + w^{-1}\varphi^{-1}\varphi_x w + (\varphi w)^{-1}u(\varphi w) = \\ &= (\varphi w)^{-1}(\varphi w)_x + (\varphi w)^{-1}u(\varphi w).\end{aligned} \tag{5.9}$$

Thus, the composition map

$$u \rightarrowtail \bar{u} \rightarrowtail \bar{\bar{u}} \tag{5.10}$$

is effected by the cumulative parameter

$$\varphi w = (v^{-1}\bar{v})\ (\bar{v}^{-1}\bar{\bar{v}})\ = v^{-1}\ \bar{\bar{v}}. \tag{5.11}$$

6 Intrinsic Proof

The symmetry formulae

$$\bar{u} = \varphi^{-1}\varphi_x + \varphi^{-1}u\varphi, \tag{6.1}$$

$$X_n(u) = (\partial + ad_u)(Q_n(u)), \tag{6.2}$$

$$X_n(\varphi) = ((\partial + \hat{L}_u)^n - \hat{L}_{Q_{n(u)}})(\varphi), \tag{6.3}$$

$$X_n(\bar{u}) = (\partial + ad_{\bar{u}})\ (Q_n(\bar{u})), \tag{6.4}$$

make no reference to the extrinsic heat flows; one therefore ought to be able to deduce formula (6.4) directly from formulae (6.1-3). Such a derivation follows. Denote $X_n(\varphi)$ by $\dot{\varphi}$, and $X_n(u)$ by $\dot{u}$. Then

$$\begin{aligned}&X_n(\bar{u}) = X_n\ (\varphi^{-1}(\varphi_x + u\varphi)) = -\varphi^{-1}\dot{\varphi}\varphi^{-1}(\varphi_x + u\varphi) + \\ &+\varphi^{-1}(\dot{\varphi}^{(1)} + u\dot{\varphi} + \dot{u}\varphi) = \\ &= \varphi^{-1}\{-\dot{\varphi}(\varphi_x + \varphi^{-1}u\varphi) + (\partial + \hat{L}_u)(\dot{\varphi}) + \dot{u}\varphi\}.\end{aligned} \tag{6.5}$$

Proposition 6.6

$$Q_n(\varphi^{-1}\varphi_x + \varphi^{-1}u\varphi) = \varphi^{-1}(\partial + \hat{L}_u)^n(\varphi). \tag{6.7}$$

Proof. We have:

$$\begin{aligned}&\partial + \hat{L}_{\bar{u}} = \partial + \hat{L}_{\varphi^{-1}\varphi^{(1)}} + \hat{L}_{\varphi^{-1}u\varphi} = \hat{L}_{\varphi-1}(\hat{L}_\varphi\partial + \hat{L}_{\varphi^{(1)}} + \\ &+\hat{L}_{u\varphi}) = \hat{L}_\varphi^{-1}(\partial\hat{L}_\varphi + \hat{L}_u\hat{L}_\varphi) = \hat{L}_\varphi^{-1}(\partial + \hat{L}_u)\hat{L}_\varphi.\end{aligned} \tag{6.8}$$

Therefore,

$$(\partial + \hat{L}_{\bar{u}})^n = \hat{L}_{\varphi}^{-1}(\partial + \hat{L}_u)^n \hat{L}_{\varphi}, \tag{6.9}$$

and formula (6.7) follows.

■

Thus, the RHS of formula (6.4) is:

$$\begin{aligned}
&(\partial + ad_{\bar{u}})(Q_n(\bar{u})) = \\
&= (\partial + \hat{L}_{\bar{u}} - \hat{R}_{\bar{u}})(\varphi^{-1}(\partial + \hat{L}_u)^n)(\varphi) = \\
&= (\varphi^{-1}(\partial + \hat{L}_u)\varphi - \hat{R}_{\varphi^{-1}\varphi^{(1)}+\varphi^{-1}u\varphi})(\varphi^{-1}(\partial + \hat{L}_u)^n(\varphi)) = \\
&= \varphi^{-1}(\partial + \hat{L}_u)^{n+1}(\varphi) - \varphi^{-1}(\partial + \hat{L}_u)^n(\varphi) \cdot \varphi^{-1}(\partial + \hat{L}_u)(\varphi).
\end{aligned} \tag{6.10}$$

Formula (6.4) therefore becomes:

$$-\dot{\varphi}(\varphi^{-1}\varphi_x + \varphi^{-1}u\varphi) + (\partial + \hat{L}_u)(\dot{\varphi}) + \dot{u}\varphi \stackrel{?}{=} \tag{6.11a}$$

$$\stackrel{?}{=} (\partial + \hat{L}_u)^{n+1}(\varphi) - (\partial + \hat{L}_u)^n(\varphi) \cdot \varphi^{-1}(\partial + \hat{L}_u)(\varphi). \tag{6.11b}$$

By formulae (6.2,3), the LHS of this identity is:

$$\begin{aligned}
&-((\partial + \hat{L}_u)^n(\varphi) - Q_n\varphi)\varphi^{-1}(\varphi_x + u\varphi) + \\
&+(\partial + \hat{L}_u)((\partial + \hat{L}_u)^n(\varphi) - Q_n\varphi) + (Q_n^{(1)} + uQ_n - Q_n u)\varphi.
\end{aligned} \tag{6.12}$$

Canceling the like-terms, formula (6.11) reduces to

$$Q_n\varphi_x - (\partial + \hat{L}_u)(Q_n\varphi) + (Q_n^{(1)} + uQ_n)\varphi \stackrel{?}{=} 0, \tag{6.13}$$

or

$$Q_n\varphi_x - (Q_n^{(1)}\varphi + Q_n\varphi_x) - uQ_n\varphi + (Q_n^{(1)} + uQ_n)\varphi \stackrel{?}{=} 0, \tag{6.14}$$

which is obviously true. ■

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