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Darboux transformations for q-discretizations of 2D second order differential equations

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Abstract

We present q-discretizations of a second order differential equation in two independent variables that not only go to the differential counterpart as q goes to 1 but admit Moutard-Darboux transformations as well.

1 Introduction

One of the important components of the theory of nonlinear S-integrable differential equations [1] is Darboux transformations, the story of which starts with the paper by Moutard [2] and therefore, contrary to the common tendency, we will call them Moutard-Darboux transformations. An example of an equation which admits Moutard-Darboux transformations is the second order linear differential equation in two independent variables

$$
\begin{aligned}
&L\psi = 0\\
&L := a\partial_x^2 + b\partial_y^2 + 2c\partial_x\partial_y + (a_{,x} + c_{,y} + w)\partial_x + (c_{,x} + b_{,y} + z)\partial_y - f
\end{aligned}
\tag{1.1}
$$

where ψ is dependent variable $\psi : \mathbb{R}^2 \supset \mathcal{D} \ni (x, y) \mapsto \psi(x, y) \in \mathbb{R}$ while a, b, c, w, z and w are given $\mathcal{C}^1$ functions of independent variables x and y (we use the standard notation $a_{,x} := \frac{\partial a}{\partial x}$, $b_{,y} := \frac{\partial b}{\partial y}$ etc.). The goal of this paper is to introduce such q-discretizations of the equation (1.1) that not only go to the equation (1.1) as q goes to 1 but admit Moutard-Darboux transformations as well. The literature concerning such discretizations is rich, see e.g. [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15], but almost all of the mentioned papers concern 4-point difference schemes. The main result of this paper is the Moutard-Darboux transformations introduced for a 7-point self-adjoint scheme [16] and a 6-point difference scheme [17] can easily be extended to q-difference equations.

The q-difference schemes we discuss here are the 6-point scheme

$$
\begin{aligned}
&L_6\Psi = 0\\
&L_6 := \mathbf{a}D_x^qD_x^q + \mathbf{b}D_y^qD_y^q + \mathbf{c}D_x^qD_y^q + \mathbf{g}D_x^q + \mathbf{h}D_y^q - \mathbf{f}
\end{aligned}
\tag{1.2}
$$

and the self-adjoint 7-point schemes (a q-discretization of formally self-adjoint eq. (1.1) i.e. equation with $w = 0 = z$)

$$\begin{gathered} L_7 \Psi = 0 \\ L_7 := D_x^q \mathfrak{a} D_x^{\frac{1}{q}} + D_y^q \mathfrak{b} D_y^{\frac{1}{q}} + D_x^q \mathfrak{c} D_y^{\frac{1}{q}} + D_y^q \mathfrak{c} D_x^{\frac{1}{q}} - \mathfrak{f} \end{gathered} \tag{1.3}$$

where the q-derivative of a function $f \colon \mathbb{R} \ni x \mapsto f(x) \in \mathbb{R}$ is defined by

$$D^q f \colon \mathbb{R} \ni x \mapsto D^q f(x) := \frac{f(qx) - f(x)}{(q-1)x}$$

with $\mathbb{R} \ni q \neq \pm 1, 0$. We will express formulas mostly in terms of q-shift operators

$$T_x^q f(x,y) := f(qx,y) \qquad T_y^q f(x,y) := f(x,qy)$$

It is convenient to use the following notation for them

$$f_1 := T_1 f := T_x^q f \qquad f_{12} := T_{12} f := T_x^q T_y^q f$$

$$f_{-1} := T_{-1} f := T_x^{\frac{1}{q}} f \qquad \text{etc.}$$

The paper is organized as follows. In section 2 we derive Moutard-Darboux transformations for the 2D second order differential equation (1.1). Applying the procedure from the continuous case we introduce Moutard-Darboux transformations for the q-difference 6-point scheme in subsection 3.1 and for the q-difference 7-point scheme in subsection 3.2. We end the paper with concluding remarks (section 4) indicating differences between the discrete and continuous cases and raising some open problems.

2 Moutard-Darboux Transformation for 2D second order differential equation

The goal of this section is to derive Moutard-Darboux transformations for 2D general second order differential equation

$$\begin{gathered} L\psi = 0 \\ L := a\partial_x^2 + b\partial_y^2 + 2c\partial_x\partial_y + (a_{,x} + c_{,y} + w)\partial_x + (c_{,x} + b_{,y} + z)\partial_y - f \end{gathered} \tag{2.1}$$

where a, b, c, w, z and w are given $\mathcal{C}^1$ functions of independent variables x and y such that $\forall (x,y) \in \mathcal{D}$ $a^2 + b^2 + c^2 \neq 0$. To do that we need a solution ϕ of the equation formally adjoint to eq. (2.1)

$$\begin{gathered} L^\dagger \phi = 0 \\ L^\dagger := \partial_x a \partial_x + \partial_y b \partial_y + \partial_x c \partial_y + \partial_y c \partial_x - \partial_x w - \partial_y z - f \end{gathered} \tag{2.2}$$

Then the equation

$$\phi L \psi - \psi L^\dagger \phi = 0 \tag{2.3}$$

which can be written explicitly in the form

$$[a\phi\psi_{,x} + c\phi\psi_{,y} + (w\phi - a\phi_{,x} - c\phi_{,y})\psi]_{,x} + [b\phi\psi_{,y} + c\phi\psi_{,x} + (z\phi - b\phi_{,y} - c\phi_{,x})\psi]_{,y} = 0 \tag{2.4}$$

guarantees the existence of a potential r

$$\begin{aligned} r_{,x} &= b\phi\psi_{,y} + c\phi\psi_{,x} + (z\phi - b\phi_{,y} - c\phi_{,x})\psi \\ r_{,y} &= -a\phi\psi_{,x} - c\phi\psi_{,y} - (w\phi - a\phi_{,x} - c\phi_{,y})\psi \end{aligned} \tag{2.5}$$

When we write the potential r in the form $r = \bar{\psi}\gamma - \phi\theta p\frac{\psi}{\theta}$ (the role of new functions $\bar{\psi}$, γ, θ and p will become clear in a moment) then we obtain

$$\begin{aligned} (\bar{\psi}\gamma)_{,x} &= \theta[(p+c)\phi(\tfrac{\psi}{\theta})_{,x} + b\phi(\tfrac{\psi}{\theta})_{,y}] + [(\phi\theta p)_{,x} - \theta^2(b(\tfrac{\phi}{\theta})_{,y} + c(\tfrac{\phi}{\theta})_{,x} - z\tfrac{\phi}{\theta})]\tfrac{\psi}{\theta} \\ (\bar{\psi}\gamma)_{,y} &= \theta[(p-c)\phi(\tfrac{\psi}{\theta})_{,y} - a\phi(\tfrac{\psi}{\theta})_{,x}] + [(\phi\theta p)_{,y} + \theta^2(a(\tfrac{\phi}{\theta})_{,x} + c(\tfrac{\phi}{\theta})_{,y} - w\tfrac{\phi}{\theta})]\tfrac{\psi}{\theta} \end{aligned} \tag{2.6}$$

The crucial point is to assure that the map $\psi \mapsto \bar{\psi}$ is an invertible map between two equations of the second order. We demand that the factor multiplying $\frac{\psi}{\theta}$ in eq.(2.6) vanish i.e.

$$\begin{aligned} &(\phi\theta p)_{,x} = \theta^2[b(\tfrac{\phi}{\theta})_{,y} + c(\tfrac{\phi}{\theta})_{,x} - z\tfrac{\phi}{\theta}] \\ &(\phi\theta p)_{,y} = -\theta^2[a(\tfrac{\phi}{\theta})_{,x} + c(\tfrac{\phi}{\theta})_{,y} - w\tfrac{\phi}{\theta}] \end{aligned} \tag{2.7}$$

and $\phi^2\theta^2(p^2 - c^2 + ab)$ does not vanish in the domain. In virtue of (2.2) compatibility conditions of eqs. (2.7) reads

$$L\theta = 0 \tag{2.8}$$

So finally we have theorem

Theorem 1. *Assume that $L\theta = 0$, $L^{\dagger}\phi = 0$, p is defined by eqs. (2.7), $d := \phi\theta(p^2 - c^2 + ab) \neq 0$ in the domain and ψ is the kernel of the operator L then formulas*

$$\begin{bmatrix} (\gamma\bar{\psi})_{,x} \\ (\gamma\bar{\psi})_{,y} \end{bmatrix} = \phi\theta \begin{bmatrix} c+p & b \\ -a & p-c \end{bmatrix} \begin{bmatrix} \left(\frac{\psi}{\theta}\right)_{,x} \\ \left(\frac{\psi}{\theta}\right)_{,y} \end{bmatrix} \tag{2.9}$$

constitute transformation $\psi \mapsto \bar{\psi}$ from solution space of the equation (2.1) to the solution space of the equation

$$\begin{gathered} \bar{L}\bar{\psi} = 0 \\ \bar{L} := \bar{a}\partial_x^2 + \bar{b}\partial_y^2 + 2\bar{c}\partial_x\partial_y + (\bar{a}_{,x} + \bar{c}_{,y} + \bar{w})\partial_x + (\bar{c}_{,x} + \bar{b}_{,y} + \bar{z})\partial_y - \bar{f} \end{gathered} \tag{2.10}$$

Where the coefficients of (2.10) are given by

$$\bar{a} = \frac{a\gamma\delta}{d} \quad \bar{b} = \frac{b\gamma\delta}{d} \quad \bar{c} = \frac{c\gamma\delta}{d}$$

$$\bar{w} = (\tfrac{a\gamma_x + c\gamma_y}{d} - (\tfrac{p}{d})_y\gamma)\delta - \tfrac{a\gamma}{d}\delta_x - \tfrac{c\gamma}{d}\delta_y \quad \bar{z} = (\tfrac{b\gamma_y + c\gamma_x}{d} + (\tfrac{p}{d})_x\gamma)\delta - \tfrac{b\gamma}{d}\delta_y - \tfrac{c\gamma}{d}\delta_x \tag{2.11}$$

$$\bar{f} = \{-\tfrac{a\gamma_{xx} + b\gamma_{yy} + 2c\gamma_{xy}}{d} - [(\tfrac{a}{d})_x + (\tfrac{c-p}{d})_y]\gamma_x - [(\tfrac{b}{d})_y + (\tfrac{c+p}{d})_x]\gamma_y\}\delta$$

where γ and δ are arbitrary (of class $\mathcal{C}^2$) functions.

3 Moutard-Darboux transformation for q-difference schemes

3.1 The 6-point case

We consider the following q-difference equation

$$\begin{gathered} L_6\Psi = 0 \\ L_6 := AT_{11} + BT_{22} + 2CT_{12} + GT_1 + HT_2 - F \end{gathered} \tag{3.1}$$

which relates the points of the rectangular lattice shown on the Figure 1.

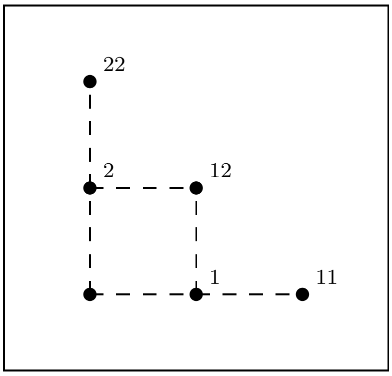

Figure 1. The 6-point scheme

Similarly like in theorem 1 we think of Ψ as the kernel of the operator L_6 while Θ and Φ are given elements of the kernels of L^6 and $L_6^\dagger$ respectively i.e.

$$L_6\Theta = 0 \tag{3.2}$$

$$\begin{gathered} L_6^\dagger\Phi = 0 \\ L_6^\dagger := \tfrac{1}{q^2}A_{-1-1}T_{-1-1} + \tfrac{1}{q^2}B_{-2-2}T_{-2-2} \\ +2\tfrac{1}{q^2}C_{-1-2}T_{-1-2} + \tfrac{1}{q}G_{-1}T_{-1} + \tfrac{1}{q}H_{-2}T_{-2} - F \end{gathered} \tag{3.3}$$

One can consider the operator $L_6^\dagger$ as the operator formally adjoint to the operator L_6 since the Green's identity $fL_6g - gL_6^\dagger f = D_x^qM(f,g) + D_y^qN(f,g)$ holds. From the above we conclude that there exists a function S such that

$$\begin{aligned} D_y^qS &= x\{\tfrac{1}{q}(G\Theta_1 + A\Theta_{11} + C\Theta_{12})_{-1}\Phi_{-1} + \tfrac{1}{q^2}A_{-1-1}\Theta\Phi_{-1-1} + \tfrac{1}{q^2}C_{-1-2}\Theta\Phi_{-1-2}\} \\ D_x^qS &= -y\{\tfrac{1}{q}(H\Theta_2 + B\Theta_{22} + C\Theta_{12})_{-2}\Phi_{-2} + \tfrac{1}{q^2}B_{-2-2}\Theta\Phi_{-2-2} + \tfrac{1}{q^2}C_{-1-2}\Theta\Phi_{-1-2}\} \end{aligned}$$

Next we define

$$P := \frac{1}{xy(q-1)}\frac{S_{12}}{\Theta_{12}\Phi} \tag{3.4}$$

Starting with the equation

$$\Phi L_6\Psi - \Psi L_6^\dagger\Phi = 0 \tag{3.5}$$

one obtains

Theorem 2. *Let Ψ be the kernel of the operator L_6 and Θ and Φ be given solutions of the equations $L_6\Theta = 0$ and $L_6^{\dagger}\Phi = 0$ respectively, P is given by (3.4) and $D := x^2y^2\Phi_{-1}\Phi_{-2}\Theta_1\Theta_2[A_{-1}B_{-2} - (C+P)_{-2}(C-P)_{-1}] \neq 0$ in a domain, then the equation*

$$\begin{bmatrix} D_x^q(\bar{\Psi}\Gamma) \\ D_y^q(\bar{\Psi}\Gamma) \end{bmatrix} = \begin{bmatrix} xy(C+P)_{-2}\Phi_{-2}\Theta_1 & y^2B_{-2}\Phi_{-2}\Theta_2 \\ -x^2A_{-1}\Phi_{-1}\Theta_1 & xy(P-C)_{-1}\Phi_{-1}\Theta_2 \end{bmatrix} \begin{bmatrix} D_x^q(\frac{\Psi}{\Theta}) \\ D_y^q(\frac{\Psi}{\Theta}) \end{bmatrix}$$

defines a map $\Psi \mapsto \bar{\Psi}$ and $\bar{\Psi}$ is the kernel of bared $\bar{L}_6$ operator i.e.

$$\bar{A}\bar{\Psi}_{11} + \bar{B}\bar{\Psi}_{22} + 2\bar{C}\bar{\Psi}_{12} + \bar{G}\bar{\Psi}_1 + \bar{H}\bar{\Psi}_2 - \bar{F}\bar{\Psi} = 0$$

with the new potentials

$$\bar{A} = \frac{\Delta\Gamma_{11}\Theta_{11}\Phi A}{D_1} \qquad \bar{B} = \frac{\Delta\Gamma_{22}\Theta_{22}\Phi B}{D_2}$$

$$\bar{C} = \left(\frac{\Theta_{11}\Phi_{1-2}(C+P)_{1-2}}{D_1} + \frac{\Theta_{22}\Phi_{-12}(C-P)_{-12}}{D_2}\right)\frac{\Delta\Gamma_{12}}{2}$$

$$\bar{G} = \left(-\frac{\Theta_{11}\Phi_{1-2}(C+P)_{1-2} + \Theta_{11}\Phi A}{D_1} - \frac{\Theta_2\Phi_{-1}(C-P)_{-1} + \Theta_1\Phi_{-1}A_{-1}}{D}\right)\Delta\Gamma_1$$

$$\bar{H} = \left(-\frac{\Theta_{22}\Phi_{-12}(C-P)_{-12} + \Theta_{22}\Phi B}{D_2} - \frac{\Theta_1\Phi_{-2}(C+P)_{-2} + \Theta_2\Phi_{-2}B_{-2}}{D}\right)\Delta\Gamma_2$$

$$\frac{\bar{F}}{\Gamma} = \frac{\bar{A}}{\Gamma_{11}} + \frac{\bar{B}}{\Gamma_{22}} + 2\frac{\bar{C}}{\Gamma_{12}} + \frac{\bar{G}}{\Gamma_1} + \frac{\bar{H}}{\Gamma_2}$$

where Δ and Γ are arbitrary functions.

3.2 The 7-point self-adjoint case

A similar procedure can be applied to the following 7-point scheme

$$\begin{gathered} L_7\Psi = 0 \\ L_7 := \mathcal{A}T_1 + \frac{1}{q}\mathcal{A}_{-1}T_{-1} + \mathcal{B}T_2 + \frac{1}{q}\mathcal{B}_{-2}T_{-2} + \mathcal{C}_1T_{1-2} + \mathcal{C}_2T_{-12} - \mathcal{F} \end{gathered} \tag{3.6}$$

which relates the points of a rectangular lattice shown on the Figure 2. We take only one

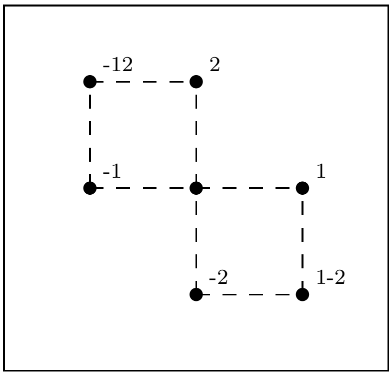

Figure 2. The 7-point scheme

function satisfying $L_7\Phi = 0$, instead of two functions. Actually, we may define the second function, for example: $L_7\Theta = 0$, but the reasoning from the 6-point case can be applied here if $\Theta \equiv \Phi$. The continuous limit of the equation is Moutard (self-adjoint) reduction [16] of the general Moutard-Darboux transformation we have presented in section 2.

Starting with the equation

$$\Phi L_7\Psi - \Psi L_7\Phi = 0 \tag{3.7}$$

one obtains

Theorem 3. *Let Ψ be the kernel of the operator L_7, Φ be a particular solution of $L_7\Phi = 0$ and function $\mathcal{D} = x^2y^2\Phi\mathcal{A}_{-1}\mathcal{B}_{-2} + x^2y^2\Phi_{-1}\mathcal{C}\mathcal{A}_{-1} + x^2y^2\Phi_{-2}\mathcal{C}\mathcal{B}_{-2}$ do not vanish in a domain then the equation*

$$\begin{bmatrix} D_x^q(\bar{\Psi}\Phi) \\ D_y^q(\bar{\Psi}\Phi) \end{bmatrix} = \begin{bmatrix} xy\mathcal{C}\Phi_{-1}\Phi_{-2} & -y^2(\mathcal{B}_{-2}\Phi\Phi_{-2} + \mathcal{C}\Phi_{-1}\Phi_{-2}) \\ x^2(\mathcal{A}_{-1}\Phi\Phi_{-1} + \mathcal{C}\Phi_{-1}\Phi_{-2}) & -xy\mathcal{C}\Phi_{-1}\Phi_{-2} \end{bmatrix} \begin{bmatrix} D_x^{\frac{1}{q}}(\frac{\Psi}{\Phi}) \\ D_y^{\frac{1}{q}}(\frac{\Psi}{\Phi}) \end{bmatrix}$$

defines a map

$$\Psi \mapsto \bar{\Psi}$$

where $\bar{\Psi}$ is the kernel of bared $\bar{L}_7$ operator i.e.

$$\bar{\mathcal{A}}\bar{\Psi}_1 + \frac{1}{q}\bar{\mathcal{A}}_{-1}\bar{\Psi}_{-1} + \bar{\mathcal{B}}\bar{\Psi}_2 + \frac{1}{q}\bar{\mathcal{B}}_{-2}\bar{\Psi}_{-2} + \bar{\mathcal{C}}_1\bar{\Psi}_{1-2} + \bar{\mathcal{C}}_2\bar{\Psi}_{-12} - \bar{\mathcal{F}}\bar{\Psi} = 0$$

with the new potentials

$$\bar{\mathcal{A}} = \frac{\Phi\Phi_1\mathcal{A}_{-1}}{q\Phi_{-2}\mathcal{D}} \qquad \bar{\mathcal{B}} = \frac{\Phi\Phi_2\mathcal{B}_{-2}}{q\Phi_{-1}\mathcal{D}} \qquad \bar{\mathcal{C}} = \frac{\mathcal{C}_{-1-2}\Phi_{-1}\Phi_{-2}}{\Phi_{-1-2}\mathcal{D}_{-1-2}}$$

$$\bar{\mathcal{F}} = \Phi(\bar{\mathcal{A}}\tfrac{1}{\Phi_1} + \tfrac{1}{q}\bar{\mathcal{A}}_{-1}\tfrac{1}{\Phi_{-1}} + \bar{\mathcal{B}}\tfrac{1}{\Phi_2} + \tfrac{1}{q}\bar{\mathcal{B}}_{-2}\tfrac{1}{\Phi_{-2}} + \bar{\mathcal{C}}_1\tfrac{1}{\Phi_{1-2}} + \bar{\mathcal{C}}_2\tfrac{1}{\Phi_{-12}})$$

4 Concluding remarks

The q-discretizations of Moutard-Darboux transformations we have shown in this paper do not differ essentially from discretizations of Moutard-Darboux transformations. But the world of both discretizations is essentially different from their continuous counterpart. We will end this paper with a brief review of the differences indicating open problems.

The Moutard-Darboux transformations we have just presented can be reduced or specified i.e. one can impose such constraints on transformations that allow the preservation of particular forms of the equations. In the continuous case one can:

- **specify the gauge** e.g. one can choose the affine gauge i.e. put $f = 0$ in the eq. (1.1) and demand that $\bar{f}$ in eq. (2.11) to be equal zero as well. It can be achieved by imposing constraints on function γ. The simplest way (but not the only one) is to put $\gamma = 1$

- **specify the operator**; due to simple transformation rules for coefficients a, b and c one can put e.g. $a = 0 = b$ or $c = 0$

- **reduce the transformation**; in the case of a reduction it is necessary to relate functions ψ and θ e.g. in the Moutard reduction $p = 0$, $w = 0 = z$ and $\psi = \theta$

Both for 6-point scheme and for 7-point scheme specifications to affine gauge ($F = A + B + 2C + G + H$ and $\mathcal{F} = \mathcal{A} + \mathcal{A}_{-1} + \mathcal{B} + \mathcal{B}_{-2} + \mathcal{C}_1 + \mathcal{C}_2$ respectively) are admissible. For the 6-point scheme there is a specification of Moutard-Darboux transformation to 4-point scheme ($A = 0 = B$) and $C = 0$ does not yield specification while for the 7-point scheme there exist specification to 5-point scheme ($\mathcal{C} = 0$) and one can not put ($\mathcal{A} = 0 = \mathcal{B}$). The situation is less investigated as far as reductions are concerned. The self-adjoint 7-point scheme obviously can not be a reduction of the 6-point scheme. A Moutard reduction for 6-point scheme is not known (there exists Moutard reduction for 4-point scheme specification [8] and it is not self-adjoint reduction anymore [18]). A generalization of the 7-point scheme to a scheme that goes to the general 2D second order differential equation is not known as well. We finally observe that 6-point scheme is appropriate to solve quite different boundary problems (Figure 3.) than 7-point scheme does (Figure 4). It makes the problems of existence of Moutard reduction of the 6-point scheme and generalizations of 7-point scheme especially interesting.

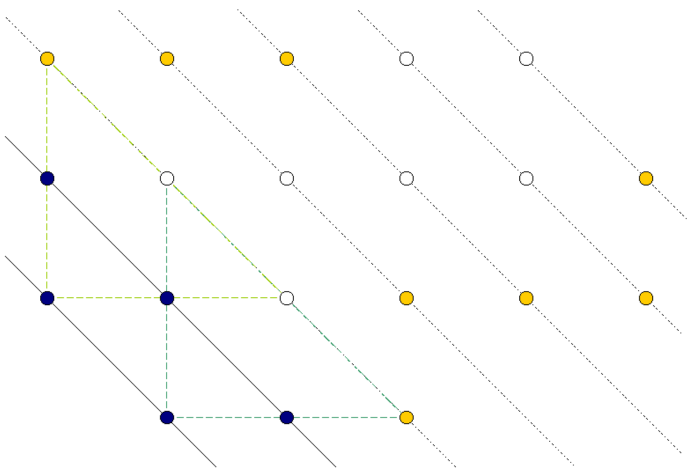

Figure 3.
An initial-boundary problem for the 6-point scheme.

Having given the initial conditions at dark blue (black) points on two solid lines and boundary conditions at yellow (grey) points one can propagate the solution inside the domain (white points). Wave fronts are drawn with the doted lines

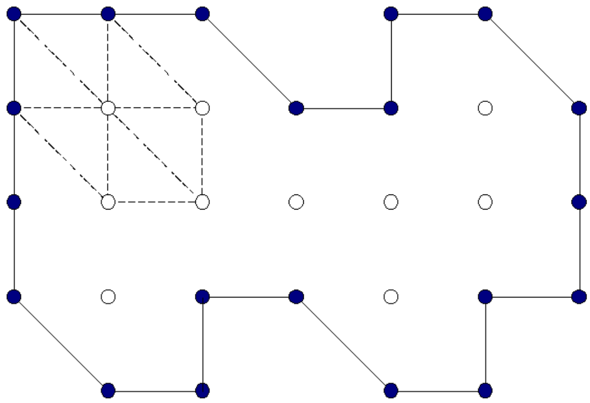

Figure 4.
A Dirichlet problem for the 7-point scheme.

Having given boundary conditions for the hexagonal lattice at dark blue (black) points one can find unique solution at all internal (white) points.

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