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Classification of Fully Nonlinear Integrable Evolution Equations of Third Order

Rafael HERNÁNDEZ HEREDERO

Departamento de Matemática Aplicada, EUIT de Telecomunicación
Universidad Politécnica de Madrid, 28031 Madrid, Spain.
E-mail: rafahh@euitt.upm.es

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Abstract

A fully nonlinear family of evolution equations is classified. Nine new integrable equations are found, and all of them admit a differential substitution into the Korteweg-de Vries or Krichever-Novikov equations. One of the equations contains hyperelliptic functions, but it is transformable into the Krichever-Novikov equation by a differential substitution that only involves elliptic functions.

1 Introduction

The family of partial differential equations of the form

$$w_t = H(t, x, w, w_x, w_{xx}, w_{xxx}), \tag{1.1}$$

includes the mathematical object that led to the birth of modern integrability and soliton theory [4], the Korteweg-de Vries (KdV) equation (cf. (1.3)). Many efforts (just some of them are [1–3, 5, 8, 10, 14, 16]) have been dedicated to study (1.1) during the last 30 years. Today there is the impression that this research is almost complete: many specific subfamilies have been fully classified and the result has been always the same, namely, that the following conjecture [6] holds:

Conjecture. *All integrable equations in the family* (1.1) *are related via either a classical (pointorcontact) transformation or a more complex differential substitution, to the three fundamental integrable equations*

$$w_t = w_{xxx} + \alpha(x)w_x + \beta(x)w, \tag{1.2}$$

$$w_t = w_{xxx} + ww_x, \tag{1.3}$$

$$w_t = w_{xxx} - \tfrac{3}{2}w_x^{-1}w_{xx}^2 + (4w^3 + g_1w + g_2)w_x^{-1} + cw_x, \tag{1.4}$$

that is, the linear equation, the KdV equation and the Krichever-Novikov (KN) equation.

Nevertheless the conjecture is far from being proven and there is still plenty of room to find interesting results within this family of equations, may be even a new integrable equation. The computational complexity of the calculations can be treated by contemporary computer capabilities. The full classification is very long and its sheer length forces us to perform it in several pieces. It appears though that every new piece is more interesting than the previous one.

As in the Conjecture we assume equivalence relations between equations to be classical transformations, i.e. special pointlike or contact transformations

$$\bar{t} = \chi(t), \quad \bar{x} = \phi(x, w, w_x), \quad \bar{w} = \psi(x, w, w_x) \tag{1.5}$$

with $D_x(\phi)\partial\psi/\partial w_x = D_x(\psi)\partial\phi/\partial w_x$ relating equivalent evolution equations

$$w_t = F(x, w, w_x, w_{xx}, w_{xxx}) \text{ and } \bar{w}_t = \bar{F}(\bar{x}, \bar{w}, \bar{w}_{\bar{x}}, \bar{w}_{\bar{x}\bar{x}}, \bar{w}_{\bar{x}\bar{x}\bar{x}}). \tag{1.6}$$

When one considers some subfamily of equations, there exists a subclass of (1.5), called "allowed transformations", that preserves the functional form of the subfamily. For example, considering evolution equations of the form (1.1) we cannot make the new time depend on x or u, because the evolution character of the resulting equation would be lost. Thus, the most general form of allowed transformation for (1.1) is of the form (1.5).

We also use more general *differential substitutions*

$$\bar{t} = \chi(t), \quad \bar{x} = \phi(x, w, w_x, w_{xx}, \ldots), \quad \bar{w} = \psi(x, w, w_x, w_{xx}, \ldots), \tag{1.7}$$

which may relate different integrable equations, although in a noninvertible manner. For example Miura-type or Cole-Hopf-type [15] transformations are of this kind. The existence of a differential substitution relating two equations is not a trivial fact because it can be considered a restricted type of Bäcklund transformation, an object typical of integrable equations. If we are able to relate an equation with an integrable equation by means of a differential substitution, then the original equation is integrable [13]. Ample information about differential substitutions can be found in [12, 15].

In this paper we continue the classification [5,7] of integrable equations of the form (1.1). In the papers cited we established a scheme to perform the classification and in Sec. 2 we sketch that analysis and the underlying theory of integrability (the formal symmetry approach). We divide the family (1.1) into three types of equations comprising quasilinear and fully nonlinear equations in the highest derivative w_{xxx}. The quasilinear integrable equations have been classified, listed in [10] and the most complicated cases studied in [7]. All three totally nonlinear categories are basically unknown, except for some special equations of Harry-Dym type or results like as [9]. In Sec. 4 we study one of the three categories and find nine new fully nonlinear equations that satisfy necessary integrability conditions. Some of these equations would be very difficult to pinpoint using alternative integrability techniques, but the one we use here is able to do it. In Sec. 5 we prove that all the new equations are related to the KdV or the KN equations through (quite complicated) differential substitutions, thus proving their integrability and reinforcing the conjecture.

One of the new integrable equations, (4.8), is remarkable. It possesses hyperelliptic functions of the variables x and w.

2 The symmetry approach and non-standard variables

The symmetry approach to integrability, developed by Shabat et al [10], defines an integrable equation in family (1.1) as one admitting an infinite set of higher symmetries. This ultimately translates into a set of integrability conditions that are an infinite overdetermined system of partial differential equations over the function H in the rhs of (1.1). These integrability conditions are expressible in the form of so-called *canonical conservation laws*

$$D_t\rho_i = D_x\sigma_i, \qquad i = 1, 2, 3, \dots . \tag{2.1}$$

For example the first two canonical densities, ρ_i, are

$$\rho_1 = \left(\frac{\partial H}{\partial w_{xxx}}\right)^{-1/3}, \quad \rho_2 = \left(\frac{\partial H}{\partial w_{xxx}}\right)^{-1} \frac{\partial H}{\partial w_{xx}} \tag{2.2}$$

With the use of just two conditions $D_t\rho_i = D_x\sigma_i$, $i = 1, 2$, the first classification result was obtained [9]: the integrable equations of the form (1.1) must have one of the following dependencies on the third-order derivative w_{xxx}:

$$w_t = f_1 w_{xxx} + f_2, \tag{2.3}$$

$$w_t = (f_1 w_{xxx} + f_2)^{-2} + f_3, \tag{2.4}$$

$$w_t = (2f_1 w_{xxx} + f_2)(f_1 w_{xxx}^2 + f_2 w_{xxx} + f_3)^{-1/2} + f_4, \tag{2.5}$$

where $f_i = f_i(x, w, w_x, w_{xx})$.

To find all the solutions of (2.1) it is necessary to analyze a whole casuistic tree of parameters appearing in intermediate calculations. In [5] we gave an approach that cuts significantly the number of possibilities, merging intermediate subfamilies of equations and effectively delaying the branching process. The idea is to exploit the fact that the integrability conditions are almost evolution equations themselves and one may use them as an alternative representation of the original equation. The first necessary condition for integrability is that the separant[1],

$$u = u(x, w, w_x, \dots, w^{(n)}) = \rho_1 = \left(\frac{\partial}{\partial w_{xxx}} H(x, w, w_x, w_{xx}, w_{xxx})\right)^{-1/3}, \tag{2.6}$$

of (1.1) must be a conserved density, i. e., there must exist a differential function, $\tilde\sigma_1(x, w, \dots, w^{(n+2)})$, such that

$$u_t = D_x[\tilde\sigma_1(x, w, \dots, w^{(n+2)})]. \tag{2.7}$$

This is also an evolution relation. Using (2.6) we can express $w^{(n)}$ as a function of u, x, w, …, $w^{(n-1)}$. Often, when one uses this substitution (and its consequences for $w^{(n+1)}$, $w^{(n+2)}$ and $w^{(n+3)}$), (2.7) becomes a genuine evolution equation for u because all dependencies on w,…,$w^{(n-1)}$ disappear. For example the equation

$$w_t = -2w_x \left(\frac{w_{xxx}}{w_x} - \frac{3}{2}\frac{w_{xx}^2}{w_x^2}\right)^{-1/2} \text{transforms onto } u_t = D_x\left(\frac{u_{xx}}{u^3} - \frac{3}{2}\frac{u_x^2}{u^4}\right).$$

[1] $w^{(i)}$ denotes $\partial^i w/\partial x^i$. Note that, if $n < 3$, then the equation is *quasilinear.*

If the expression of u_t after substituting $w^{(n)}$,...,$w^{(n+3)}$ still depends on $w, \ldots, w^{(n-1)}$, we can resort to the theory of constrained dynamical systems as explained in [11]. Equation (2.7) is a system

$$(v_i)_x = \Phi_i(x, v_1, \ldots, v_n, u), \tag{2.8}$$
$$(v_i)_t = G_i(x, v_1, \ldots, v_n, u, u_x, \ldots), \tag{2.9}$$
$$u_t = F(x, v_1, \ldots, v_n, u, u_x, u_{xx}, u_{xxx}), \tag{2.10}$$

where $v_1, \ldots, v_n$ denote $w, \ldots, w^{(n-1)}$ respectively. Equation (2.10) is interpreted as an evolution equation over the main variable u, with some additional low-order variables, $v_1, \ldots, v_n$, subjected to the differential constraints (2.8,2.9). The variables $(x, v_1, \ldots, v_n, u)$ are what we call *nonstandard variables*. Now an interesting fact is that the symmetry approach provides almost the same integrability conditions [11] for a constrained equation (2.10) as for a pure evolution equation. The conditions that we are going to use in this paper are the first five canonical laws with densities

$$\begin{aligned}
&\rho_1 = u, \qquad \rho_2 = u^3 \frac{\partial F}{\partial u_{xx}}, \\
&\rho_3 = \left(2u^{-2}u_x + u^2 \frac{\partial F}{\partial u_{xx}}\right)_x + u^{-3}u_x^2 + \frac{1}{3}u^5\left(\frac{\partial F}{\partial u_{xx}}\right)^2 + uu_x\frac{\partial F}{\partial u_{xx}} - \\
&\quad - u^2\frac{\partial F}{\partial u_x} + u\sigma_1, \\
&\rho_4 = -\frac{1}{3}u_{xx}\frac{\partial F}{\partial u_{xx}} - u_x\frac{\partial F}{\partial u_x} + u\frac{\partial F}{\partial u} + u^{-1}u_x^2\frac{\partial F}{\partial u_{xx}} - \frac{1}{3}u^4\frac{\partial F}{\partial u_x}\frac{\partial F}{\partial u_{xx}} + \\
&\quad + \frac{1}{3}u^3u_x\left(\frac{\partial F}{\partial u_{xx}}\right)^2 + \frac{2}{27}u^7\left(\frac{\partial F}{\partial u_{xx}}\right)^3 + \frac{1}{3}u\sigma_2, \\
&\rho_5 = u\sigma_3 - \rho_3\sigma_1 - 3\frac{\partial \mathbf{\Phi}}{\partial u}(F).
\end{aligned} \tag{2.11}$$

Note that the first two densities are the same as (2.2) and that we have used the fact that the equation in the new variables, (2.10), has precisely the form

$$u_t = \frac{u_{xxx}}{u^3} + \text{lower order terms}.$$

This is already a very big advantage of the new representation because the separant u is of low order and we have to study equations of the type (2.3) (albeit with constraints).

Using the new representation and the first integrability conditions $D_t\rho_i = D_x\sigma_i$, $i = 1, 2, 3$, we found [5] that all integrable equations (2.8)-(2.10) have the general form

$$\begin{aligned}
&\mathbf{v}_x = \mathbf{\Phi}(\mathbf{v}), \\
&\mathbf{v}_t = \left(\frac{u_{xx}}{u^3} - \frac{3}{2}\frac{u_x^2}{u^4} - \frac{3}{4}P^{-1}\frac{\partial P}{\partial u}\frac{u_x^2}{u^3} - \frac{3}{2}P^{-1}\mathbf{\Phi}(P)\frac{u_x}{u^3} + E\frac{u_x}{u^3}\right)\frac{\partial \mathbf{\Phi}}{\partial u} - \\
&\quad - \frac{1}{2}\frac{u_x^2}{u^3}\frac{\partial^2 \mathbf{\Phi}}{\partial u^2} - \frac{u_x}{u^3}\left[\mathbf{\Phi}, \frac{\partial \mathbf{\Phi}}{\partial u}\right] + \mathbf{r}(\mathbf{v}, u), \\
&u_t = D_x\left(\frac{u_{xx}}{u^3} - \frac{3}{2}\frac{u_x^2}{u^4} - \frac{3}{4}P^{-1}\frac{\partial P}{\partial u}\frac{u_x^2}{u^3} - \frac{3}{2}P^{-1}\mathbf{\Phi}(P)\frac{u_x}{u^3} + E\frac{u_x}{u^3} + q(\mathbf{v}, u)\right),
\end{aligned} \tag{2.12}$$

where $\mathbf{v}$ is the vector $(x, v_1, v_2, \ldots, v_n)$, $\boldsymbol{\Phi}$ denotes the vector field

$$\boldsymbol{\Phi} = \varphi_n(x, \mathbf{v}, u)\frac{\partial}{\partial v_{n-1}} + v_{n-1}\frac{\partial}{\partial v_{n-2}} + \cdots + v_2\frac{\partial}{\partial v_1}$$

and $P = P(\mathbf{v}, u)$, $E = E(\mathbf{v}, u)$. There are compatibility conditions:

$$\begin{aligned}
&\frac{\partial^3\boldsymbol{\Phi}}{\partial u^3} + \frac{3}{2}P^{-1}\frac{\partial P}{\partial u}\frac{\partial^2\boldsymbol{\Phi}}{\partial u^2} = 0,\\
&\left[\boldsymbol{\Phi}, \frac{\partial^2\boldsymbol{\Phi}}{\partial u^2}\right] + \left(\frac{1}{2}P^{-1}\frac{\partial P}{\partial u} - \frac{1}{u}\right)\left[\boldsymbol{\Phi}, \frac{\partial\boldsymbol{\Phi}}{\partial u}\right] + \left(P^{-1}\frac{\partial P}{\partial u} - \frac{2}{3}E\right)\frac{\partial^2\boldsymbol{\Phi}}{\partial u^2} = 0,\\
&\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial q}{\partial u}\frac{\partial\boldsymbol{\Phi}}{\partial u} + \frac{1}{u^3}\left[\boldsymbol{\Phi}, \left[\boldsymbol{\Phi}, \frac{\partial\boldsymbol{\Phi}}{\partial u}\right]\right] + \frac{3}{2u^3}\left(P^{-1}\frac{\partial P}{\partial u} - \frac{2}{3}E\right)\left[\boldsymbol{\Phi}, \frac{\partial\boldsymbol{\Phi}}{\partial u}\right],\\
&[\boldsymbol{\Phi}, \mathbf{r}] - \boldsymbol{\Phi}(q)\frac{\partial\boldsymbol{\Phi}}{\partial u} = 0.
\end{aligned} \tag{2.13}$$

Studying the compatibility equations, as in [5], one finds several families of integrable equations. We proved in [7] that for classifying (1.1) it is enough to study the three types of equations characterized by ($\boldsymbol{\Phi}_2 \neq 0$)

$$\boldsymbol{\Phi} = u^2\boldsymbol{\Phi}_2(\mathbf{v}) + \boldsymbol{\Phi}_0(\mathbf{v}), \quad P = 1, \quad E = 0, \tag{2.14}$$

$$\boldsymbol{\Phi} = S\boldsymbol{\Phi}_2(\mathbf{v}) + \boldsymbol{\Phi}_0(\mathbf{v}), \quad P = \alpha(\mathbf{v})u^2 - 1, \quad S = P^{1/2}, \quad E = 0, \tag{2.15}$$

$$\boldsymbol{\Phi} = u\,\boldsymbol{\Phi}_1(\mathbf{v}) + \boldsymbol{\Phi}_0(\mathbf{v}), \quad P = \alpha(\mathbf{v})u^2, \quad [\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_0] \neq 0, \quad E = E(\mathbf{v}), \tag{2.16}$$

for which n can take any value of 0, 1, 2 or 3. These equations give the relationship between the highest derivative $w^{(n)}$ in the separant u and u because $w^{(n)} = D_x w^{(n-1)} = D_x v_n = \boldsymbol{\Phi}(v_n)$. To $n = 0$, 1 or 2 there correspond different types of integrable quasilinear equations (2.3), which have been all listed in [10]. In this paper we are interested in starting to research fully nonlinear equations and so $n = 3$. In this case type (2.14) corresponds to equations (2.5) with $f_1 = 0$, type (2.15) to equations (2.5) with $f_1 \neq 0$ and type (2.16) to equations (2.4).

3 Integrable equations of the form (2.14)

To find all the fully nonlinear ($n = 3$) integrable differential equations of type (2.14) we write them in the form

$$w_t = -\frac{2}{\sqrt{A^{-3/2}w_{xxx} - B}} + C \tag{3.1}$$

where A, B and C are functions of $\mathbf{v} = (x, w, w_x, w_{xx})$. Thus

$$u = (A^{-1/2}w_{xxx} - AB)^{1/2}$$

$$\boldsymbol{\Phi}_2 = A^{1/2}\frac{\partial}{\partial w_{xx}}, \qquad \boldsymbol{\Phi}_0 = A^{3/2}B\frac{\partial}{\partial w_{xx}} + w_{xx}\frac{\partial}{\partial w_x} + w_x\frac{\partial}{\partial w} + \frac{\partial}{\partial x}$$

The third integrability condition in ρ_3 together with the compatibility conditions imply that now our system of equations takes the form

$$u_t = D_x\left(\frac{u_{xx}}{u^3} - \frac{3}{2}\frac{u_x^2}{u^4} + \frac{Q(\mathbf{v})}{u^2} + T(\mathbf{v}) + l(\mathbf{v})u^2 + m(\mathbf{v})u\right), \tag{3.2}$$

$$\mathbf{v}_t = \left(2\frac{u_{xx}}{u^2} - 4\frac{u_x^2}{u^3}\right)\mathbf{\Phi}_2 + 2\,\frac{u_x}{u^2}\,[\mathbf{\Phi}_2, \mathbf{\Phi}_0] + \left(4\frac{Q(\mathbf{v})}{u} + m(\mathbf{v})u^2 + \frac{4}{3}l(\mathbf{v})u^3\right)\mathbf{\Phi}_2 + \\ + 2\,[\mathbf{\Phi}_0, [\mathbf{\Phi}_2, \mathbf{\Phi}_0]]\,\frac{1}{u} - 2\,[\mathbf{\Phi}_2, [\mathbf{\Phi}_2, \mathbf{\Phi}_0]]\,u + \mathbf{s}(\mathbf{v}) \tag{3.3}$$

and the compatibility conditions, (2), are

$$\mathbf{\Phi}_2(l) = 0, \tag{3.4}$$

$$\mathbf{\Phi}_0(T) - \mathbf{\Phi}_2(Q) = 0, \tag{3.5}$$

$$\mathbf{\Phi}_2(m) = 0. \tag{3.6}$$

$$[\mathbf{\Phi}_2, [\mathbf{\Phi}_2, [\mathbf{\Phi}_2, \mathbf{\Phi}_0]]] + \frac{2}{3}l[\mathbf{\Phi}_2, \mathbf{\Phi}_0] + \left(\mathbf{\Phi}_2(T) + \frac{1}{3}\mathbf{\Phi}_0(l)\right)\mathbf{\Phi}_2 = 0, \tag{3.7}$$

$$[\mathbf{\Phi}_0, [\mathbf{\Phi}_0, [\mathbf{\Phi}_2, \mathbf{\Phi}_0]]] - 2Q[\mathbf{\Phi}_2, \mathbf{\Phi}_0] + \mathbf{\Phi}_0(Q)\mathbf{\Phi}_2 = 0, \tag{3.8}$$

$$[\mathbf{\Phi}_2, \mathbf{s}] - m[\mathbf{\Phi}_2, \mathbf{\Phi}_0] - \mathbf{\Phi}_0(m)\mathbf{\Phi}_2 = 0, \tag{3.9}$$

$$[\mathbf{\Phi}_0, \mathbf{s}] = 0. \tag{3.10}$$

The integrability condition of ρ_3 is far from being satisfied. The following conditions are necessary and sufficient for that:

$$0 = \mathbf{\Phi}_2\mathbf{\Phi}_2\mathbf{\Phi}_2(T) - 3\mathbf{\Phi}_2\mathbf{\Phi}_2\mathbf{\Phi}_0(l) + \frac{8}{3}\mathbf{\Phi}_2(T)l, \tag{3.11}$$

$$0 = 15\mathbf{\Phi}_0\mathbf{\Phi}_2\mathbf{\Phi}_0(Q) - 10\mathbf{\Phi}_0\mathbf{\Phi}_0\mathbf{\Phi}_2(Q) - 6\mathbf{\Phi}_2\mathbf{\Phi}_0\mathbf{\Phi}_0(Q) + 8\mathbf{\Phi}_2(Q)Q, \tag{3.12}$$

$$\mathbf{\Phi}_2(E) = \mathbf{\Phi}_0\mathbf{\Phi}_2\mathbf{\Phi}_2(T) + \mathbf{\Phi}_0\mathbf{\Phi}_2\mathbf{\Phi}_0(l), \tag{3.13}$$

$$\mathbf{\Phi}_0(E) = 2\mathbf{\Phi}_0\mathbf{\Phi}_0\mathbf{\Phi}_2(T) + \mathbf{\Phi}_2\mathbf{\Phi}_2\mathbf{\Phi}_0(Q) - 2\mathbf{\Phi}_2\mathbf{\Phi}_0\mathbf{\Phi}_2(Q) + \frac{2}{3}\mathbf{\Phi}_0\mathbf{\Phi}_0\mathbf{\Phi}_0(l), \tag{3.14}$$

$$\mathbf{s}(l) = -3\mathbf{\Phi}_2\mathbf{\Phi}_2\mathbf{\Phi}_0(m) - 2\mathbf{\Phi}_0(m)l + m\mathbf{\Phi}_0(l), \tag{3.15}$$

$$\mathbf{s}(T) = 3\mathbf{\Phi}_0\mathbf{\Phi}_2\mathbf{\Phi}_0(m) - 3\mathbf{\Phi}_2\mathbf{\Phi}_0\mathbf{\Phi}_0(m) + m\mathbf{\Phi}_2(Q), \tag{3.16}$$

$$\mathbf{s}(Q) = -\mathbf{\Phi}_0\mathbf{\Phi}_0\mathbf{\Phi}_0(m) + \mathbf{\Phi}_0(Q)m + 2Q\mathbf{\Phi}_0(m), \tag{3.17}$$

where $E = E(\mathbf{v}, u)$ is the function

$$E(\mathbf{v}, u) = \frac{5}{3}\mathbf{\Phi}_0\mathbf{\Phi}_0(l) + \frac{7}{3}\mathbf{\Phi}_0\mathbf{\Phi}_2(T) + \frac{4}{3}\mathbf{\Phi}_2\mathbf{\Phi}_0(T) + \frac{1}{6}T^2 - lQ.$$

The integrability conditions written above suffice for the classification we intend to make here.

4 Nine nonlinear integrable equations

Theorem. *Any equation of the form* (3.1) *satisfying* (3.4-3.17) *can be transformed, using a contact transformation,* $\bar{x} = \phi(x, w, w_x)$, $\bar{u} = \psi(x, w, w_x)$, *into one of the following nine* *equations.*

$$w_t = -2w_{xxx}^{-1/2} + c_1(x^2 w_x - 2xw) + c_2(xw_x - w) + \\ + c_3 w_x + c_4 x^2 + c_5 x + c_6. \quad (4.1)$$

$$w_t = -2\lambda^{3/2}\left(w_x^{-3}w_{xxx} - \frac{3}{2}w_x^{-4}w_{xx}^2\right)^{-1/2} + c_1x^2 + c_2xw_x + c_3w_x + c_4w^2 + \\ + c_5w + c_6. \quad (4.2)$$

$$w_t = -2\left(z^{-3}w_{xxx} - B\right)^{-1/2} + c_1w_x + c_2. \quad (4.3)$$

$$B = 8w_x - 12w_x^3z^{-1} - 6\lambda_2w_xz^{-1} - 3\lambda_1z^{-1} + \\ + (4w_x^7 + 6\lambda_2w_x^5 + 5\lambda_1w_x^4 + 2\lambda_2^2w_x^3 + 4\lambda_0w_x^3 + \\ + 3\lambda_1\lambda_2w_x^2 + \lambda_1^2w_x + 2\lambda_0\lambda_2w_x + \lambda_1\lambda_0)z^{-3},$$

$$z = (w_{xx} + w_x^4 + \lambda_2w_x^2 + \lambda_1w_x + \lambda_0)^{1/2}.$$

$$w_t = -2\left(z^{-3}w_{xxx} - 2\lambda_1^2w_x^3z^{-3} + 6\lambda_1w_xz^{-1} - \lambda\right)^{-1/2} + \\ + c_1\left(\lambda_1x^2w_x - x\right) + c_2xw_x + c_3w_x + c_4. \quad (4.4)$$

$$z = (w_{xx} + \lambda_1w_x^2)^{1/2}.$$

$$w_t = -2\left(z^{-3}w_{xxx} - 2\lambda^2w_x^3z^{-3} - 6\lambda\exp(-4\lambda w)w_xz^{-3} + 6\lambda w_xz^{-1}\right)^{-1/2} + \\ + c_1\left(\lambda x^2w_x - x\right) + c_2(2\lambda xw_x - 1) + c_3w_x. \quad (4.5)$$

$$z = (w_{xx} + \lambda w_x^2 + \exp(-4\lambda w))^{1/2}, \qquad \lambda \neq 0.$$

$$w_t = -2\left(z^{-3}w_{xxx} + w_xz^{-3}\right)^{-1/2} + c_1w_x + c_2\sin x + c_3\cos x, \quad (4.6)$$

$$z = (w_{xx} + w)^{1/2}.$$

Below $z = (w_{xx} + S)^{1/2}$ *and* $\nabla = S\dfrac{\partial}{\partial w_x} - w_x\dfrac{\partial}{\partial w} - \dfrac{\partial}{\partial x}$

$$w_t = -2\left(A^{-3/2}w_{xxx} - B\right)^{-1/2} + c_1w_x + c_2, \quad (4.7)$$

$$A = w_x^{1/2}z^2,$$

$$B = \lambda_2w_x^{-2} + \lambda_1 + \frac{3}{2}w_x^{-7/4}z + \frac{3}{2}Sw_x^{-7/4}z^{-1} - \\ - 3w_x^{-3/4}\frac{\partial S}{\partial w_x}z^{-1} + w_x^{-3/4}\nabla(S)z^{-3}$$

$$S = \gamma_1w_x^{1/2} + \gamma_2w_x^{3/2} + \gamma_3w_x^{5/2} + \frac{1}{16}\lambda_1^2w_x^{7/2} + \frac{1}{16}\lambda_2^2w_x^{-1/2}.$$

For the last two equations $A = \beta^{-1/2} w_x^{1/2} z^2$ *and*

$$B = \frac{3}{2}\beta^{3/4} w_x^{-7/4} z + \frac{3}{2}\beta^{3/4}\left[S w_x^{-7/4} + \frac{\beta_w}{\beta} w_x^{1/4} + \frac{\beta_x}{\beta} w_x^{-3/4} - 2\frac{\partial S}{\partial w_x} w_x^{-3/4}\right] z^{-1} + \\ + \nabla(S)\beta^{3/4} w_x^{-3/4} z^{-3}.$$

$$w_t = -2\left(A^{-3/2} w_{xxx} - B\right)^{-1/2}, \tag{4.8}$$

$$S = \beta^{3/2} w_x^{1/2} + 10(f+g)\beta^{1/2} w_x^{3/2} + \beta^{3/2} w_x^{5/2} - \\ - \beta_x \beta^{-1} w_x + \beta_w \beta^{-1} w_x^2,$$

$$\beta = f - g, \qquad f = f(x+w), \quad g = g(x-w),$$

where f and g satisfy: $(y')^2 = -16y^5 - c_1 y^3 - c_2 y^2 - c_3 y - c_4$.

$$w_t = -2\left(A^{-3/2} w_{xxx} - B\right)^{-1/2} + c_1(x^2 w_x + w^2) + c_2(x w_x - w) + c_3(w_x + 1), \tag{4.9}$$

$$S = \beta^{1/2}\left(\lambda w_x^{3/2} + 2w_x - 2w_x^2\right),$$

$$\beta = (x+w)^{-2}.$$

Proof. The coefficient in $\partial/\partial w_x$ of (3.7) together with (3.4) imply

$$l = -\frac{3}{4}\frac{\partial^2 A}{\partial w_{xx}^2}, \qquad A = \gamma_2 w_{xx}^2 + \gamma_1 w_{xx} + \gamma_0. \tag{4.10}$$

A contact transformation $\bar{x} = \phi(x, w, w_x)$, $\bar{w} = \psi(x, w, w_x)$ can linearize the function A, so $\gamma_2 = l = 0$. The coefficient in $\partial/\partial w_{xx}$ of (3.7) gives

$$\frac{\partial T}{\partial w_{xx}} = A^{-1}\mathbf{\Phi}_0\mathbf{\Phi}_2\mathbf{\Phi}_2(A^{1/2}) + 3A^{-1}\mathbf{\Phi}_2\mathbf{\Phi}_2\mathbf{\Phi}_0(A^{1/2}) - \\ - 3A^{-1}\mathbf{\Phi}_2\mathbf{\Phi}_0\mathbf{\Phi}_2(A^{1/2}) - A^{-1}\mathbf{\Phi}_2\mathbf{\Phi}_2\mathbf{\Phi}_2(A^{3/2}B). \tag{4.11}$$

¿From the coefficients of relation (3.8) in $\partial/\partial w_{xx}$, $\partial/\partial w_x$ and $\partial/\partial w$ we obtain

$$3\mathbf{\Phi}_0\mathbf{\Phi}_2\mathbf{\Phi}_0(A^{3/2}B) - \mathbf{\Phi}_2\mathbf{\Phi}_0\mathbf{\Phi}_0(A^{3/2}B) - 8\mathbf{\Phi}_0\mathbf{\Phi}_0\mathbf{\Phi}_0(A^{1/2}) + \\ + 4Q\mathbf{\Phi}_0(A^{1/2}) - \mathbf{\Phi}_0(Q)A^{1/2} = 0, \tag{4.12}$$

$$Q = \frac{1}{2}A^{-1/2}\mathbf{\Phi}_2\mathbf{\Phi}_0(A^{3/2}B) - 3A^{-1/2}\mathbf{\Phi}_0\mathbf{\Phi}_0(A^{1/2}), \tag{4.13}$$

$$\mathbf{\Phi}_2(A^{3/2}B) - 3\mathbf{\Phi}_0(A^{1/2}) = 0. \tag{4.14}$$

¿From (3.9), (3.10) and (3.6) we obtain

$$\mathbf{s} = H\frac{\partial}{\partial w} + \mathbf{\Phi}_0(H)\frac{\partial}{\partial w_x} + \mathbf{\Phi}_0(\mathbf{\Phi}_0(H))\frac{\partial}{\partial w_{xx}} \tag{4.15}$$

with $H = H(x, w, w_x)$. From the remaining conditions we find the relations:

$$m = \frac{\partial H}{\partial w_x}, \tag{4.16}$$

$$\mathbf{s}(A^{3/2}B) = \mathbf{\Phi}_0\mathbf{\Phi}_0\mathbf{\Phi}_0(H), \tag{4.17}$$

$$\mathbf{s}(A^{1/2}) = \mathbf{\Phi}_2\mathbf{\Phi}_0\mathbf{\Phi}_0(H) - 2m\mathbf{\Phi}_0(A^{1/2}) - \mathbf{\Phi}_0(m)A^{1/2}. \tag{4.18}$$

The classification now starts to branch in subcases. Using the general conditions (4.11) to (4.18) we proceed to find a "classification tree" of integrable equations. The first division depends on the following fact. The function A was found (4.10) to be a quadratic polynomial in w_{xx} and linearizable by means of a contact transformation. If A has a double root the linearization can bring it to the form $A = a(x, w, w_x)$. The resulting equations are called **type 1** equations. If A has not a double root, we obtain **type 2** equations with $A = a(x, w, w_x)^2 \left[w_{xx} + S(x, w, w_x)\right]$.

4.1 Type 1: $A = a(x, w, w_x)$

Conditions (4.14), (4.13), (4.11) and (3.5) imply that

$$
\begin{aligned}
B &= \frac{3}{4} a^{-5/2} \frac{\partial a}{\partial w_x} w_{xx}^2 + \frac{3}{2} a^{-5/2} \left(w_x \frac{\partial a}{\partial w} + \frac{\partial a}{\partial x} \right) w_{xx} + b(x, w, w_x), \\
Q &= -\frac{3}{8} \left[\frac{1}{a} \frac{\partial^2 a}{\partial w_x^2} - \frac{1}{2a^2} \left(\frac{\partial a}{\partial w_x} \right)^2 \right] w_{xx}^2 - \\
&\quad - \frac{3}{4} \left[\frac{1}{a} \left(w_x \frac{\partial^2 a}{\partial w_x \partial w} + \frac{\partial^2 a}{\partial w_x \partial x} \right) - \frac{1}{2a^2} \frac{\partial a}{\partial w_x} \left(w_x \frac{\partial a}{\partial w} + \frac{\partial a}{\partial x} \right) \right] w_{xx} + \\
&\quad + \frac{1}{2} a^{3/2} \frac{\partial b}{\partial w_x} + \frac{9}{8a^2} \left(w_x \frac{\partial a}{\partial w} + \frac{\partial a}{\partial x} \right)^2 - \frac{3}{4a} \left(w_x^2 \frac{\partial^2 a}{\partial w^2} + 2 w_x \frac{\partial^2 a}{\partial w \partial x} + \frac{\partial^2 a}{\partial x^2} \right), \\
T &= -\frac{3}{4} a^{-1/2} \frac{\partial a}{\partial w_x}.
\end{aligned}
$$

Formula (4.12) implies that a takes the form $a = (\gamma_2 w_x^2 + \gamma_1 u_x + \gamma_0)^2$, where $\gamma_i = \gamma_i(x, w)$ and $\gamma_1^2 - 4\gamma_0\gamma_2 = \text{const}$. Point transformations $\bar{x} = \phi(x, w)$, $\bar{w} = \psi(x, w)$ allow us to transform a to $a = 1$ (**class 1A** equations) if the polynomial $\gamma_2 w_x^2 + \gamma_1 w_x + \gamma_0$ has a double root. If the roots are different, we can put $a = \lambda^2 w_x^2$ (**class 1B** equations).

4.1.1 Class 1A: $a = 1$

In this case we have $B = b(x, w, w_x)$, $Q = \frac{1}{2} \partial b / \partial w_x$ and $T = 0$. From (4.12) it follows that $b = 2\alpha(x) w_x + \alpha'(x) w + \beta(x)$. Gauging with point transformations $\bar{x} = \phi(x)$, $\bar{w} = \phi' x w + \psi x$ we can make $b = 0$. From (4.17) and (4.18) we can see that $H = C_1(x^2 w_x - 2xw) + C_2(x w_x - w) + C_3 w_x + C_4 x^2 + C_5 x + C_6$ and we have obtained Eq. (4.1). ■

4.1.2 Class 1B: $a = \lambda^2 w_x^2$

We have $B = \frac{3}{2} \lambda^{-3} w_{xx}^2 w_x^{-4} + b(x, w, w_x)$ and $Q = \frac{1}{2} \lambda^3 w_x^3 + \frac{\partial b}{\partial w_x}$, $T = -\frac{3}{2}\lambda$. From (4.12) it follows that $b = \alpha(x) w_x^{-2} + \beta(w)$. A point transformation of the form $\bar{x} = \phi(x)$, $\bar{w} = \psi(w)$ allows to take $b = 0$. ¿From (4.17) and (4.18) we obtain that for class 1B $H = (C_1 x^2 + C_2 x + C_3) w_x + C_4 w^2 + C_5 w + C_6$ and the resulting equation is Eq. (4.2). ■

4.2 Type 2: $A = a(x, w, w_x)^2\,[w_{xx} + S(x, w, w_x)]$

Let $z = (w_{xx} + S)^{1/2}$. Then $A = a^2 z^2$. From (4.14), (4.13), (4.11) and (3.5) we find that

$$B = 6\frac{\partial a}{\partial w_x} a^{-4} z + b(x, w, w_x) + 6\nabla(a) a^{-4} z^{-1} - 3\frac{\partial S}{\partial w_x} a^{-3} z^{-1} + \nabla S\, a^{-3} z^{-3}$$
$$Q = -\frac{3}{a}\left[\frac{\partial^2 a}{\partial w_x^2} + \frac{3}{a}\left(\frac{\partial a}{\partial w_x}\right)^2\right] z^4 - a^2\left[3b\frac{\partial a}{\partial w_x} + \frac{a}{4}\frac{\partial b}{\partial w_x}\right] z^3 + q_2 z^2 + q_1 z + q_0$$
$$T = -6\frac{\partial a}{\partial w_x} z - \frac{3}{8} a^4 b,$$

where $\nabla = S\partial/\partial w_x - w_x \partial/\partial w - \partial/\partial x$ and the functions q_i are expressed in terms of a, b and S. The condition (3.11) is automatically fulfilled and now our task is to determine the functions $a(x, w, w_x)$, $b(x, w, w_x)$ and $S(x, w, w_x)$. It is enough to use conditions (3.12), (3.13), (3.14) and (4.12). Starting with (3.13), we easily see that $\mathbf{\Phi}_2(E) = 0$, i.e. $E = E(x, w, w_x)$. Then $\mathbf{\Phi}_0(E) = z^2\,\partial E/\partial w_x - \nabla(E)$. Using the conditions derived from the coefficients of z^4, z^3 and z of (3.14) we obtain the following system of equations:

$$\frac{\partial^3 a}{\partial w_x^3} + \frac{9}{a}\frac{\partial a}{\partial w_x}\frac{\partial^2 q}{\partial w_x^2} + \frac{6}{a^2}\left(\frac{\partial a}{\partial w_x}\right)^3 = 0, \tag{4.19}$$
$$\frac{\partial^2 b}{\partial w_x^2} + \frac{12}{a}\frac{\partial q}{\partial w_x}\frac{\partial b}{\partial w_x} + \frac{12}{a} b\frac{\partial^2 a}{\partial w_x^2} + \frac{36}{a^2} b\left(\frac{\partial a}{\partial w_x}\right)^2 = 0, \tag{4.20}$$
$$\frac{\partial a}{\partial w_x}\left(\frac{\partial b}{\partial w} w_x + \frac{\partial b}{\partial x}\right) - \frac{\partial b}{\partial w_x}\left(\frac{\partial a}{\partial w} w_x + \frac{\partial a}{\partial x}\right) - 2b\frac{\partial a}{\partial w} - \frac{1}{2} a\frac{\partial b}{\partial w} = 0. \tag{4.21}$$

The general solution of the first equation is $a = (\gamma_2 w_x^2 + \gamma_1 w_x + \gamma_0)^{1/4}$, $\gamma_i = \gamma_i(x, w)$. With point transformations $\bar{x} = \phi(x, w)$, $\bar{w} = \psi(x, w)$ we can simplify a. If the polynomial $\gamma_2 w_x^2 + \gamma_1 w_x + \gamma_0$ has a multiple root, we can put $a = 1$, featuring equations of **class 2A**, and, if it has two different roots, we can put $a = \alpha(x, w) w_x^{1/4}$, which characterizes equations of **class 2B**.

4.2.1 Class 2A: $a = 1$

Equations (4.20) and (4.21) are rewritten as $\partial^2 b/\partial w_x^2 = 0$, $b_w = 0$. Allowed point transformations are $\bar{x} = \phi(x)$, $\bar{w} = w + \psi(x)$ and we write $b = \lambda w_x + n(x)$, where $\lambda n = 0$, $n'' = 0$. Conditions (3.13) and (3.14) imply $E = E(x, w)$ and

$$\frac{\partial E}{\partial w} w_x + \frac{\partial E}{\partial x} = S\left(\frac{3}{16} b\frac{\partial b}{\partial w_x} - \frac{1}{2}\frac{\partial^3 S}{\partial w_x^3}\right) - \frac{3}{16} b\frac{\partial b}{\partial x} +$$
$$+ \frac{1}{4}\frac{\partial^2 S}{\partial w_x \partial w} + \frac{1}{2}\left(\frac{\partial^3 S}{\partial w_x^2 \partial w} w_x + \frac{\partial S}{\partial w_x^2 \partial x}\right).$$

Condition (4.12) implies that $\partial^5 S/\partial w_x^5 = 0$ and

$$5\frac{\partial^2 S}{\partial w_x \partial w} + 2\nabla\left(\frac{\partial^2 S}{\partial w_x^2}\right) - \frac{3}{4}b\nabla(b) = 0, \qquad (4.22)$$

$$\nabla^2(b) - \frac{\partial S}{\partial w_x}\nabla(b) + 4b\frac{\partial S}{\partial w} = 0, \qquad (4.23)$$

$$\nabla\left(\frac{\partial S}{\partial w}\right) - 2\frac{\partial S}{\partial w}\frac{\partial S}{\partial w_x} = 0. \qquad (4.24)$$

If we write $S = \frac{1}{64}\lambda^2 w_x^4 + \alpha(x,w)w_x^2 + \beta(x,w)w_x + \gamma(x,w)$ relation (3.12) is equivalent to $\beta_{ww} - 2\alpha_{wx} = 0$. Dividing class 2A into subcases, we firstly find that for $\lambda \neq 0$ there is only one equation that satisfies all the previous conditions. This is Eq. (4.3). ■

The remaining subcases have $b = b(x)$ and either $b = \lambda$ or we can use an appropriate point transformation to make $b = x$. This last subcase cannot satisfy conditions (4.22)-(4.24). Consider then $b = \lambda \neq 0$. From (4.23) and (4.22) it follows that $\partial S/\partial w = 0$ and $\alpha' = 0$. Allowed point transformations, $\bar{x} = \phi(x)$, $\bar{w} = w + \psi(x)$, put the equation in the form Eq. (4.4). ■

When $b = 0$ and $\partial S/\partial w \neq 0$ (if $\partial S/\partial w = 0$ then we can transform the equation to (4.4)), conditions (4.22)-(4.24) and the allowed transformations lead to two different equations. The first with $b = 0$, $S = \lambda w_x^2 + \exp(-4\lambda w)$, is Eq. (4.5). ■

The second has $b = 0$ and $S = w$ and is equation Eq. (4.6). ■

4.2.2 Class 2B: $a = \alpha(x,w)w_x^{1/4}$

¿From condition (4.20) we obtain that $b = \lambda_2(x,w)w_x^{-2} + \lambda_1(x,w)$. Then from the coefficients of $w_x^{-11/4}$, $w_x^{-7/4}$, $w_x^{-3/4}$ and $w_x^{1/4}$ in (4.21) we obtain

$$8\alpha_x\lambda_2 + (\lambda_2)_x\alpha = 0, \qquad (4.25)$$

$$(\lambda_2)_w = 0, \qquad (\lambda_1)_x = 0, \qquad (4.26)$$

$$8\alpha_w\lambda_1 + (\lambda_1)_w\alpha = 0. \qquad (4.27)$$

Thus (4.26) implies that $b = \lambda_2(x)w_x^{-2} + \lambda_1(w)$. Condition (4.12) is a polynomial of fifth order in z. Equating to zero the coefficient of z^5 we obtain that

$$S = \gamma_1 w_x^{5/2} + \gamma_2 w_x^{3/2} + \gamma_3 w_x^{1/2} + \frac{1}{16}\alpha^6\lambda_1^2 w_x^{7/2} + \frac{1}{16}\alpha^6\lambda_2^2 w_x^{-1/2} + 4\frac{\alpha_x}{\alpha}w_x - 4\frac{\alpha_w}{\alpha}w_x^2$$

with $\gamma_i = \gamma_i(x,w)$. Now we can distinguish two subcases.

Subcase $(\log\alpha)_{xw} = 0$**.** Using the transformations $\bar{x} = \phi(x)$, $\bar{u} = \psi(u)$, we can put $\alpha = 1$. Then from (4.25)-(4.27) and (4.12) λ_1, λ_2, γ_1, γ_2 and γ_3 must be constants. Using (4.17) and (4.18) we find the equation Eq. (4.7):

$$w_t = -2(A^{-3/2} - B)^{-\frac{1}{2}} + c_1 w_x + c_2 \qquad (4.28)$$

with $A = w_x^{1/2}z^2$, $z = (w_{xx} + S)^{1/2}$ and

$$B = \frac{3}{2}w_x^{-7/4}z + \frac{3}{2}\left(Sw_x^{-7/4} - 2\frac{\partial S}{\partial w_x}w_x^{-3/4}\right)z^{-1} + S\frac{\partial S}{\partial w_x}w_x^{-3/4}z^{-3} + \lambda_2 w_x^{-2} + \lambda_1,$$

$$S = \gamma_1 w_x^{5/2} + \gamma_2 w_x^{3/2} + \gamma_3 w_x^{1/2} + \frac{1}{16}\lambda_1^2 w_x^{7/2} + \frac{1}{16}\lambda_2^2 w_x^{-1/2}.$$ ■

Subcase $(\log\alpha)_{xw}\neq 0$**.** ¿From (4.25)-(4.27) we have that $\lambda_1=\lambda_2=0$ and besides $A=w_x^{1/2}\alpha^2z^2$, $z=(w_{xx}+S)^{1/2}$ and

$$
\begin{aligned}
B={}&\frac{3}{2}w_x^{-7/4}\alpha^{-3}z+\Bigg(6\alpha_w\alpha^{-4}w_x^{1/2}+6\alpha_x\alpha^{-4}w_x^{-1/2}-\\
&-\frac{3}{2}S\alpha^{-3}w_x^{-7/4}+3\frac{\partial S}{\partial w_x}\alpha^{-3}w_x^{-3/4}\Bigg)z^{-1}+\alpha^{-3}w_x^{-3/4}S\nabla(S)z^{-3},\\
S={}&\gamma_1w_x^{5/2}+\gamma_2w_x^{3/2}+\gamma_3w_x^{1/2}-4\frac{\alpha_w}{\alpha}w_x^2+4\frac{\alpha_x}{\alpha}w_x.
\end{aligned}
$$

¿From the coefficient in z^3 of (4.12) we obtain that

$$
\begin{aligned}
&(\gamma_1)_w+6\alpha^{-1}\alpha_w\gamma_1=0,\qquad (\gamma_3)_w+6\alpha^{-1}\alpha_x\gamma_3=0,\\
&(\gamma_2)_w-5(\gamma_1)_x+2\gamma_2\alpha^{-1}\alpha_w+10\gamma_1\alpha^{-1}\alpha_x=0,\\
&(\gamma_2)_x-5(\gamma_3)_w+2\gamma_2\alpha^{-1}\alpha_x+10\gamma_3\alpha^{-1}\alpha_w=0.
\end{aligned}
\tag{4.29}
$$

Solving the first equation we have $\gamma_1=k_1(x)\alpha^{-6}$. The second gives $\gamma_3=k_3(w)\alpha^{-6}$ with k_1 and k_3 arbitrary functions. Using allowed point transformations we can make both constant. There are then three possible cases: $k_1=k_3=1$; $k_1=1$ and $k_3=0$; $k_1=k_3=0$. We study these three cases together, considering only that k_1 and k_3 are arbitrary constants. It is useful to define $\alpha=\beta^{-1/4}$ and $\gamma_2=\gamma\beta^{1/2}$. Thus we have

$$
S=k_1\beta^{3/2}w_x^{5/2}+\gamma\beta^{1/2}w_x^{3/2}+k_3\beta^{3/2}w_x^{1/2}+\beta_w\beta^{-1}w_x^2-\beta_x\beta^{-1}w_x \tag{4.30}
$$

and system (4.2.2) is equivalent to

$$
\gamma_x=10k_3\beta_w,\qquad \gamma_w=10k_1\beta_x. \tag{4.31}
$$

The evident compatibility condition for this system is

$$
k_1\beta_{xx}-k_3\beta_{ww}=0. \tag{4.32}
$$

Using these last three equations (4.12) is equivalent to

$$
\left(\beta^{-1}\beta_{xx}-\frac{3}{2}\beta^{-2}\beta_x^2\right)_w+4k_3\gamma\beta\beta_w+8k_1k_3\beta^2\beta_x=0, \tag{4.33}
$$

$$
\left(\beta^{-1}\beta_{ww}-\frac{3}{2}\beta^{-2}\beta_w^2\right)_x+4k_1\gamma\beta\beta_x+8k_1k_3\beta^2\beta_w=0. \tag{4.34}
$$

We can solve the system formed by these last five equations and find the expressions for β and γ as follows.

Consider the **case** $k_1=k_3=1$. From (4.32) we have that $\beta=f(w+x)-g(w-x)$, where f and g are arbitrary functions. ¿From (4.31) we find that $\gamma=10f(w+x)+10g(w-x)$. We introduce the notations $z_1=w+x$ and $z_2=w-x$. The system (4.33)-(4.34) is now

$$
\begin{aligned}
&\frac{f'''}{f-g}-4\frac{f'f''}{(f-g)^2}-2\frac{f'g''}{(f-g)^2}+3\frac{(f'^2-g'^2)f'}{(f-g)^3}+\\
&\qquad+40(f-g)(f+g)f'+8(f-g)^2f'=0,\\
&\frac{g'''}{f-g}+4\frac{g'g''}{(f-g)^2}+2\frac{g'f''}{(f-g)^2}-3\frac{(f'^2-g'^2)g'}{(f-g)^3}+\\
&\qquad+40(f-g)(f+g)g'-8(f-g)^2g'=0.
\end{aligned}
$$

Multiply the first equation by $(f-g)^{-1}$. It becomes

$$\frac{\partial}{\partial z_1}\left(\frac{f''+g''}{(f-g)^2}-\frac{f'^2-g'^2}{(f-g)^3}+24f^2+32fg\right)=0$$

so

$$\frac{f''+g''}{(f-g)^2}-\frac{f'^2-g'^2}{(f-g)^3}+24f^2+32fg+\kappa_1(z_2)=0$$

with $\kappa_1(z_2)$ being an arbitrary function of z_2. Analogously we obtain

$$\frac{f''+g''}{(f-g)^2}-\frac{f'^2-g'^2}{(f-g)^3}+24g^2+32fg+\kappa_2(z_1)=0$$

with $\kappa_1(z_1)$ being an arbitrary function of z_1. Adding these two equations we obtain that $\kappa_1=24g^2+\mu$ and $\kappa_2=24f^2+\mu$, where μ is an arbitrary constant. Then the system of two equations is equivalent to one equation:

$$\frac{f''+g''}{(f-g)^2}-\frac{f'^2-g'^2}{(f-g)^3}+24f^2+32fg+24g^2+\mu=0. \tag{4.35}$$

Consider the case $f'\neq 0$, $g'\neq 0$. Multiplying (4.35) by f' and integrating with respect to z_1 we obtain

$$\begin{aligned}(f')^2&+16f^5+2\mu f^3+(2\theta_1-64g^3-4\mu g)f^2+\\&+(48g^4+2\mu g^2-4\theta_1 g-2g'')f+2gg''-(g')^2+2\theta_1 g^2=0,\end{aligned} \tag{4.36}$$

where θ_1 is an arbitrary function of z_2. Analogously multiplying (4.35) by g' and integrating with respect to z_2 we have

$$\begin{aligned}(g')^2&+16g^5+2\mu g^3+(2\theta_2-64f^3-4\mu f)g^2+\\&+(48f^4+2\mu f^2-4\theta_2 f-2f'')g+2ff''-(f')^2+2\theta_2 f^2=0,\end{aligned} \tag{4.37}$$

where θ_2 is an arbitrary function of z_1. Differentiating (4.2.2) with respect z_1 and (4.2.2) with respect z_2 we obtain, quotienting respectively by f' and g':

$$f''+40f^4+3\mu f^2+(2\theta_1-64g^2-4\mu g)f+24g^4+\mu g^2-g''-2\theta_1 g=0, \tag{4.38}$$

$$g''+40g^4+3\mu g^2+(2\theta_2-64f^2-4\mu f)g+24f^4+\mu f^2-f''-2\theta_2 f=0. \tag{4.39}$$

Adding we have $(f-g)\left\{(\theta_1-\theta_2)+2\mu(f-g)+32(f^3-g^3)\right\}=0$. ¿From here we find that $\theta_1=32g^3+2\mu g+\mu_1$, $\theta_2=32f^3+2\mu f+\mu_1$, where μ_1 is an arbitrary constant. Equations (4.38) and (4.39) are equivalent to

$$f''+40f^4+3\mu f^2+2\mu_1 f-g''-40g^4-3\mu g^2-2\mu_1 g=0.$$

As f is a function of z_1 only, and g a function of z_2 only, it must be

$$f''+40f^4+3\mu f^2+2\mu_1 f+\mu_2=0,$$
$$g''+40g^4+3\mu g^2+2\mu_1 g+\mu_2=0,$$

with μ_2 an arbitrary constant, thus obtaining a pair of *ordinary* differential equations. Integrating once we obtain

$$(f')^2 + 16f^5 + 2\mu f^3 + 2\mu_1 f^2 - 2\mu_2 f + \mu_3 = 0,$$
$$(g')^2 + 16g^5 + 2\mu f^3 + 2\mu_1 g^2 - 2\mu_2 g + \mu_4 = 0.$$

Using these four relations in (4.35) we can see that $\mu_3 = \mu_4$. So finally we have found that

$$\beta = f(w+x) - g(w-x), \qquad \gamma = 10f(w+x) + 10g(w-x),$$

where f and g are any solutions of

$$(y')^2 + 16y^5 + \lambda_3 y^3 + \lambda_2 y^2 + \lambda_1 y + \lambda_0 = 0$$

with λ_i arbitrary constants. This is Eq. (4.8). ∎

Although we assumed that $f' \neq 0$ and $g' \neq 0$, it is possible to prove that these expressions are also true when $f' = 0$ or $g' = 0$. The case $f' = g' = 0$ was studied above.

The **case** $k_1 = 1$**,** $k_3 = 0$. From (4.31) and (4.32) we can put $\gamma = 10g(w)$, $\beta = g'(w)x + f(w)$, with $g(w)$ and $f(w)$ arbitrary functions, with $g'(w) \neq 0$ in order not to fall into cases already studied. Equation (4.33) becomes $g''f - g'f' = 0$ and it must be that $f = \lambda g'$, that is, $\beta = g'(x+\lambda)$. Using point transformations we can put $\beta = 1$ and this is a case studied already.

For **case** $k_1 = k_3 = 0$, from (4.31) it follows that γ is an arbitrary constant, while (4.33) and (4.34) imply that

$$\frac{\beta_{xx}}{\beta} - \frac{3}{2}\frac{\beta_x^2}{\beta^2} = \iota_1(x), \qquad \frac{\beta_{ww}}{\beta} - \frac{3}{2}\frac{\beta_w^2}{\beta^2} = \iota_2(w).$$

Using point transformations $\bar{x} = \phi(x)$ and $\bar{w} = \psi(w)$, we can make $\iota_1 = \iota_2 = 0$. The solution of the equations yields $\beta_x = \kappa_1(w)\beta^{3/2}$, $\beta_w = \kappa_2(x)\beta^{3/2}$. Compatibility conditions imply that $\kappa_1' - \kappa_2' = 0$. If $\kappa_1' = 0$, allowed transformations permit to put $\beta = 1/(x+w)^2$, yielding Eq. (4.9). ∎

If $\kappa_1' \neq 0$, we can put $\beta = 1/(xw+\lambda)^2$, but using allowed transformations this case is the previous one.

5 Differential substitutions into the KdV and KN equations

All the nine differential equations of the previous section are related to the KdV or KN equations through a highly nontrivial differential substitution (1.7) (and thus are integrable). When an evolution equation admits a classical symmetry, then there exists a differential substitution to an equation with a smaller symmetry group. These are the so-called *group transformations* [15]. The original KdV (1.3) admits classical symmetries, and the substitution $w = 1/u_x$, $y = u$ plus a scaling transforms KdV into

$$w_t = D_y\left(\frac{w_{yy}}{w^3} - 3\frac{w_y^2}{w^4} - 3y\right), \tag{5.1}$$

that has a null symmetry group. Similarly the nonsymmetric version of Krichever--Novikov (1.4) is

$$w_t = D_y\left(\frac{w_{yy}}{w^3} - \frac{3}{2}\frac{w_y^2}{w^4} + R(y)w^2\right), \quad R(y) = y^3 + \rho_1 y + \rho_0. \tag{5.2}$$

Differential substitutions (1.7) transform conserved densities ρ (so $\rho_t = \sigma_x$) into conserved densities $\bar{\rho} = \rho/D_x\phi$. Consider differential substitutions (in non-standard variables) of the general equation (3.2). The variable u is a conserved density of (3.2). Requiring it to transform into the conserved density w of (5.1) or (5.2), the substitution must be of the form

$$y = Z, \qquad w = \frac{u}{D_x Z}. \tag{5.3}$$

5.1 Differential substitutions to KdV

We give here a way to find differential substitutions from equation (3.2)-(3.3) to (5.1). Imposing that the differential substitution (5.3) must transform (3.2) into (5.1) we see that

$$Z_t = \frac{D_x Z}{u}\left(\frac{Q}{u^2} + T + lu^2 + m\,u\right) + \frac{D_x^3 Z}{u^3} - 3D_x^2 Z\frac{u_x}{u^4} + \frac{3}{2}D_x Z\frac{u_x^2}{u^5} + 3\frac{Z\,D_x Z}{u}$$

and there are two types of substitutions, denoted as KdV1 and KdV2:

$$Z = \frac{1}{u}D_x\left(-\frac{u_x}{u^2} + N(\mathbf{v})u\right) - \frac{1}{2}\frac{u_x^2}{u^4} - \frac{Q}{u^2} - \frac{1}{3}T + \frac{1}{3}lu^2, \tag{5.4}$$

$$Z = \frac{1}{u}D_x\left(\frac{u_x}{u^2} + \frac{N(\mathbf{v})}{u}\right) - \frac{1}{2}\frac{u_x^2}{u^4} - \frac{Q}{u^2} - \frac{1}{3}T + \frac{1}{3}lu^2. \tag{5.5}$$

Necessary and sufficient compatibility conditions are, for KdV1:

$$\mathbf{\Phi}_2(N) = -\frac{2}{3}l - \frac{1}{4}N^2, \tag{5.6}$$

$$0 = 6\mathbf{\Phi}_2\mathbf{\Phi}_2\mathbf{\Phi}_0(N) + 3N\mathbf{\Phi}_2\mathbf{\Phi}_0(N) - 2\mathbf{\Phi}_2(T)N + 2\mathbf{\Phi}_2\mathbf{\Phi}_2(T) + 6\mathbf{\Phi}_2\mathbf{\Phi}_0(l), \tag{5.7}$$

$$0 = 3\mathbf{\Phi}_0\mathbf{\Phi}_2\mathbf{\Phi}_0(N) - 3\mathbf{\Phi}_2\mathbf{\Phi}_0\mathbf{\Phi}_0(N) + \frac{3}{2}N\mathbf{\Phi}_0\mathbf{\Phi}_0(N) + 2\mathbf{\Phi}_2(Q)N - \\ -\frac{3}{2}QN^2 - 2\mathbf{\Phi}_0\mathbf{\Phi}_2(T) - \frac{4}{3}Ql + E, \tag{5.8}$$

$$0 = \mathbf{\Phi}_0\mathbf{\Phi}_0\mathbf{\Phi}_0(N) - 2Q\mathbf{\Phi}_0(N) - \mathbf{\Phi}_0(Q)N + 2\mathbf{\Phi}_0\mathbf{\Phi}_2(Q) - \mathbf{\Phi}_2\mathbf{\Phi}_0(Q), \tag{5.9}$$

$$\mathbf{s}(N) = \mathbf{\Phi}_0(N)m - \mathbf{\Phi}_0(m)N + 2\mathbf{\Phi}_2\mathbf{\Phi}_0(m) \tag{5.10}$$

and for KdV2:

$$\mathbf{\Phi}_0(N) = 2Q - \frac{1}{4}N^2, \tag{5.11}$$

$$0 = \mathbf{\Phi}_2\mathbf{\Phi}_2\mathbf{\Phi}_2(N) + \frac{2}{3}l\mathbf{\Phi}_2(N) - \mathbf{\Phi}_2\mathbf{\Phi}_2(T) - \mathbf{\Phi}_2\mathbf{\Phi}_0(l), \tag{5.12}$$

$$0 = 3\mathbf{\Phi}_2\mathbf{\Phi}_0\mathbf{\Phi}_2(N) - 3\mathbf{\Phi}_0\mathbf{\Phi}_2\mathbf{\Phi}_2(N) + \frac{3}{2}N\mathbf{\Phi}_2\mathbf{\Phi}_2(N) - N\mathbf{\Phi}_2(T) - \\ - N\mathbf{\Phi}_0(l) + \frac{1}{2}lN^2 - 2\mathbf{\Phi}_2\mathbf{\Phi}_2(Q) - \frac{4}{3}Ql + E, \tag{5.13}$$

$$0 = 3\mathbf{\Phi}_0\mathbf{\Phi}_0\mathbf{\Phi}_2(N) + \frac{3}{2}N\mathbf{\Phi}_0\mathbf{\Phi}_2(N) - N\mathbf{\Phi}_2(Q) + 3\mathbf{\Phi}_2\mathbf{\Phi}_0(Q) - 8\mathbf{\Phi}_0\mathbf{\Phi}_2(Q), \tag{5.14}$$

$$\mathbf{s}(N) = -\frac{1}{4}N^2m + N\mathbf{\Phi}_0(m) + 2Qm - 2\mathbf{\Phi}_0\mathbf{\Phi}_0(m). \tag{5.15}$$

5.2 Differential substitutions to KN

The differential substitution is also of the form (5.3) with

$$Z_t = \frac{D_xZ}{u}\left(\frac{Q}{u^2} + T + lu^2 + mu\right) + \frac{1}{u^3}\left(D_x^3Z - \frac{3}{2}\frac{(D_x^2Z)^2}{D_xZ}\right) - \frac{R(Z)}{D_xZ}u.$$

One can see that $Z = Z(\mathbf{v})$ and there are again two types of differential substitution. For the first one, KN1, there must be a nonconstant function, $Z(\mathbf{v})$, satisfying

$$\mathbf{\Phi}_2(Z) = 0, \tag{5.16}$$

$$0 = \mathbf{\Phi}_2\mathbf{\Phi}_2\mathbf{\Phi}_0(Z) - \frac{1}{2}\frac{(\mathbf{\Phi}_2\mathbf{\Phi}_0(Z))^2}{\mathbf{\Phi}_0(Z)} + \frac{1}{3}l\mathbf{\Phi}_0(Z) - \frac{1}{3}\frac{R(Z)}{\mathbf{\Phi}_0(Z)}, \tag{5.17}$$

$$0 = \mathbf{\Phi}_2\mathbf{\Phi}_0\mathbf{\Phi}_0(Z) - \mathbf{\Phi}_0\mathbf{\Phi}_2\mathbf{\Phi}_0(Z) - \frac{\mathbf{\Phi}_2\mathbf{\Phi}_0(Z)\mathbf{\Phi}_0\mathbf{\Phi}_0(Z)}{\mathbf{\Phi}_0(Z)} + \frac{1}{3}T\mathbf{\Phi}_0(Z), \tag{5.18}$$

$$0 = \mathbf{\Phi}_0\mathbf{\Phi}_0\mathbf{\Phi}_0(Z) - \frac{3}{2}\frac{(\mathbf{\Phi}_0\mathbf{\Phi}_0(Z))^2}{\mathbf{\Phi}_0(Z)} + Q\mathbf{\Phi}_0(Z), \tag{5.19}$$

$$\mathbf{s}(Z) = m\,\mathbf{\Phi}_0(Z). \tag{5.20}$$

For KN2 $Z(\mathbf{v})$ must satisfy

$$\mathbf{\Phi}_0(Z) = 0, \tag{5.21}$$

$$0 = \mathbf{\Phi}_0\mathbf{\Phi}_0\mathbf{\Phi}_2(Z) - \frac{1}{2}\frac{(\mathbf{\Phi}_0\mathbf{\Phi}_2(Z))^2}{\mathbf{\Phi}_2(Z)} - Q\mathbf{\Phi}_2(Z) - \frac{1}{3}\frac{R(Z)}{\mathbf{\Phi}_2(Z)}, \tag{5.22}$$

$$0 = \mathbf{\Phi}_0\mathbf{\Phi}_2\mathbf{\Phi}_2(Z) - \mathbf{\Phi}_2\mathbf{\Phi}_0\mathbf{\Phi}_2(Z) - \frac{\mathbf{\Phi}_2\mathbf{\Phi}_2(Z)\mathbf{\Phi}_0\mathbf{\Phi}_2(Z)}{\mathbf{\Phi}_2(Z)} + \frac{1}{3}T\mathbf{\Phi}_2(Z), \tag{5.23}$$

$$0 = \mathbf{\Phi}_2\mathbf{\Phi}_2\mathbf{\Phi}_2(Z) - \frac{3}{2}\frac{(\mathbf{\Phi}_2\mathbf{\Phi}_2(Z))^2}{\mathbf{\Phi}_2(Z)} - \frac{1}{3}l\mathbf{\Phi}_2(Z), \tag{5.24}$$

$$\mathbf{s}(Z) = 0. \tag{5.25}$$

5.3 The differential substitutions of equations (4.1)–(4.9)

In some case there is more than one type of substitution that transforms any of equations (4.1)–(4.9) to either KdV or KN. We give here just one transformation for each equation.

For equations (4.1)–(4.6) a KdV1 substitution with $N = 0$ suffices to transform them into KdV.

Equation (4.7) admits KdV2 with $N = -\lambda_1 w_x^{3/4} z + \frac{1}{4}\lambda_1^2 w_x^{5/2}$.

Equation (4.9) admits many substitutions into the KN equation, of type KN1 and KN2. One of type KN1 has

$$Z = Z(x, w, w_x) = -\frac{3}{4}\frac{x-\alpha}{w+\alpha} - \frac{\lambda}{8},$$

where α is a root of $c_1\alpha^2 + c_2\alpha + c_3 = 0$.

The calculation of a transformation for equation (4.8) is nontrivial and it is the subject of the following section. It is remarkable that, in spite of the appearance of general hyperelliptic functions, the substitution relates this equation to KN, that is related to standard elliptic functions (there is a point transformation from the form given here (5.2) to an equation with a Weierstrassian $\wp$).

6 Derivation of the differential substitution for eq. (4.8)

We have found a KN-1 substitution for eq. (4.8). Condition (5.16) implies that $N(x, w, w_x, w_{xx}) = N(x, w, w_x)$. Now condition (5.19) is a polynomial of fourth order in z^2. The coefficient in z^8 implies that either

$$N = \frac{d(x, w)}{\sqrt{w_x} + k(x, w)} + n(x, w) \qquad (6.1)$$

or $N = n(x, w)\sqrt{w_x} + m(x, w)$. This latter possibility is discarded because then (5.18) implies that $n(x, w) = 0$ and $m(x, w) = k$ so that $N = k$ constant, which does not constitute a proper differential substitution.

Consider the form (6.1) of N. The coefficient of z^6 in (5.19) implies

$$d = -\frac{1}{\beta^{3/2}}\frac{\partial n}{\partial w}$$

(which discards $\partial n/\partial w = 0$ as a valid solution). The conditions now translate into relations between different partial derivatives of $n(x, w)$ and $k(x, w)$, involving the parameters in the equation (4.9). For example the previous condition also yields (using subscript notation for partial derivatives)

$$n_{xw} = \beta^{3/2} k n_x - \left(\frac{\beta^{3/2}}{k} - \frac{3}{2}\frac{\beta_x}{\beta} - \frac{1}{k}k_x\right) n_w.$$

The coefficient in (5.19) of z^4 implies a relationship between n_x and n_{www}, n_{ww}, n_w, k_w. Condition (5.18) gives relations: between n_{www} and n_{ww}, n_w, k_w, between k_x and k_{ww}, k_w and finally a relation between k_{www} and k_{ww}, k_w. After we consider all the relations, conditions (5.18)–(5.19) are satisfied.

Equation (5.17) provides four independent conditions. Denoting $R(n) = n^3 + \rho_1 n + \rho_0$ (recall (5.2)) we can write them as

$$3n_w^2 = 8\beta^2(n^3 + \rho_1 n + \rho_0), \qquad (6.2)$$

$$3\beta(3\beta k^4 + 24nk^2 - 3\beta)^2(n^3 + \rho_1 n + \rho_0) = $$
$$= 8[3\beta(3n^2 + \rho_1)k^2 + 8(n^3 + \rho_1 n + \rho_0)]^2 k^2. \qquad (6.3)$$

The first relation is a differential equation that defines the function $n(x, w)$ as an elliptic function. The second relation is a polynomial relating the functions $n(x, w)$ and $k(x, w)$: of eighth order in k and sixth order in n! These equations define a differential substitution from equation (4.8) into the KN equation (5.2). A major computational difficulty was to prove that they are compatible with all the previous differential relations, but modern computer algebra systems made this possible.

7 Conclusions and further work

We have researched and classified an interesting family of fully nonlinear evolution equations of order three. We have found that any integrable equation in the family can be put, using a contact transformation, in one of nine "normal" forms corresponding to nine given evolution equations. One of these equations (eq. (4.8)) involves hyperelliptical functions and that could be a sign of being a new integrable equation, because the only ones known in this family involve polynomials (KdV) and elliptic functions (some form of KN). However the same happened with some simpler integrable quasilinear equations studied in [7], but they finally turned out to be transformable into KN. We have shown that the same situation arises in the fully nonlinear case.

The types of equations (2.15) and (2.16) await classification. They will surely provide computational challenges and a good, final test to our conjecture.

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