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Bäcklund transformations for the rational Lagrange chain

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Abstract

We consider a long–range homogeneous chain where the local variables are the generators of the direct sum of N $\mathfrak{e}(3)$ interacting Lagrange tops. We call this classical integrable model rational "Lagrange chain" showing how one can obtain it starting from $\mathfrak{su}(2)$ rational Gaudin models. Moreover we construct one- and two–point integrable maps (Bäcklund transformations).

1 Introduction

In [6] we have performed a Inönü–Wigner contraction on Gaudin models [3, 4], showing that the integrability property is preserved by this algebraic procedure. Starting from rational, trigonometric and elliptic Gaudin models it is possible to obtain new integrable long–range chains, associated to the same linear r-matrix structure.

The first natural extension [6, 7] of the N-sites $\mathfrak{su}(2)$ Gaudin model is obtained contracting two copies of the Lie-Poisson algebra $\mathfrak{su}(2)$, namely

$$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \simeq \mathfrak{o}(4) \to \mathfrak{e}(3)$$

where $\mathfrak{e}(3)$ is the real euclidean Lie–Poisson algebra in the 3-space. In this way one obtains new integrable chains where local variables are the generators of the direct sum of N $\mathfrak{e}(3)$ interacting Lagrange tops. From this feature comes the name "Lagrange chains".

Other interesting structures can be inherited from the Gaudin models as well. For example in [7] it is shown how to obtain Bäcklund transformations for the Lagrange top from those of the $\mathfrak{su}(2)$ Gaudin model. It turns out that it is possible to generalize this approach to the construction of Bäcklund transformations for Lagrange chains. This is the main goal of the present paper.

2 Algebraic contractions of $\mathfrak{su}(2)$ Gaudin models

In this section we illustrate the contraction procedure performed on the rational $\mathfrak{su}(2)$ Gaudin model. We stress the fact that this construction is also possible when considering trigonometric and elliptic solutions of the classical Yang–Baxter equation and considering a generic finite–dimensional simple Lie algebra instead of $\mathfrak{su}(2)$. See [6] for details.

The rational $\mathfrak{su}(2)$ Gaudin model is derived from the following 2×2 Lax matrix

$$L_{\mathcal{G}}(\lambda) = \tau + \sum_{i=1}^{N} \sum_{\alpha=1}^{3} \sigma^{\alpha} \, \frac{x_i^{\alpha}}{(\lambda - \lambda_i)}, \tag{2.1}$$

where τ can be chosen as any $\mathfrak{su}(2)$-matrix and $\sigma^1, \sigma^2, \sigma^3$ as a basis of the fundamental representation of $\mathfrak{su}(2)$:

$$\sigma^1 \doteq \frac{\mathrm{i}\,\sigma_x}{2} = \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^2 \doteq \frac{\mathrm{i}\,\sigma_y}{2} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma^3 \doteq \frac{\mathrm{i}\,\sigma_z}{2} = \frac{1}{2}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix},$$

where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices. The constants $\lambda_i \in \mathbb{C}$ are parameters of the model and $\lambda \in \mathbb{C}$ is the spectral parameter. The local variables of the model x_i^{α}, $\alpha = 1, 2, 3$, $i = 1, ..., N$ are the real generators of the direct sum of N $\mathfrak{su}(2)$ spins with the following Lie–Poisson brackets:

$$\left\{ x_i^{\alpha}, x_j^{\beta} \right\} = \delta_{ij}\, x_i^{\gamma} \qquad i, j = 1, ..., N$$

where $\alpha\beta\gamma$ is the cyclic permutation of 123.

The Lax matrix (2.1) satisfies the linear r-matrix Poisson algebra:

$$\{L_{\mathcal{G}}(\lambda) \otimes \mathbb{1}, \mathbb{1} \otimes L_{\mathcal{G}}(\mu)\} - [r(\lambda - \mu), L_{\mathcal{G}}(\lambda) \otimes \mathbb{1} + \mathbb{1} \otimes L_{\mathcal{G}}(\mu)] = 0,$$

where $\mathbb{1}$ is the 2×2 identity matrix and the r-matrix is given by

$$r(\lambda) = \frac{\mathrm{i}}{\lambda} \sum_{\alpha=1}^{3} \sigma^{\alpha} \otimes \sigma^{\alpha}.$$

Performing a Inönü–Wigner contraction to the direct sum of 2 copies of $\mathfrak{su}(2)$ we obtain from (2.1) the following new Lax matrix [6]:

$$L(\lambda) = \tau + \sum_{\alpha=1}^{3} \sigma^{\alpha} \left[\frac{y^{\alpha}}{\lambda} + \frac{z^{\alpha}}{\lambda^2} \right], \tag{2.2}$$

where the new real generators (y^{α}, z^{α}), $\alpha = 1, 2, 3$ satisfy the $\mathfrak{e}(3)$ Lie–Poisson brackets:

$$\left\{ y^{\alpha}, y^{\beta} \right\} = y^{\gamma}, \qquad \left\{ y^{\alpha}, z^{\beta} \right\} = z^{\gamma}, \qquad \left\{ z^{\alpha}, z^{\beta} \right\} = 0, \qquad \alpha, \beta, \gamma = 1, 2, 3,$$

where $\alpha\beta\gamma$ is the cyclic permutation of 123. Note that in the rational case the Lax matrix (2.2) is exactly the Lax matrix of the Lagrange top [1, 11]. This particular contraction procedure on the two–site $\mathfrak{su}(2)$ rational Gaudin model has been considered in [7].

We now extend the Lax matrix (2.2) to the N-bodies case. Namely we consider the Lax matrix associated to the Lie–Poisson algebra of the direct sum of N copies of $\mathfrak{e}(3)$:

$$\mathcal{L}(\lambda) = \tau + \sum_{i=1}^{N} \sum_{\alpha=1}^{3} \sigma^\alpha \left[\frac{y_i^\alpha}{\lambda - \lambda_i} + \frac{z_i^\alpha}{(\lambda - \lambda_i)^2} \right],$$

where the local variables of the model are generators of the direct sum of N $\mathfrak{e}(3)$ tops, (y_i^α, z_i^α), $i = 1, ..., N$, $\alpha = 1, 2, 3$ with the following Lie–Poisson brackets:

$$\left\{ y_i^\alpha, y_j^\beta \right\} = \delta_{ij}\, y_j^\gamma, \qquad \left\{ y_i^\alpha, z_j^\beta \right\} = \delta_{ij}\, z_j^\gamma, \qquad \left\{ z_i^\alpha, z_j^\beta \right\} = 0, \qquad \alpha, \beta, \gamma = 1, 2, 3. \quad (2.3)$$

3 The rational Lagrange chain

As we have shown in the previous section the $\mathfrak{e}(3)$ Lagrange chain is derived from the following 2×2 Lax matrix

$$\mathcal{L}(\lambda) = w\, \sigma^3 + \sum_{i=1}^{N} \sum_{\alpha=1}^{3} \sigma^\alpha \left[\frac{y_i^\alpha}{\lambda - \lambda_i} + \frac{z_i^\alpha}{(\lambda - \lambda_i)^2} \right]. \tag{3.1}$$

The parameter w denotes the intensity of an external field, taken as uniform (along the chain) and constant (in time).

Recall that a generic element $\xi \in \mathfrak{su}(2)$ may be written as

$$\xi = \frac{1}{2} \begin{pmatrix} \mathrm{i}\,\xi^3 & \mathrm{i}\,\xi^1 + \xi^2 \\ \mathrm{i}\,\xi^1 - \xi^2 & -\mathrm{i}\,\xi^3 \end{pmatrix}.$$

If we associate to this matrix the vector

$$\xi = (\xi^1, \xi^2, \xi^3)^T \in \mathbb{R}^3$$

then it is easy to verify that this correspondence is an isomorphism between $\mathfrak{su}(2)$ and the Lie algebra $(\mathbb{R}^3, [\cdot,\cdot])$, where the Lie bracket $[\cdot,\cdot]$ is realized with the vector product.

Let us now fix the following notation:

$$\mathbf{Y}_i \doteq (y_i^1, y_i^2, y_i^3)^T \in \mathbb{R}^3, \qquad \mathbf{Z}_i \doteq (z_i^1, z_i^2, z_i^3)^T \in \mathbb{R}^3, \qquad i = 1, ...N,$$

$$\mathbf{y}^\alpha \doteq (y_1^\alpha, ..., y_N^\alpha)^T \in \mathbb{R}^N, \qquad \mathbf{z}^\alpha \doteq (z_1^\alpha, ..., z_N^\alpha)^T \in \mathbb{R}^N, \qquad \alpha = 1, 2, 3.$$

Namely $\mathbf{Y}_i$ denotes the i-th angular momentum and $\mathbf{Z}_i$ denotes the vector pointing from the fixed point to the center of mass of the i-th top; $\mathbf{y}^\alpha$ and $\mathbf{z}^\alpha$ denote respectively the sets of N angular momenta and position vectors with α fixed.

The Lie–Poisson brackets (2.3) have $2N$ Casimir functions:

$$C_i^{(1)} = \langle \mathbf{Y}_i, \mathbf{Z}_i \rangle \doteq \sum_{\alpha=1}^{3} y_i^\alpha\, z_i^\alpha, \qquad C_i^{(2)} = \langle \mathbf{Z}_i, \mathbf{Z}_i \rangle \doteq \sum_{\alpha=1}^{3} (z_i^\alpha)^2 \qquad i = 1, ..., N.$$

Fixing their values one gets a $2N$-dimensional symplectic leaf

$$\mathcal{O} \doteq \left\{ (\mathbf{Y}_i, \mathbf{Z}_i) \in \mathbb{R}^3 \times \mathbb{R}^3, i = 1, .., N \mid C_i^{(1)} = \ell_i,\ C_i^{(2)} = 1 \right\}.$$

As we have shown in [6] the Lax matrix (3.1) satisfies the linear r-matrix algebra

$$\{\mathcal{L}(\lambda) \otimes \mathbb{1}, \mathbb{1} \otimes \mathcal{L}(\mu)\} - [r(\lambda - \mu), \mathcal{L}(\lambda) \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{L}(\mu)] = 0, \tag{3.2}$$

where $\mathbb{1}$ is the 2×2 identity matrix and the r-matrix is given by

$$r(\lambda) = \frac{\mathrm{i}}{\lambda} \sum_{\alpha=1}^{3} \sigma^\alpha \otimes \sigma^\alpha.$$

The spectral curve Γ,

$$\Gamma : \quad \det(\mathcal{L}(\lambda) - \mu \mathbb{1}) = 0, \tag{3.3}$$

is an hyperelliptic curve of genus $g = 2N - 1$, reading

$$-\mu^2 = w^2 + \sum_{i=1}^{N} \frac{R_i}{\lambda - \lambda_i} + \frac{S_i}{(\lambda - \lambda_i)^2} + \frac{C_i^{(1)}}{(\lambda - \lambda_i)^3} + \frac{C_i^{(2)}}{(\lambda - \lambda_i)^4},$$

with the Hamiltonians R_i and S_i given by:

$$\begin{aligned} R_i &= \langle \mathbf{W}, \mathbf{Y}_i \rangle + \sum_{k \neq i}^{N} \left(\frac{\langle \mathbf{Y}_i, \mathbf{Y}_k \rangle}{\lambda_i - \lambda_k} + \frac{\langle \mathbf{Y}_i, \mathbf{Z}_k \rangle - \langle \mathbf{Y}_k, \mathbf{Z}_i \rangle}{(\lambda_i - \lambda_k)^2} - 2 \frac{\langle \mathbf{Z}_i, \mathbf{Z}_k \rangle}{(\lambda_i - \lambda_k)^3} \right), \\ S_i &= \langle \mathbf{W}, \mathbf{Z}_i \rangle + \frac{\langle \mathbf{Y}_i, \mathbf{Y}_i \rangle}{2} + \sum_{k \neq i}^{N} \left(\frac{\langle \mathbf{Y}_k, \mathbf{Z}_i \rangle}{\lambda_i - \lambda_k} + \frac{\langle \mathbf{Z}_i, \mathbf{Z}_k \rangle}{(\lambda_i - \lambda_k)^2} \right), \end{aligned}$$

where $\mathbf{W} \doteq (0, 0, w)^T \in \mathbb{R}^3$ is the external field vector. These are integrals of motion of the rational Lagrange chain, which are Poisson commuting due to (3.2):

$$\{R_i, R_j\} = \{S_i, S_j\} = \{R_i, S_j\} = 0, \quad i, j = 1, ..., N.$$

Notice that the linear integral

$$\sum_{i=1}^{N} R_i = \sum_{i=1}^{N} \langle \mathbf{W}, \mathbf{Y}_i \rangle,$$

yields the third component of the total angular momentum of the chain.

We can bring the curve Γ into the canonical form by the scaling

$$\mu \longmapsto \hat{\mu} = \mu \prod_{i=1}^{N} (\lambda - \lambda_i)^2.$$

The equation of the spectral curve becomes

$$\begin{aligned} -\hat{\mu}^2 &= \left[w^2 + \sum_{i=1}^{N} \frac{R_i}{\lambda-\lambda_i} + \frac{S_i}{(\lambda-\lambda_i)^2} + \frac{C_i^{(1)}}{(\lambda-\lambda_i)^3} + \frac{C_i^{(2)}}{(\lambda-\lambda_i)^4} \right] \prod_{i=1}^{N} (\lambda-\lambda_i)^4 = \\ &= w^2\lambda^{4N} + s_1\lambda^{4N-1} + s_2\lambda^{4N-2} + ... + s_{4N}, \end{aligned}$$

where the coefficients s_j, $j = 1, ..., 4N$ are linear combinations of the Hamiltonians and the Casimir functions.

In the following we will use complex conjugated generators

$$y_i^{\pm} \doteq y_i^1 \pm \mathrm{i}\, y_i^2 \qquad i = 1, ..., N,$$

which have the brackets:

$$\begin{aligned} &\left\{ y_i^3, y_j^{\pm} \right\} = \mp\delta_{ij}\, y_j^{\pm}, \qquad \left\{ y_i^+, y_j^- \right\} = -2\mathrm{i}\, \delta_{ij}\, y_j^3, \\ &\left\{ y_i^3, z_j^{\pm} \right\} = \left\{ z_i^3, y_j^{\pm} \right\} = \mp\mathrm{i}\, \delta_{ij}\, z_j^{\pm}, \qquad \left\{ y_i^+, z_j^- \right\} = \left\{ z_i^+, y_j^- \right\} = -2\mathrm{i}\, \delta_{ij}\, z_j^3, \\ &\left\{ y_i^\beta, z_j^\beta \right\} = 0, \qquad \left\{ z_i^\alpha, z_j^\beta \right\} = 0, \qquad \alpha, \beta, \gamma = \pm, 3. \end{aligned}$$

In terms of these generators the Lax matrix (3.1) has the following explicit form

$$\mathcal{L}(\lambda) = \mathcal{W} + \sum_{i=1}^{N} \left[\frac{\mathcal{Y}_i}{\lambda-\lambda_i} + \frac{\mathcal{Z}_i}{(\lambda-\lambda_i)^2} \right], \tag{3.4}$$

where $\mathcal{W} \doteq w\,\sigma^3 \in \mathfrak{su}(2)$ and

$$\mathcal{Y}_i \doteq \frac{\mathrm{i}}{2} \begin{pmatrix} y_i^3 & y_i^- \\ y_i^+ & -y_i^3 \end{pmatrix} \in \mathfrak{su}(2), \qquad \mathcal{Z}_i \doteq \frac{\mathrm{i}}{2} \begin{pmatrix} z_i^3 & z_i^- \\ z_i^+ & -z_i^3 \end{pmatrix} \in \mathfrak{su}(2).$$

4 A Lax formulation

A possible choice for a physical Hamiltonian describing the dynamics of the model is the following one:

$$\mathcal{H} = \sum_{i=1}^{N} (\lambda_i\, R_i + S_i) = \sum_{i=1}^{N} \langle \mathbf{W}, \lambda_i \mathbf{Y}_i + \mathbf{Z}_i \rangle + \frac{1}{2} \sum_{i,k=1}^{N} \langle \mathbf{Y}_i, \mathbf{Y}_k \rangle. \tag{4.1}$$

Let us remark that if $N = 1$ the Lagrange chain degenerates into the well-known $\mathfrak{e}(3)$ symmetric Lagrange top, whose Hamiltonians are given by y^3 and $[(y^1)^2+(y^2)^2+(y^3)^2]/2+wz^3$ [1, 11, 7].

In the case of the Hamiltonian (4.1) the equations of motion are given by

$$\begin{cases} \dot{\mathbf{Y}}_i = \mathbf{W} \wedge \mathbf{Z}_i + \left(\lambda_i \mathbf{W} + \sum_{k=1}^{N} \mathbf{Y}_k \right) \wedge \mathbf{Y}_i\,, \\ \dot{\mathbf{Z}}_i = \left(\lambda_i \mathbf{W} + \sum_{k=1}^{N} \mathbf{Y}_k \right) \wedge \mathbf{Z}_i\,, \end{cases} \qquad i = 1, ..., N, \tag{4.2}$$

where $\wedge$ is the standard vector product in $\mathbb{R}^3$. Note again that if $N = 1$ equations (4.2) coincide with the equations of motion of the symmetric Lagrange top [1, 7, 2] provided that $\lambda_1 = 0$.

We can write the equation of motion in the following form:

$$\begin{cases} \dot{\mathcal{Y}}_i = [\mathcal{W}, \mathcal{Z}_i] + \left[\lambda_i \mathcal{W} + \sum_{k=1}^N \mathcal{Y}_k, \mathcal{Y}_i\right], \\ \dot{\mathcal{Z}}_i = \left[\lambda_i \mathcal{W} + \sum_{k=1}^N \mathcal{Y}_k, \mathcal{Z}_i\right], \end{cases} \qquad i = 1, ..., N. \tag{4.3}$$

One has the following statement:

Proposition 1. The Lax representation for equations (4.3) is given by

$$\dot{\mathcal{L}}(\lambda) = [\mathcal{L}(\lambda), M(\lambda)]$$

where the two matrices from the loop algebra $\mathfrak{su}(2)[\lambda]$ are the following ones:

$$\mathcal{L}(\lambda) = \mathcal{W} + \sum_{i=1}^N \left[\frac{\mathcal{Y}_i}{\lambda - \lambda_i} + \frac{\mathcal{Z}_i}{(\lambda - \lambda_i)^2}\right],$$

$$M(\lambda) = \sum_{i=1}^N \frac{1}{\lambda - \lambda_i} \left[\lambda_i \mathcal{Y}_i + \frac{\lambda \mathcal{Z}_i}{\lambda - \lambda_i}\right].$$

Proof: A direct calculation.

□

Let us notice that in the case $N = 1$ we recover the well–known Lax representation for the symmetric Lagrange top [1, 11] provided that $\lambda_1 = 0$:

$$\dot{\mathcal{L}}(\lambda) = [\mathcal{L}(\lambda), M(\lambda)]$$

where

$$\mathcal{L}(\lambda) = \mathcal{W} + \frac{\mathcal{Y}}{\lambda} + \frac{\mathcal{Z}}{\lambda^2}, \qquad M(\lambda) = \frac{\mathcal{Z}}{\lambda}.$$

5 Separation of variables

In this section we construct the simplest separation of variables for the Lagrange chain with the Lax matrix (3.4). The details of the approach can be found in [12, 8].

The basic separation has only $N - 1$ pairs of separation variables belonging to the spectral curve Γ (3.3). It corresponds to the standard normalization vector $\alpha_0 = (1, 0)$ and it is defined by the equations

$$(1, 0)\ (\mathcal{L}(u) - v\mathbb{1})^\wedge = 0, \qquad u, v \in \mathbb{C} \tag{5.1}$$

where $(\cdot)^\wedge$ denotes the adjoint matrix. Equation (5.1) is the equation for the poles (u_k, v_k) of the Baker-Akhiezer function Ψ, which is defined as a properly normalized eigenfunction of the Lax matrix:

$$\mathcal{L}(u)\, \Psi = v\, \Psi, \qquad (u, v) \in \Gamma,$$

$$\langle \alpha_0, \Psi \rangle = 1.$$

It is easy to see that the equation (5.1) gives the following separation variables:

$$\mathcal{L}_{12}(u_k) = 0, \qquad v_k = -\mathcal{L}_{11}(u_k),$$

where $\mathcal{L}_{jk}(\lambda)$ denotes the entry jk of the Lax matrix $\mathcal{L}(\lambda)$. Explicitly, the first $N-1$ components u_k of the separation variables are defined as zeros of the element $\mathcal{L}_{12}$ of the Lax matrix (3.4):

$$\mathcal{L}_{12}(u_k) = \mathrm{i}\sum_{i=1}^{N}\left[\frac{y_i^-}{u_k-\lambda_i} + \frac{z_i^-}{(u_k-\lambda_i)^2}\right] = 0 \qquad k = 1, ..., N-1, \tag{5.2}$$

while the second components are the values of $-\mathcal{L}_{11}(u)$ in those zeros:

$$v_k = -\mathcal{L}_{11}(u_k) = -\mathrm{i}\,w - \mathrm{i}\sum_{i=1}^{N}\left[\frac{y_i^3}{u_k-\lambda_i} + \frac{z_i^3}{(u_k-\lambda_i)^2}\right] \qquad k = 1, ..., N-1. \tag{5.3}$$

Let us show the canonicity of these separation variables. We obtain

$$0 \equiv \{\mathcal{L}_{12}(u_k), v_l\} = \frac{d\mathcal{L}_{12}(u_k)}{du_k}\{u_k, v_l\} + \{\mathcal{L}_{12}(u), v_l\}_{u=u_k}.$$

Using the linear r-matrix Poisson algebra (3.2) we obtain

$$\{u_k, v_l\} = -\left(\frac{d\mathcal{L}_{12}(u_k)}{du_k}\right)^{-1}\{\mathcal{L}_{12}(u), v_l\}_{u=u_k} = \delta_{kl} \qquad k, l = 1, ..., N-1.$$

For the completeness, one has to add an extra pair of canonical variables which Poisson commute with all separation variables, in order to make the total number of new Darboux variables equal twice the number of degrees of freedom. This pair is taken from the asymptotics of the elements $\mathcal{L}_{11}(u)$ and $\mathcal{L}_{12}(u)$:

$$\mathcal{L}_{11}(u) = \mathrm{i}\,w + \mathrm{i}\sum_{i=1}^{N}\frac{y_i^3}{u} + O\left(\frac{1}{u^2}\right), \quad \mathcal{L}_{12}(u) = \mathrm{i}\sum_{i=1}^{N}\frac{y_i^-}{u} + O\left(\frac{1}{u^2}\right) \qquad u \to \infty.$$

Thus we can choose as the last pair of separation variables the following ones:

$$u_N = \mathrm{i}\sum_{i=1}^{N} y_i^-, \qquad v_N = -\frac{\sum_{i=1}^{N} y_i^3}{\sum_{i=1}^{N} y_i^-}.$$

Indeed, it is easy to check that u_N, v_N commute with the separation variables defined in (5.2) and (5.3).

6 Bäcklund transformations

In this paper, following the approach of [9, 10], we look at the Bäcklund transformations (BTs) for finite-dimensional (Liouville) integrable systems as special canonical transformations. Such BTs are defined as symplectic, or more generally Poisson, integrable maps which are explicit maps (rather than implicit multivalued correspondences) and which can be viewed as time discretizations of particular continuous flows.

The most characteristic properties of such maps are:

1. a BT preserves the same set of integrals of motion as does the continuous flow which it discretizes;
2. it depends on a Bäcklund parameter η that specifies the corresponding shift on a Jacobian or on a generalized Jacobian [10];
3. a spectrality property holds with respect to η and to the conjugate variable μ, which means that the point (η, μ) belongs to the spectral curve [9, 10].

Explicitness makes these maps purely iterative, while the importance of the parameter η is that it allows for an adjustable discrete time step. The spectrality property is related with the simplecticity of the map [10].

6.1 One–point BTs

A one-point Bäcklund transformation for the $\mathfrak{e}(3)$ rational Lagrange chain can be defined as the following similarity transform on the Lax matrix $\mathcal{L}(\lambda)$ (3.4):

$$\mathcal{L}(\lambda) \longmapsto M(\lambda;\eta)\,\mathcal{L}(\lambda)\,M^{-1}(\lambda,\eta) \qquad \forall\,\lambda \in \mathbb{C},$$

with some generally non-degenerate 2×2 matrix $M(\lambda, \eta)$, simply because a BT should preserve the spectrum of $\mathcal{L}(\lambda)$. The parameter $\eta \in \mathbb{C}$ is called a Bäcklund parameter of the transformation. We use $\tilde{}$-notations for the updated variables, so that

$$\tilde{\mathcal{L}}(\lambda) = \mathcal{W} + \sum_{i=1}^{N} \left[\frac{\tilde{\mathcal{Y}}_i}{\lambda - \lambda_i} + \frac{\tilde{\mathcal{Z}}_i}{(\lambda - \lambda_i)^2} \right],$$

where

$$\tilde{\mathcal{Y}}_i \doteq \frac{\mathrm{i}}{2} \begin{pmatrix} \tilde{y}_i^3 & \tilde{y}_i^- \\ \tilde{y}_i^+ & -\tilde{y}_i^3 \end{pmatrix}, \qquad \tilde{\mathcal{Z}}_i \doteq \frac{\mathrm{i}}{2} \begin{pmatrix} \tilde{z}_i^3 & \tilde{z}_i^- \\ \tilde{z}_i^+ & -\tilde{z}_i^3 \end{pmatrix}.$$

We are looking for a Poisson map that intertwines two Lax matrices $\mathcal{L}(\lambda)$ and $\tilde{\mathcal{L}}(\lambda)$:

$$M(\lambda;\eta)\,\mathcal{L}(\lambda) = \tilde{\mathcal{L}}(\lambda)\,M(\lambda;\eta) \qquad \forall\,\lambda \in \mathbb{C}, \quad \eta \in \mathbb{C}. \tag{6.1}$$

Let us take

$$M(\lambda;\eta) = \begin{pmatrix} \lambda - \eta + p\,q & p \\ q & 1 \end{pmatrix}, \qquad \det M(\lambda;\eta) = \lambda - \eta. \tag{6.2}$$

We stress that the number of zeros of $\det M$ is the number of essential Bäcklund parameters. Here the variables p and q are indeterminate dynamical variables. The ansatz (6.2) for the matrix M comes from the simplest L-operator of the quadratic r-matrix algebra with the same r-matrix of the model [5, 7].

Comparing the asymptotics in $u \to \infty$ in both sides of (6.1) we readily get

$$p = \frac{1}{2w} \sum_{i=1}^{N} y_j^-, \qquad q = \frac{1}{2w} \sum_{i=1}^{N} \tilde{y}_j^+. \tag{6.3}$$

If we want an explicit single-valued map from $\mathcal{L}(\lambda)$ to $\tilde{\mathcal{L}}(\lambda)$ we must express $M(\lambda;\eta)$, and therefore p and q in term of the old variables. To solve this problem we use the spectrality of the BT [9, 10]. Equation (6.1) defines a map $\mathcal{B}_P$ parametrized by the point point $P=(\eta,\mu)\in\Gamma$. Notice that there are two points on Γ, $P=(\eta,\mu)$ and $Q=(\eta,-\mu)$, corresponding to the same λ and sitting one above the other because of the hyperelliptic involution:

$$(\eta,\mu)\in\Gamma: \qquad \det(\mathcal{L}(\eta)-\mu\mathbb{1})=0.$$

This spectrality property, used as a new datum, produces the formula

$$q=\frac{\mathcal{L}_{11}(\eta)-\mu}{\mathcal{L}_{12}(\eta)}=-\frac{\mathcal{L}_{21}(\eta)}{\mathcal{L}_{11}(\eta)+\mu}\,. \tag{6.4}$$

Now the equation (6.1) gives an integrable Poisson map from $\mathcal{L}(\lambda)$ to $\tilde{\mathcal{L}}(\lambda)$. We recall here the notation $\mathbf{y}^\alpha \doteq (y_1^\alpha,...,y_N^\alpha)^T\in\mathbb{R}^N$ and $\mathbf{z}^\alpha\doteq(z_1^\alpha,...,z_N^\alpha)^T\in\mathbb{R}^N$, with $\alpha=\pm,3$.

The following statement shows how the one-point BT can be written in a symplectic form through a generating function.

Proposition 2. The one–point BT for the rational Lagrange chain is defined by

$$\begin{aligned}
z_i^3 &= \mathrm{i}\, z_i^- \,\frac{\partial F_\eta(\mathbf{z}^-,\mathbf{y}^-\mid\tilde{\mathbf{z}}^+,\tilde{\mathbf{y}}^+)}{\partial y_i^-}\,,\\
y_i^3 &= \mathrm{i}\, z_i^- \,\frac{\partial F_\eta(\mathbf{z}^-,\mathbf{y}^-\mid\tilde{\mathbf{z}}^+,\tilde{\mathbf{y}}^+)}{\partial z_i^-}+\mathrm{i}\, y_i^-\,\frac{\partial F_\eta(\mathbf{z}^-,\mathbf{y}^-\mid\tilde{\mathbf{z}}^+,\tilde{\mathbf{y}}^+)}{\partial y_i^-}\,,\\
\tilde{z}_i^3 &= \mathrm{i}\, \tilde{z}_i^+ \,\frac{\partial F_\eta(\mathbf{z}^-,\mathbf{y}^-\mid\tilde{\mathbf{z}}^+,\tilde{\mathbf{y}}^+)}{\partial \tilde{y}_i^+}\,,\\
\tilde{y}_i^3 &= \mathrm{i}\, \tilde{z}_i^+ \,\frac{\partial F_\eta(\mathbf{z}^-,\mathbf{y}^-\mid\tilde{\mathbf{z}}^+,\tilde{\mathbf{y}}^+)}{\partial \tilde{z}_i^+}+\mathrm{i}\, \tilde{y}_i^+\,\frac{\partial F_\eta(\mathbf{z}^-,\mathbf{y}^-\mid\tilde{\mathbf{z}}^+,\tilde{\mathbf{y}}^+)}{\partial \tilde{y}_i^+}\,,
\end{aligned} \tag{6.5}$$

with

$$\begin{aligned}
F_\eta(\mathbf{z}^-,\mathbf{y}^-\mid\tilde{\mathbf{z}}^+,\tilde{\mathbf{y}}^+) &= -\frac{\mathrm{i}}{2w}\sum_{i,j=1}^N y_i^-\tilde{y}_j^+ -\mathrm{i}\sum_{i=1}^N k_i\left(\frac{y_i^-}{z_i^-}+\frac{\tilde{y}_i^+}{\tilde{z}_i^+}-\frac{1}{\eta-\lambda_i}\right)+\\
&\quad +\mathrm{i}\log\prod_{i=1}^N\left(\frac{1+k_i}{1-k_i}\right)^{\ell_i}-\mathrm{i}\,w\,N\,\eta\,,
\end{aligned} \tag{6.6}$$

where

$$k_i^2=1+(\eta-\lambda_i)z_i^-\,\tilde{z}_i^+.$$

Proof: First, because the Casimir functions $C_i^{(1)},C_i^{(2)}, i=1,...,N$ do not change under the map, namely

$$y_i^3z_i^3+\frac{1}{2}\left(y_i^-z_i^++y_i^+z_i^-\right)=\tilde{y}_i^3\tilde{z}_i^3+\frac{1}{2}\left(\tilde{y}_i^-\tilde{z}_i^++\tilde{y}_i^+\tilde{z}_i^-\right)=\ell_i,$$

$$(z_i^3)^2+z_i^-z_i^+=(\tilde{z}_i^3)^2+\tilde{z}_i^-\tilde{z}_i^+=1,$$

we can exclude $4N$ variables z_i^+, y_i^+ and $\tilde z_i^-, \tilde y_i^-$, with $i = 1, ..., N$, using the following substitutions:

$$z_i^+ = \frac{1-(z_i^3)^2}{z_i^-}, \qquad y_i^+ = \frac{2\ell_i}{z_i^-} - \frac{2y_i^3 z_i^3}{z_i^-} - \frac{y_i^-}{(z_i^-)^2}\left[1-(z_i^3)^2\right],$$

$$\tilde z_i^- = \frac{1-(\tilde z_i^3)^2}{\tilde z_i^+}, \qquad \tilde y_i^+ = \frac{2\ell_i}{\tilde z_i^+} - \frac{2\tilde y_i^3 \tilde z_i^3}{\tilde z_i^+} - \frac{\tilde y_i^+}{(\tilde z_i^+)^2}\left[1-(\tilde z_i^3)^2\right].$$

Now we have only $4N+4N$ (old and new) independent variables: z_i^-, z_i^3, y_i^-, y_i^3 and $\tilde z_i^+$, $\tilde z_i^3$, $\tilde y_i^+$, $\tilde y_i^3$, $i = 1, ..., N$.

The map (6.1) explicitly reads

$$\tilde{\mathcal L}_{11}(\lambda) = \frac{(\lambda-\eta+2\,p\,q)[\mathcal L_{11}(\lambda) - q\,\mathcal L_{12}(\lambda)] + p\,\mathcal L_{21}(\lambda)}{\lambda-\eta},$$
$$\tilde{\mathcal L}_{12}(\lambda) = \frac{(\lambda-\eta+2\,p\,q)^2\mathcal L_{12}(\lambda) - 2\,p\,(\lambda-\eta+2\,p\,q)\mathcal L_{11}(\lambda) - p^2\mathcal L_{21}(\lambda)}{\lambda-\eta},$$
$$\tilde{\mathcal L}_{21}(\lambda) = \frac{\mathcal L_{21}(\lambda) + 2\,q\,\mathcal L_{11}(\lambda) - q^2\mathcal L_{12}(\lambda)}{\lambda-\eta}$$

Equating residues at $\lambda = \lambda_i$ in both sides of the above equations we obtain, after a straightforward computation,

$$\begin{aligned}
z_i^3 &= \frac{z_i^-}{2w}\sum_{j=1}^N \tilde y_j^+ + k_i,\\
y_i^3 &= \frac{\ell_i}{k_i} + \frac{\eta-\lambda_i}{2k_i}\left(\tilde z_i^+ y_i^- + z_i^- \tilde y_i^+\right) - \frac{z_i^- \tilde z_i^+}{2k_i} + \frac{y_i^-}{2w}\sum_{j=1}^N \tilde y_j^+,\\
\tilde z_i^3 &= \frac{\tilde x_+}{2w}\sum_{j=1}^N y_j^- + k_i,\\
\tilde y_i^3 &= \frac{\ell_i}{k_i} + \frac{\eta-\lambda_i}{2k_i}\left(\tilde z_i^+ y_i^- + z_i^- \tilde y_i^+\right) - \frac{z_i^- \tilde z_i^+}{2k_i} + \frac{\tilde y_i^+}{2w}\sum_{j=1}^N y_j^-.
\end{aligned} \tag{6.7}$$

where

$$k_i^2 = 1 + (\eta-\lambda_i) z_i^- \tilde z_i^+.$$

It is now easy to check that the function $F_\eta(\mathbf z^-, \mathbf y^- \mid \tilde{\mathbf z}^+, \tilde{\mathbf y}^+)$ (6.6) satisfies equations (6.5).

□

The spectrality property of a Bäcklund transformation means that the two coordinates η and μ of the point $P \in \Gamma$ parametrizing the map are conjugated variables, namely

$$\mu = -\frac{\partial F}{\partial \eta},$$

where F is the generating function of the BT.

We now show the spectrality property for the one-point constructed BT. Using the equations (6.3), (6.4), (6.6) and (6.7) we obtain

$$
\begin{aligned}
\mu &= \mathcal{L}_{11}(\eta) - \left(\frac{1}{2w}\sum_{i=1}^{N}\tilde{y}_j^+\right)\mathcal{L}_{12}(\eta) = \\
&= \mathrm{i}\sum_{i=1}^{N}\frac{1}{k_i}\left[\frac{1}{(\eta-\lambda_i)^2}+\frac{\ell_i}{\eta-\lambda_i}+\frac{z_i^-\,\tilde{z}_i^+}{2}\left(\frac{y_i^-}{z_i^-}+\frac{\tilde{y}_i^+}{\tilde{z}_i^+}+\frac{1}{\eta-\lambda_i}\right)\right]+\mathrm{i}\,w\,N = \\
&= -\frac{\partial F_\eta(\mathbf{z}^-,\mathbf{y}^-\mid\tilde{\mathbf{z}}^+,\tilde{\mathbf{y}}^+)}{\partial\eta}\,.
\end{aligned}
$$

Notice that the above one-point BT is a complex map, so it is a non-physical Bäcklund transformation. In order to obtain a physical map we will construct a two–point BT in the next section.

6.2 Two–point BTs

According to [5, 7], we now construct a composite map which is a product of the map $\mathcal{B}_{P_1} \equiv \mathcal{B}_{(\eta_1,\mu_1)}$ and $\mathcal{B}_{Q_2} \equiv \mathcal{B}_{(\eta_2,-\mu_2)}$:

$$
\mathcal{B}_{P_1,Q_2} = \mathcal{B}_{Q_2}\circ\mathcal{B}_{P_1}:\ \mathcal{L}(\lambda)\overset{\mathcal{B}_{P_1}}{\longmapsto}\tilde{\mathcal{L}}(\lambda)\overset{\mathcal{B}_{Q_2}}{\longmapsto}\overset{\approx}{\mathcal{L}}(\lambda).
$$

The two maps are inverse to each other when $\eta_1=\eta_2$ and $\mu_1=\mu_2$. This two-point BT for the rational Lagrange chain is defined by the following “discrete-time” Lax equation:

$$
M(\lambda;\eta_1,\eta_2)\,\mathcal{L}(\lambda) = \overset{\approx}{\mathcal{L}}(\lambda)\,M(\lambda;\eta_1,\eta_2) \qquad \forall\lambda\in\mathbb{C}, \qquad \eta_1,\eta_2\in\mathbb{C}, \tag{6.8}
$$

where the matrix $M(\lambda;\eta_1,\eta_2)$ is [5, 7]

$$
M(\lambda;\eta_1,\eta_2) = \begin{pmatrix} \lambda-\eta_1+x\,X & X \\ -x^2\,X+(\eta_1-\eta_2)\,x & \lambda-\eta_2-x\,X \end{pmatrix}, \tag{6.9}
$$

$$
\det M(\lambda;\eta_1,\eta_2) = (\lambda-\eta_1)(\lambda-\eta_2).
$$

The spectrality property with respect to two fixed points $(\eta_1, \mu_1) \in \Gamma$ and $(\eta_2, \mu_2) \in \Gamma$ give

$$
\begin{aligned}
x &= \frac{\mathcal{L}_{11}(\eta_1) - \mu_1}{\mathcal{L}_{12}(\eta_1)} = -\frac{\mathcal{L}_{21}(\eta_1)}{\mathcal{L}_{11}(\eta_1) + \mu_1} = \frac{\overset{\approx}{\mathcal{L}}_{11}(\eta_2) - \mu_2}{\overset{\approx}{\mathcal{L}}_{12}(\eta_2)} = -\frac{\overset{\approx}{\mathcal{L}}_{21}(\eta_2)}{\overset{\approx}{\mathcal{L}}_{11}(\eta_2) + \mu_2} = \\
&= \frac{1}{2w}\sum_{i=1}^{N} \tilde{y}_j^{+}. \qquad (6.10) \\
X &= \frac{(\eta_2 - \eta_1)\mathcal{L}_{12}(\eta_1)\mathcal{L}_{12}(\eta_2)}{\mathcal{L}_{12}(\eta_1)(\mathcal{L}_{11}(\eta_2) + \mu_2) - \mathcal{L}_{12}(\eta_2)(\mathcal{L}_{11}(\eta_1) - \mu_1)} = \\
&= \frac{(\eta_1 - \eta_2)(\mathcal{L}_{11}(\eta_1) + \mu_1)(\mathcal{L}_{11}(\eta_2) - \mu_2)}{(\mathcal{L}_{11}(\eta_1) + \mu_1)\mathcal{L}_{21}(\eta_2) - (\mathcal{L}_{11}(\eta_2) - \mu_2)\mathcal{L}_{21}(\eta_1)} = \\
&= \frac{(\eta_2 - \eta_1)\overset{\approx}{\mathcal{L}}_{12}(\eta_1)\overset{\approx}{\mathcal{L}}_{12}(\eta_2)}{\overset{\approx}{\mathcal{L}}_{12}(\eta_2)(\overset{\approx}{\mathcal{L}}_{11}(\eta_1) + \mu_1) - \overset{\approx}{\mathcal{L}}_{12}(\eta_1)(\overset{\approx}{\mathcal{L}}_{11}(\eta_2) - \mu_2)} = \\
&= \frac{(\eta_1 - \eta_2)(\overset{\approx}{\mathcal{L}}_{11}(\eta_1) - \mu_1)(\overset{\approx}{\mathcal{L}}_{11}(\eta_2) + \mu_2)}{(\overset{\approx}{\mathcal{L}}_{11}(\eta_2) + \mu_2)\overset{\approx}{\mathcal{L}}_{21}(\eta_1) - (\overset{\approx}{\mathcal{L}}_{11}(\eta_1) - \mu_1)\overset{\approx}{\mathcal{L}}_{21}(\eta_2)} = \\
&= \frac{1}{2w}\sum_{i=1}^{N}\left(y_j^{-} - \overset{\approx}{y}{}_j^{+}\right). \qquad (6.11)
\end{aligned}
$$

Now we have two Bäcklund parameters $\eta_1, \eta_2 \in \mathbb{C}$. The above formulae give several equivalent expressions for the variables x and X since the points (η_1, μ_1) and (η_2, μ_2) belong to the spectral curve Γ, i.e., are bound by the following relations

$$
\mu_k^2 = \mathcal{L}_{11}^2(\eta_k) + \mathcal{L}_{12}(\eta_k)\mathcal{L}_{21}(\eta_k) = \overset{\approx}{\mathcal{L}}{}_{11}^{2}(\eta_k) + \overset{\approx}{\mathcal{L}}_{12}(\eta_k)\overset{\approx}{\mathcal{L}}_{21}(\eta_k), \qquad k = 1, 2.
$$

Together with (6.10) and (6.11), the formula (6.8) give an explicit two-point Poisson integrable map from $\mathcal{L}(\lambda)$ to $\overset{\approx}{\mathcal{L}}(\lambda)$ (as well as its inverse, i.e the map from $\overset{\approx}{\mathcal{L}}(\lambda)$ to $\mathcal{L}(\lambda)$). The map is parametrized by two points $\mathcal{B}_{P_1} \equiv \mathcal{B}_{(\eta_1,\mu_1)}$ and $\mathcal{B}_{Q_2} \equiv \mathcal{B}_{(\eta_2,-\mu_2)}$.

Obviously, when $\eta_1 = \eta_2$ (and $\mu_1 = \mu_2$) the map turns into an identity map. We want to stress the fact that the two–point BT sends real variables to real variables provided [7]

$$
\eta_1 = \bar{\eta}_2 \doteq \eta \in \mathbb{C}.
$$

Therefore, the two-point map leads to a physical Bäcklund transformation with two real parameters.

7 Concluding remarks

In the present paper we investigated a new classical integrable system, which we called “rational Lagrange chain”. It consists of a long–range homogeneous chain where the local variables are the generators of the direct sum of N $\mathfrak{e}(3)$ interacting Lagrange tops. In our previous paper [6] we have shown that this model can be derived from the N sites $\mathfrak{su}(2)$ rational Gaudin models. Moreover this construction can be generalized to trigonometric and elliptic solutions of the classical Yang–Baxter equation and to a generic finite–dimensional simple Lie algebra.

We obtained a Lax representation for the system which naturally reduces to the well–known Lax pair of the Lagrange top in the one–body case.

Following the approach proposed by V.B. Kuznetsov and E.K. Sklyanin for the construction of Bäcklund transformations for finite–dimensional systems [9, 10], we obtained one- and two–point integrable symplectic maps for the Lagrange chain. Our explicit maps are the natural generalization to the N bodies system of the BTs for the symmetric Lagrange top [7].

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