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The Relation between a 2D Lotka-Volterra equation and a 2D Toda Lattice

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Abstract

It is shown that the 2-discrete dimensional Lotka-Volterra lattice, the two dimensional Toda lattice equation and the recent 2-discrete dimensional Toda lattice equation of Santini et al can be obtained from a 2-discrete 2-continuous dimensional Lotka-Volterra lattice.

1 Introduction

Recently two new integrable systems, a 2-discrete dimensional Lotka-Volterra system [5] and a 2-discrete dimensional Toda lattice system [9] have been investigated. Both these systems lie on 2-dimensional square lattices and additionally they both have one continuous independent variable. For both these systems, multisoliton solutions can be found by using a Darboux transformation approach and indeed the solutions can be expressed very compactly in terms of pfaffians. The aim of this paper is to investigate the relationship between these two systems and other more well known Toda lattice equations. The main result is that all the systems investigated can be obtained from a 2-discrete 2-continuous dimensional Lotka-Volterra lattice system. The solutions of these systems fall into two basic classes, (i) lattice solutions and (ii) molecule solutions. The lattice solutions are defined over the whole of the lattice while the molecule solutions only exist on some finite part of the lattice. In the cases where we have a two dimensional lattice the solutions could fall into both classes by being lattice like in one of the discrete dimensions and molecule like in the other.

In Section 2 we shall look at some different Toda lattice equations and their solutions. In Section 3 we shall investigate the 2D Lotka-Volterra lattice equation. In Section 4 we look at the 2+2 Lotka-Volterra lattice which can be regarded as a master system from which all the other systems can be obtained and shall look at the reductions which take us from this master lattice to the other systems.

2 Toda lattice equations and solutions

There are several different variations on the Toda lattice equations, here we shall briefly recall some of these systems and their solutions using a bilinear approach. The solutions to all the systems we shall consider can be expressed in terms of grammians [8] or Gram-type pfaffians [2].

2.1 The two continuous dimensional Toda lattice

The two continuous and one discrete dimensional Toda lattice equation ((2,1)-TL equation) is given by [3]

$$\frac{\partial^2 Q_n}{\partial x \partial s} = e^{Q_{n+1}} - 2e^{Q_n} + e^{Q_{n-1}}, \tag{1}$$

where $n = \ldots, -1, 0, 1, \ldots$. Using the change of variables

$$Q_n = \log\left(\alpha + \frac{d^2}{dxds}(\log \tau_n)\right) \tag{2}$$

equation (1) can be integrated with respect to x and s to obtain

$$\frac{\partial^2 \tau_n}{\partial x \partial s}\tau_n - \frac{\partial \tau_n}{\partial x}\frac{\partial \tau_n}{\partial s} = \tau_{n+1}\tau_{n-1} - \alpha\,\tau_n^2. \tag{3}$$

Here we have set integration constants equal to zero and the α is a constant which will play an important role in the choice of solutions. The two basics cases are $\alpha = 1$ or $\alpha = 0$. The former gives *lattice* solutions and the latter gives *molecule* solutions. In terms of the Hirota D-operator the equation becomes

$$D_x D_s \tau_n \cdot \tau_n = 2(\tau_{n+1}\tau_{n-1} - \alpha\,\tau_n^2). \tag{4}$$

The $\alpha = 1$ case:

Choose a change of variables $s = -x_{-1}$, then the equation becomes

$$-\frac{1}{2}D_x D_{x_{-1}}\tau_n \cdot \tau_n = \tau_{n+1}\tau_{n-1} - \tau_n^2. \tag{5}$$

Solutions can be written in terms of grammian determinants [8]

$$\tau_n = \left| c_{ij} + (-)^n \int^x f_i^{(n)} g_j^{(-n)} dx \right| \quad \text{for} \quad 1 \le i, j \le N, \tag{6}$$

where $f_i^{(n)}$ and $g_j^{(n)}$ are n^{th} derivatives with respect to x if $n \ge 0$ (or $-n^{th}$ antiderivatives if $n < 0$) and f_i and g_i satisfy

$$\frac{\partial f_i}{\partial x_{-1}} = f_i^{(-1)}, \qquad \frac{\partial g_j}{\partial x_{-1}} = g_j^{(-1)}. \tag{7}$$

Typically soliton solutions can be obtained by taking

$$f = e^{px + \frac{1}{p}x_{-1}}, \quad g = e^{qx + \frac{1}{q}x_{-1}}.$$

With some simple algebraic manipulation we can recast the solutions in terms of determinants and bordered determinants, for simplicity we will only consider τ_0, τ_1 and τ_{-1}.

$$\begin{aligned}
\tau_0 &= \left| c_{ij} + \int^x f_i^{(0)} g_j^{(0)} dx \right| = |\Upsilon| \quad \text{for} \quad 1 \le i, j \le N, \\
\tau_1 &= \begin{vmatrix} 1 & \underline{g}^{(-1)} \\ \underline{f}^{(0)} & \Upsilon \end{vmatrix}, \\
\tau_{-1} &= \begin{vmatrix} 1 & \underline{g}^{(0)} \\ \underline{f}^{(-1)} & \Upsilon \end{vmatrix},
\end{aligned}$$

where $\underline{f} = (f_1, f_2, f_3, \ldots, f_N)^T$, $\underline{g} = (g_1, g_2, g_3, \ldots, g_N)$ and $\Upsilon_{ij} = c_{ij} + \int^x f_i^{(0)} g_j^{(0)} dx$. In this form it is easy to show that the τ-functions satisfy the bilinear form of the (2,1)-TL equation (5) by calculating the derivatives with respect to x and x_{-1}:

$$\frac{\partial \tau_0}{\partial x} = -\begin{vmatrix} 0 & \underline{g}^{(0)} \\ \underline{f}^{(0)} & \Upsilon \end{vmatrix}, \qquad \frac{\partial \tau_0}{\partial x_{-1}} = -\begin{vmatrix} 0 & \underline{g}^{(-1)} \\ \underline{f}^{(-1)} & \Upsilon \end{vmatrix}, \tag{8}$$

$$\frac{\partial^2 \tau_0}{\partial x \partial x_{-1}} = \begin{vmatrix} 0 & 0 & \underline{g}^{(0)} \\ 0 & 0 & \underline{g}^{(-1)} \\ \underline{f}^{(0)} & \underline{f}^{(-1)} & \Upsilon \end{vmatrix} - \begin{vmatrix} 0 & \underline{g}^{(0)} \\ \underline{f}^{(-1)} & \Upsilon \end{vmatrix} - \begin{vmatrix} 0 & \underline{g}^{(-1)} \\ \underline{f}^{(0)} & \Upsilon \end{vmatrix}. \tag{9}$$

Then it is possible to see that equation (5) is equivalent to the Jacobi identity

$$\begin{vmatrix} 0 & 0 & \underline{g}^{(0)} \\ 0 & 0 & \underline{g}^{(-1)} \\ \underline{f}^{(0)} & \underline{f}^{(-1)} & \Upsilon \end{vmatrix} |\Upsilon| - \begin{vmatrix} 0 & \underline{g}^{(0)} \\ \underline{f}^{(0)} & \Upsilon \end{vmatrix} \begin{vmatrix} 0 & \underline{g}^{(-1)} \\ \underline{f}^{(-1)} & \Upsilon \end{vmatrix} + \begin{vmatrix} 0 & \underline{g}^{(0)} \\ \underline{f}^{(-1)} & \Upsilon \end{vmatrix} \begin{vmatrix} 0 & \underline{g}^{(-1)} \\ \underline{f}^{(0)} & \Upsilon \end{vmatrix} = 0. \tag{10}$$

The $\alpha = 0$ case:

In this case we carry out a change of variables $s = -y$, giving the equation

$$-\frac{1}{2} D_x D_y \tau_n \cdot \tau_n = \tau_{n+1}\tau_{n-1}. \tag{11}$$

The solutions to this system can also be written down as (bordered) grammians, but this time they are *2-component* grammians (a 2-component wronskian version of this can be found in [3]). We will just write out $\tau_{-1}, \tau_0, \tau_1$ explicitly.

$$\begin{aligned} \tau_0 &= \left| c_{ij} + \int^x \phi_i \psi_j \, dx - \int^x \widetilde{\phi}_i \widetilde{\psi}_j \, dy \right| = |\Omega| \quad \text{for} \quad 1 \le i, j \le N, \\ \tau_1 &= -\begin{vmatrix} 0 & \underline{\widetilde{\psi}} \\ \underline{\phi} & \Omega \end{vmatrix}, \\ \tau_{-1} &= \begin{vmatrix} 0 & \underline{\psi} \\ \underline{\widetilde{\phi}} & \Omega \end{vmatrix}, \end{aligned}$$

where $\underline{\phi} = (\phi_1, \phi_2, \ldots, \phi_N)^T$, $\underline{\psi} = (\psi_1, \psi_2, \ldots, \psi_N)$ and similarly for the tilde variables. The $\phi, \psi, \widetilde{\phi}$ and $\widetilde{\psi}$ all obey linear equations of the form

$$\frac{\partial \phi_i^{(n)}}{\partial x} = \phi_i^{(n+1)}, \quad \frac{\partial \widetilde{\phi}_i^{(n)}}{\partial y} = \widetilde{\phi}_i^{(n+1)},$$

$$\frac{\partial \psi_j^{(n)}}{\partial x} = \psi_j^{(n+1)}, \quad \frac{\partial \widetilde{\psi}_j^{(n)}}{\partial y} = \widetilde{\psi}_j^{(n+1)}.$$

As with the $\alpha = 1$ case, the derivatives can be calculated and equation (11) can be shown to be equivalent to a Jacobi identity. The other τ-functions $\tau_{\pm 2}, \tau_{\pm 3}....,$ are determinants with progressively wider borders. For example τ_2 is given by

$$\tau_2 = -\begin{vmatrix} 0 & 0 & \underset{\sim}{\widetilde{\psi}}_y \\ 0 & 0 & \underset{\sim}{\widetilde{\psi}} \\ \underline{\phi}_x & \underline{\phi} & \Omega \end{vmatrix}.$$

Eventually for $M = N+1$ the block of zeros in the top left hand corner of the determinant is larger than the Ω in the bottom right corner and consequently $\tau_M = \tau_{-M} = 0$.

This demonstrates an important distinction between lattice solutions (case $\alpha = 1$) and molecule solutions (case $\alpha = 0$). The lattice solutions are defined on the whole of the lattice while the molecule solutions are just on a finite part of the lattice.

2.2 The two discrete dimensional Toda lattice equation.

In recent work of Santini et al [9] a new lattice, the (1,2)-TL equation, in one continuous dimension and two discrete dimensions is described. The system is given by

$$u_{p,q}\frac{d}{dx}\left(\frac{1}{u_{p,q}}\frac{dQ_{p,q}}{dx}\right) = \Delta_p(u_{p,q}u_{p-1,q}\, e^{\Delta_p Q_{p-1,q}}) + \Delta_q(u_{p,q}u_{p,q-1}\, e^{\Delta_q Q_{p,q-1}}) \quad (12)$$

$$\frac{u_{p+1,q+1}}{u_{p,q}} = e^{-\Delta_p\Delta_q Q_{p,q}}, \quad (13)$$

where p and q label the lattice points and

$$\Delta_p f = f_{p+1,q} - f_{p,q}, \qquad \Delta_q f = f_{p,q+1} - f_{p,q}$$

are differences in the two discrete directions. Choosing

$$Q = \log\frac{\tau_{p+1,q+1}}{\tau_{p,q}}, \qquad u = \frac{\tau_{p+1,q}\tau_{p,q+1}}{\tau_{p,q}\tau_{p+1,q+1}} \quad (14)$$

equation (13) is automatically satisfied and (12) becomes a quadrilinear form. By the judicious introduction of an auxiliary variable y, the quadrilinear form can be split into a bilinear system [6, 10]

$$D_x\tau_{p,q}\cdot\tau_{p+1,q+1} = D_y\tau_{p,q+1}\cdot\tau_{p+1,q}, \quad (15)$$

$$\frac{1}{2}D_xD_y\tau_{p,q}\cdot\tau_{p,q} = \tau_{p+1,q}\tau_{p-1,q} - \tau_{p,q+1}\tau_{p,q-1} - \alpha\tau_{p,q}^2. \quad (16)$$

Now in bilinear form we can see the similarity to the case with two continuous dimensions. As with the continuous case when carrying out the bilinearization we have the freedom to introduce an α as the coefficient of $\tau_{p,q}^2$. The value of this α will determine the type of solutions obtained. For this bilinear form it is possible to write down solutions explicitly in terms of pfaffians of two different types, DKP and DKP′. These different types relate to the Lie algebraic structure of the solutions, the first being D-type Lie algebra and the second a D′-type Lie algebra, some more information can be found about these in [7].

Recall that a pfaffian is an object $(\alpha_1, \alpha_2, \ldots, \alpha_N)$, N even, with a set of labels $\alpha_1, \ldots, \alpha_N$. It can be defined by its (recursive) expansion rule:

$$(\alpha_1, \alpha_2, \ldots, \alpha_N) = \sum_{j=2}^{N}(\alpha_1, \alpha_j)(\alpha_2, \cdots, \widehat{\alpha_j}, \cdots \alpha_N) \quad (17)$$

with skew-symmetric elements (α_i, α_j):

$$(\alpha_i, \alpha_j) = -(\alpha_j, \alpha_i), \tag{18}$$

where $\widehat{\alpha_j}$ indicates that the jth entry has been removed. Indeed if you take an $N \times N$ skew symmetric determinant with entries $a_{i,j} = (i, j)$ then

$$\det(a_{i,j}) = (1, 2, \cdots, N)^2. \tag{19}$$

The $\alpha = 0$ case:

This case will give lattice solutions, which we will present in terms of pfaffians. However, the discrete coordinates p and q, are not the most natural coordinates to express these solutions. Hence we introduce rotated coordinates, m and n such that $p = \frac{m+n}{2}$, $q = \frac{m-n}{2}$, and dependent variable F such that $\tau_{pq} = F_{p+q,p-q} = F_{mn}$. This transforms (15,16) into

$$D_x F_{m,n} \cdot F_{m+2,n} = D_y F_{m+1,n-1} \cdot F_{m+1,n+1}, \tag{20}$$

$$\frac{1}{2} D_x D_y F_{m,n} \cdot F_{m,n} = F_{m+1,n+1} F_{m-1,n-1} - F_{m+1,n-1} F_{m-1,n+1}. \tag{21}$$

Let θ_i satisfy

$$\frac{\partial}{\partial x} \theta_i(m,n) = \theta_i(m, n-1) - \theta_i(m, n+1), \tag{22}$$

$$\frac{\partial}{\partial y} \theta_i(m,n) = \theta_i(m-1, n) - \theta_i(m+1, n), \tag{23}$$

$$\theta_i(m+1, n+1) + \theta_i(m,n) = \theta_i(m+1, n) + \theta_i(m, n+1), \tag{24}$$

then the pfaffian elements are chosen to satisfy

$$\begin{aligned} (i,j)_{m+1,n} &= (i,j)_{m,n} + \theta_i(m,n)\theta_j(m+1,n) - \theta_i(m+1,n)\theta_j(m,n) \\ &= (i,j) + (i,j,c_1^0,c_0^0), \end{aligned} \tag{25}$$

$$\begin{aligned} (i,j)_{m,n+1} &= (i,j)_{m,n} - \theta_i(m,n)\theta_j(m,n+1) + \theta_i(m,n+1)\theta_j(m,n) \\ &= (i,j) + (i,j,c_0^0,c_0^1), \end{aligned} \tag{26}$$

$$\begin{aligned} \frac{\partial}{\partial x}(i,j)_{m,n} &= \theta_i(m,n-1)\theta_j(m,n+1) - \theta_i(m,n+1)\theta_j(m,n-1) \\ &= (i,j,c_0^1,c_0^{-1}), \end{aligned} \tag{27}$$

$$\begin{aligned} \frac{\partial}{\partial y}(i,j)_{m,n} &= \theta_i(m+1,n)\theta_j(m-1,n) - \theta_i(m-1,n)\theta_j(m+1,n) \\ &= (i,j,c_{-1}^0,c_1^0), \end{aligned} \tag{28}$$

where the pfaffian elements

$$(k, c_i^j) = \theta_k(m+i, n+j), \qquad (c_i^j, c_k^l) = 0. \tag{29}$$

These pfaffian elements are the same as the pfaffian elements used in the two discrete dimensional Lotka-Volterra lattice [5]. The pfaffian elements without subscripts are taken to be the elements at the lattice point (m,n). It can be shown that these above relationships define the pfaffian elements in a consistent way.

The actual pfaffian elements that we are interested in here are:

$$\begin{aligned}
(i,j)_{m+1,n+1} &= (i,j)+(i,j,c_1^0,c_0^1),\\
(i,j)_{m+1,n-1} &= (i,j)+(i,j,c_1^0,c_0^{-1}),\\
(i,j)_{m-1,n+1} &= (i,j)+(i,j,c_{-1}^0,c_0^1),\\
(i,j)_{m-1,n-1} &= (i,j)+(i,j,c_{-1}^0,c_0^{-1}),\\
(i,j)_{m+2,n} &= (i,j)+(i,j,c_0^1,c_0^2)+(i,j,c_0^0,c_0^1).
\end{aligned}$$

The form of the pfaffians to be used in equations (20,21) is given by $F_{m+i,n+j} = (1,2,3,\cdots,N)_{m+i,n+j}$. If we abbreviate $F_{m,n} = (\bullet)$ and use some addition formulae for pfaffians, see [3, §2.10] , then we obtain

$$\begin{aligned}
F_{m+1,n+1} &= (\bullet)+(\bullet\, c_1^0,c_0^1),\\
F_{m+1,n-1} &= (\bullet)+(\bullet\, c_1^0,c_0^{-1}),\\
F_{m-1,n+1} &= (\bullet)+(\bullet\, c_{-1}^0,c_0^1),\\
F_{m-1,n-1} &= (\bullet)+(\bullet\, c_{-1}^0,c_0^{-1}),\\
F_{m+2,n} &= (\bullet)+(\bullet\, c_0^1,c_0^2)+(\bullet\, c_0^0,c_0^1).
\end{aligned}$$

Derivatives of these pfaffians can be written down and the pfaffians can be shown to satisfy the equations (20,21). We will discuss these solutions later in connection with the 2D Lotka-Volterra system.

The $\alpha = 1$ case:

$$D_x\tau_{p,q}\cdot\tau_{p+1,q+1} = D_y\tau_{p,q+1}\cdot\tau_{p+1,q} \tag{30}$$

$$\frac{1}{2}D_xD_y\tau_{p,q}\cdot\tau_{p,q} = \tau_{p+1,q}\tau_{p-1,q} - \tau_{p,q+1}\tau_{p,q-1} - \tau_{p,q}^2. \tag{31}$$

Now in bilinear form this case will give us molecule solutions. The system (30,31) is satisfied by choosing a pfaffian with entries

$$(i,j)_p = c_{ij} + (-1)^p\int\left(f_i^{(p)}g_j^{(-p)} - g_i^{(-p)}f_j^{(p)}\right)dx. \tag{32}$$

This pfaffian is a skew symmetrized version of the grammian used for the standard 2D Toda lattice (6). Will will not discuss the details in this section as with judicious choice of τ-function in the q-direction on the lattice, these solutions are the same as the pfaffianized Toda lattice solutions which we discuss in the next section.

2.3 The pfaffianized Toda lattice equation

In addition to Santini el al's work [9], Hu et al and Willox have another 2-D Toda lattice equation [4, 10]. In Hu et al's paper they obtain this equation by the process of pfaffianization, in Willox's paper the equation is obtained by looking at generalised Jacobi identities. The pfaffianization procedure involves taking a bilinear system with, for instance, grammian solutions. The solutions are then replaced by pfaffians with similar entries. These pfaffians satisfy a bilinear equation quite close to that of the original grammian case but with an extra bilinear term involving some new τ-functions. This new equation makes up part of the pfaffianized Toda system, but as new τ-functions have been introduced, more equations are needed to make a closed system. Now for the case of the pfaffianized Toda lattice the terms in the pfaffian are taken to be:

$$(i,j)_p = \left[c_{ij} + (-)^p\int^x (f_i^{(p)}g_j^{(-p)} - g_i^{(-p)}f_j^{(p)})dx\right] \quad \text{for} \quad 1\le i,j\le N, \tag{33}$$

with $c_{ij} = -c_{ji}$. The f_i and g_i obey the linear equations (7) as in the standard 2-D Toda lattice case. By simple manipulation, the pfaffian entry at one lattice point can be expressed in terms of the entry at an adjoining point:

$$\begin{aligned}(i,j)_{p+1} &= c_{ij} + (-1)^{p+1}\int^x (f_i^{(p+1)}g_j^{(-p-1)} - g_i^{(-p-1)}f_j^{(p+1)})dx \\ &= (i,j)_p + (-1)^{p+1}(i,j,g^{(-p-1)},f^{(p)})\end{aligned}$$

where

$$(i,g^{(p)}) = g_i^{(p)}, \qquad (i,f^{(p)}) = f_i^{(p)}, \qquad (g^{(p)},f^{(r)}) = 0.$$

Using an addition formula for pfaffians [3, §2.10] we obtain

$$(1,2,\cdots,N)_{p+1} = (1,2,\cdots,N)_p + (-1)^{p+1}(1,2,\cdots,N,g^{(-p-1)},f^{(p)})_p,$$

which can be abbreviated to

$$\tau_{p+1} = \tau_p + (-1)^{p+1}(\bullet\ g^{(-p-1)},f^{(p)})_p.$$

Similarly

$$\tau_{p-1} = \tau_p + (-1)^{p+1}(\bullet\ g^{(-p)},f^{(p-1)})_p.$$

Derivatives of the pfaffian elements can be calculated and from these the derivatives of the pfaffians themselves can be calculated:

$$\begin{aligned}\frac{\partial}{\partial x}\tau_p &= (\bullet\ g^{(-p)},f^{(p)}) \\ \frac{\partial}{\partial x_{-1}}\tau_p &= (\bullet\ g^{(-p-1)},f^{(p-1)}), \\ \frac{\partial^2}{\partial x\partial x_{-1}}\tau_p &= (\bullet\ g^{(-p-1)},f^{(p-1)},g^{(-p)},f^{(p)}) + (\bullet\ g^{(-p)},f^{(p-1)}) + (\bullet\ g^{(-p-1)},f^{(p)}).\end{aligned}$$

The (pfaffian) Jacobi identity

$$\begin{aligned}&(\bullet\ g^{(-p-1)},f^{(p-1)},g^{(-p)},f^{(p)})(\bullet) - (\bullet\ g^{(-p-1)},f^{(p-1)})(\bullet\ g^{(-p)},f^{(p)}) + \\ &\quad(\bullet\ g^{(-p-1)},g^{(-p)})(\bullet\ f^{(p-1)},f^{(p)}) + (\bullet\ g^{(-p)},f^{(p-1)})(\bullet\ g^{(-p-1)},f^{(p)}) = 0\end{aligned} \tag{34}$$

gives us the first pfaffianised Toda lattice equation:

$$\frac{1}{2}D_xD_y\tau_p\cdot\tau_p = \tau_{p+1}\tau_{p-1} - \tau_p^2 - \sigma_p\overline{\sigma}_p, \tag{35}$$

while a second pfaffian identity gives us two further equations that complete the pfaffian system

$$D_x\tau_p\cdot\sigma_{p+1} = -D_y\tau_{p+1}\cdot\sigma_p, \tag{36}$$

$$D_x\tau_{p+1}\cdot\overline{\sigma}_p = -D_y\tau_p\cdot\overline{\sigma}_{p+1}, \tag{37}$$

where σ and $\bar{\sigma}$ are new functions introduced by the pfaffianization procedure given by

$$\sigma_p = (1,2,\ldots,N,f^{(p-1)},f^{(p)})_p, \qquad \bar{\sigma}_p = (1,2,\ldots,N,g^{(-p-1)},g^{(-p)})_p. \tag{38}$$

The equations (35-37) can be related to the 2D Toda equations of Santini et al simply by considering the the σ and $\bar{\sigma}$ to be τ but shifted in a second discrete direction. i.e. identifying

$$\sigma_p = \tau_{p,q+1}, \qquad \bar{\sigma}_p = \tau_{p,q-1}, \qquad \tau_p = \tau_{p,q}. \tag{39}$$

With this identification the equations become precisely the $\alpha = 1$ case of the (1,2)-TL equations (30,31). The only difference here is in the interpretation of the labels, for the pfaffianised case we only need $\tau_{p,-1}$, $\tau_{p,0}$ and $\tau_{p,1}$ whereas for the (1,2)-TL equations we could extend out (finitely) further in the q-direction having τ-functions

$$\cdots,\quad \tau_{p,-2},\quad \tau_{p,-1},\quad \tau_{p,0},\quad \tau_{p,1},\quad \tau_{p,2},\quad \cdots.$$

The character of these solutions is lattice like in the p-direction and molecule like in the q-direction. How far the lattice extends in the $\pm q$-direction will depend upon the size of the pfaffians involved.

3 The 2 Dimensional Lotka-Volterra System.

The two dimensional Lotka-Volterra lattice ((1,2)-LV lattice)

$$\begin{aligned}
&2u_t(m,n) + e^{\Delta_n^2\phi(m,n)+u(m,n+1)} + e^{\Delta_m^2\phi(m,n)+u(m+1,n)} \\
&- e^{\Delta_n^2\phi(m,n-1)+u(m,n)} - e^{\Delta_m^2\phi(m-1,n)+u(m,n)} \\
&\qquad\qquad = \Delta_n\left(e^{\Delta_n^2\phi(m-1,n)-u(m-1,n)}\right) + \Delta_m\left(e^{\Delta_m^2\phi(m,n-1)-u(m,n-1)}\right) \\
&u(m,n) = \Delta_m\Delta_n\phi(m,n)
\end{aligned}$$

is another system with two discrete dimensions. It has been investigated recently by Hu et al [5]. Like the (1,2)-TL system of Santini et al, this system has solutions that are lattice like in both discrete directions. By using the dependent variable transformation

$$u = \log\left(\frac{F_{m+1,n+1}F_{m,n}}{F_{m+1,n}F_{m,n+1}}\right) = \Delta_m\Delta_n \log F_{m,n}, \qquad \phi = \log F_{m,n}$$

and introduction of auxiliary variables x, y such that $D_x + D_y = 2D_t$ the 2D Lotka-Volterra system can be written in bilinear form:

$$D_x F_{m+1,n}\cdot F_{m,n} = F_{m+1,n-1}F_{m,n+1} - F_{m+1,n+1}F_{m,n-1} \tag{40}$$

$$D_y F_{m,n+1}\cdot F_{m,n} = F_{m-1,n+1}F_{m+1,n} - F_{m+1,n+1}F_{m-1,n}. \tag{41}$$

The solutions to this system can be expressed as pfaffians [5], indeed these pfaffians are exactly the same as for the (1,2)-TL system. The pfaffians can be defined using equations (22-29). As the solutions are the same for the two systems, this suggest that there may be a relationship between the systems themselves and indeed this is the case. Taking the bilinear form of the Lotka-Volterra system (40,41) we can turn it back into a nonlinear form by choosing $v = \log F$, this gives

$$(v_m - v)_x = e^{v_{m\bar{n}}+v_n-v_m-v} - e^{v_{mn}+v_{\bar{n}}-v_m-v} \tag{42}$$

$$(v_n - v)_y = e^{v_{\bar{m}n}+v_m-v_n-v} - e^{v_{mn}+v_{\bar{m}}-v_n-v}, \tag{43}$$

where here for brevity of presentation we shall take the subscripts m or n to denote an increment in that variable, and the subscripts $\bar{m}$ or $\bar{n}$ to denote a decrement in that variable. If we take (42) and add (42) shifted down in the m-direction and also (43) and add (43) shifted down in the n-direction, we find we can construct an equation

$$e^{v_m+v_{\bar{m}}}(v_m - v_{\bar{m}})_x = e^{v_n+v_{\bar{n}}}(v_n - v_{\bar{n}})_y. \tag{44}$$

Secondly taking the y-derivative of (42) and using (43) to eliminate the y-derivatives on the right hand side we obtain

$$\Delta_m v_{xy} = \Delta_m(e^{v_{mn}+v_{\bar{m}\bar{n}}-2v} - e^{v_{m\bar{n}}+v_{\bar{m}n}-2v}). \tag{45}$$

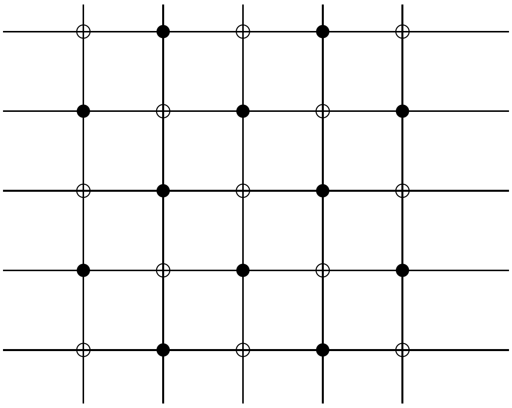

Figure 1. The (1,2)-LV lattice, incorporating two copies of the (1,2)-TL lattice, one copy consisting of the filled in lattice points, the other consisting of the open circular lattice points. The natural coordinates for the (1,2)-TL lattice are coordinates rotated by $\pi/2$ relative to the (1,2)-LV lattice.

If instead we use the same transformation $v = \log F$ on the (1,2)-TL equation we get (44) and

$$v_{xy} = (e^{v_{mn}+v_{\bar{m}\bar{n}}-2v} - e^{v_{m\bar{n}}+v_{\bar{m}n}-2v}). \tag{46}$$

which is (45) without the difference Δ_m. From this we can conclude that the (1,2)-TL system and the (1,2)-LV are equivalent.

Although we can get from the Lotka-Volterra system to the Toda lattice system the actual lattices in the two systems are not the same as the (1,2)-LV system has twice as many lattice points as the (1,2)-TL system. The (1,2)-LV system is defined on a lattice with points m, n both integers while the (1,2)-TL system is defined on a lattice with points m, n both integers and $m + n$ even, or equally possible we could take m, n both integers with $m + n$ odd. Thus we can fit two copies of this Toda lattice system on to the Lotka-Volterra lattice.

4 Reductions from a 2+2 System

Notice that both the (1,2)-LV lattice and the (1,2)-TL lattice systems have bilinear forms with two discrete variables and two continuous variables, one of the variables in each case is actually an auxiliary variable that is eliminated when the equations are returned to their nonlinear form. We may regard the equations (20, 21, 40, 41) as equations from a $2 + 2$ dimensional system which we will call the (2,2)-LV system. This $2 + 2$ dimensional system contains a whole hierarchy of equations. Equations (20), (21), (40) and (41) represent some of the simplest in the hierarchy.

In general to obtain an integral system from this (2,2)-LV system we need to pick out certain combinations of equations and carry out reductions, so as to obtain properly constructed closed systems. We have already seen two basic reductions:

The (1,2)-LV

The (1,2)-LV is obtained by choosing equations (40, 41) and then eliminating the variable y between the two equations.

The (1,2)-TL

This is obtained by first considering the (2,2)-LV system as two copies of a $2+2$ Toda lattice, so choose equations (20, 21). These lie in the (2,2)-LV system and they only involve half the lattice points. As with the (1,2)-LV system there are two bilinear equations and two continuous variables, so we need to eliminate one of these variables between the two equations.

In addition to these two reductions we can obtain the (2,1)-TL (this is the standard two continuous dimensions and one discrete dimension case). This can be achieved by again removing half the lattice points from the (2,2)-LV system to get the (2,2)-TL, then take equations that use only these lattice points, i.e. (20) and (21), now rotate back into the natural coordinates for Toda lattices, i.e. set $p = \frac{m+n}{2}$, $q = \frac{m-n}{2}$, with $\tau_{pq} = F_{p+q,p-q} = F_{mn}$. Then finally set all the τ-functions off the main horizontal axis equal to zero:

$$\tau_{i,0} \neq 0 \qquad \text{and} \qquad \tau_{i,j} = 0, \quad \forall j \neq 0,$$

leaving us with

$$\frac{1}{2} D_x D_y \tau_{p,0} \cdot \tau_{p,0} = \tau_{p+1,0}\tau_{p-1,0} - \alpha\tau_{p,0}^2. \tag{47}$$

In the $\alpha = 1$ case this can be realized by reducing the pfaffian solutions to grammian solutions by setting half the eigenfunctions f_i and g_i to zero [1]. i.e. set

$$f_i = 0 \quad \text{for} \quad 1 \leq i \leq N/2, \qquad\qquad g_i = 0 \quad \text{for} \quad N/2+1 \leq i \leq N.$$

This will give back the lattice solutions (6). For the $\alpha = 0$ case it is less clear how to proceed as in the 2-discrete dimensional lattice cases we have only looked at solutions that would reduce back to lattice solutions rather than molecule solutions.

5 Conclusions and Discussion

We have looked at both lattice solutions and molecule solutions to some integrable systems including the two discrete dimensional Toda lattice of Santini et al [9], the pfaffianized Toda lattice [4, 10] and the two discrete dimensional Lotka-Volterra equation of Hu et al [5]. With the identifications (39), the bilinear forms of the pfaffianized Toda lattice and the $\alpha = 1$ case of two discrete dimensional Toda lattice are the same.

Although the bilinearization of the two, 2-discrete dimensional systems is different the solutions are essentially the same. This is an indication that the equations in these two systems come (via reduction) from the same bilinear hierarchy. This is indeed the case as all the equations are low order equations in a 2+2 Lotka-Volterra hierarchy. The connection between the two systems is in fact closer still, as it is possible to directly transform (up to a difference operator) from one system to the other.

References

[1] Gilson C R, *Theor. Math. Phys.* **133** (2002), 1663–1674.

[2] Hirota R and Ohta Y, *J. Phys. Soc. Japan* **60** (1990), 798.

[3] Hirota R, The Direct Method in Soliton Theory, Cambridge Tracts in Mathematics **155** (2004).

[4] Hu X-B, Zhao J-X, Tam H-W, *J. Math. Analys. Appl.* **296** (2004), 256–261.

[5] Hu X-B, Li C-X, Nimmo J J C and Yu G-F, An integrable symmetric (2+1)-dimensional Lotka-Volterra equation and a family of its solutions, *J. Phys. A* **38** (2005), 195–204.

[6] Private correspondence with Hu X-B and Willox R.

[7] Jimbo M and Miwa T, Publications R.I.M.S. Kyoto University **19** (1983), 943–1001.

[8] Nakamura A, *J. Phys. Soc. Japan* **57** (1988), 3309.

[9] Santini P M, Nieszporski M and Doliwa A, An integrable generalisation of the Toda law to the square lattice *Phys. Rev. E* **70** (2004), 056615.

[10] Willox R, On a generalized Tzitzeica equation. *Glas. Math. J.* **47** (2005), to appear.