Second Order Dynamic Inclusions

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Abstract

The theory of dynamic inclusions on a time scale is introduced, hence accommodating the special cases of differential inclusions and difference inclusions. Fixed point theory for set-valued upper semicontinuous maps, Green's functions, and upper and lower solutions are used to establish existence results for solutions of second order dynamic inclusions.

1   Introduction

Let $\tilde{\mathbb{T}}$ be a time scale such that $0, T \in \tilde{\mathbb{T}}$ for some $T > 0$ (for the definition of a time scale and related notation see Section 2), put $\mathbb{T} := [0, T] \cap \tilde{\mathbb{T}}$, and consider dynamic inclusions

$$y(t) \in \int_0^T g(t, s) F(s, y^\sigma(s)) \Delta s, \quad t \in \mathbb{T}, \tag{1.1}$$

where $F : \mathbb{T} \times \mathbb{R} \to \mathrm{CK}(\mathbb{R})$ is a set-valued map and $g : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is a single-valued continuous map ($\mathrm{CK}(\mathbb{R})$ denotes the set of nonempty, closed, and convex subsets of $\mathbb{R}$). In Section 3 some general existence principles for inclusions (1.1) are derived by using fixed point theory discussed in [1]. In Section 4 we present a specific function g such that y is a solution of (1.1) if and only if y is a solution of the second order dynamic inclusion

$$y^{\Delta\Delta}(t) \in F(t, y^\sigma(t)), \quad t \in \mathbb{T}^\kappa, \quad y(0) = y(\sigma(T)) = 0 \tag{1.2}$$

(for the time scales specific notation we refer again to Section 2). Using this equivalence and the existence principles for (1.1) from Section 3, we then proceed to establish an existence result for the second order dynamic inclusion (1.2). We also offer another existence result for the dynamic inclusion (1.2) based on the notion of upper and lower solutions on time scales (see [4]). Finally, in Section 5, we present some results for time scales possessing a differentiable forward jump operator. Our investigations follow closely the arguments given by Agarwal, O'Regan, and Lakshmikantham [2] and Stehlik and Tisdell [9] for discrete second order inclusions. For more on boundary value problems on time scales we refer the reader to [7, 8, 10].

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2   Preliminaries

In this section we present some definitions and elementary results connected to the time scales calculus. For further study we refer the reader to the monographs [5, 6]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. On $\mathbb{T}$ we define the *forward* and *backward jump operators* by

$$\sigma(t) := \inf\{s \in \mathbb{T}: \ s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T}: \ s < t\} \quad \text{for} \quad t \in \mathbb{T}.$$

A point $t \in \mathbb{T}$ with $t > \inf \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The set $[a, b] \cap \mathbb{T}$ with $a, b \in \mathbb{T}$ is abbreviated by $[a, b]$, and we also shall use the notation $[a, b]^\kappa := [a, b] \setminus (\rho(b), b]$. Next, the *graininess* function μ is defined by $\mu(t) := \sigma(t) - t$ for $t \in \mathbb{T}$. For a function $f : \mathbb{T} \to \mathbb{R}$ the (delta) *derivative* $f^\Delta(t)$ at $t \in \mathbb{T}$ is defined to be the number (provided it exists) with the property such that for every $\varepsilon > 0$ there exists a neighbourhood U of t with

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \leq \varepsilon \left| \sigma(t) - s \right| \quad \text{for all} \quad s \in U.$$

A useful formula is

$$f^\sigma = f + \mu f^\Delta, \quad \text{where} \quad f^\sigma := f \circ \sigma. \tag{2.1}$$

We will use the product rule for the derivative of the product fg of two differentiable functions f and g

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma. \tag{2.2}$$

For $a, b \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^Δ is defined by

$$\int_a^b f^\Delta(t)\Delta t = f(b) - f(a). \tag{2.3}$$

Note that in the case $\mathbb{T} = \mathbb{R}$ we have

$$\sigma(t) = t, \quad \mu(t) \equiv 0, \quad f^\Delta(t) = f'(t),$$

and in the case $\mathbb{T} = \mathbb{Z}$ we have

$$\sigma(t) = t + 1, \quad \mu(t) \equiv 1, \quad f^\Delta(t) = \Delta f(t) = f(t+1) - f(t).$$

Another important time scale is $\mathbb{T} = \{q^k : k \in \mathbb{N}\}$ with $q > 1$, for which

$$\sigma(t) = qt, \quad \mu(t) = (q-1)t, \quad f^\Delta(t) = \frac{f(qt) - f(t)}{(q-1)t},$$

and this time scale gives rise to so-called q-difference equations.

3   Fixed Point Results

A map $G : \mathbb{R} \to \mathrm{CK}(\mathbb{R})$ is called *upper semicontinuous* provided $\{u_k\}_{k \in \mathbb{N}}, \{v_k\}_{k \in \mathbb{N}} \subset \mathbb{R}$ with $u_k \to u$, $v_k \to v$ $(k \to \infty)$ and $v_k \in G(u_k)$ for all $k \in \mathbb{N}$ always implies $v \in G(u)$. Throughout this paper we assume that

$$\begin{cases} F : \mathbb{T} \times \mathbb{R} \to \mathrm{CK}(\mathbb{R}) \text{ is such that} \\ F(t, \cdot) \text{ is upper semicontinuous for all } t \in \mathbb{T} \\ \mathcal{F}(y) \neq \emptyset \text{ for all } y \in \mathrm{C}(\mathbb{T}), \end{cases} \tag{3.1}$$

where $\mathrm{C}(\mathbb{T})$ abbreviates the set of all continuous functions $y : \mathbb{T} \to \mathbb{R}$, and

$$\mathcal{F}(u) := \{v \in \mathrm{C}(\mathbb{T}) : \ v(t) \in F(t, u^\sigma(t)) \text{ for all } t \in \mathbb{T}\} .$$

The existence principles presented in the remaining part of this section rely on the following two fixed point results which are extracted from [1].

Theorem 1. *Let E be a Banach space and $C \in \mathrm{CK}(E)$. If $G : C \to \mathrm{CK}(C)$ is upper semicontinuous and compact, then G has a fixed point in C.*

Theorem 2. *Let E be a Banach space. Assume $U \subset E$ is open, and let $0 \in U$. If $G : \overline{U} \to \mathrm{CK}(E)$ is upper semicontinuous and compact, then either G has a fixed point in $\overline{U}$ or there exists $u \in \partial U$ and $\lambda \in (0, 1)$ such that $u \in \lambda G(u)$.*

Now we prove the following two existence principles which shall be used to establish existence results for second order dynamic inclusions in Section 4. For a function $y \in \mathrm{C}(\mathbb{T})$ we put $\|y\| = \max_{t \in \mathbb{T}} |y(t)|$. We also use the notation $|F(t, u)| = \sup_{v \in F(t,u)} |v|$.

Theorem 3. *Assume (3.1) and suppose that $g : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is continuous. If for all $r > 0$ there exists a nonnegative function $h_r \in \mathrm{C}(\mathbb{T})$ with*

$$|F(t, u)| \leq h_r(t) \quad \text{for all} \quad t \in \mathbb{T} \quad \text{and all} \quad |u| \leq r \tag{3.2}$$

and if there exists a constant M with $\|y\| \neq M$ for all solutions y of

$$y(t) \in \lambda \int_0^T g(t, s) F(s, y^\sigma(s)) \Delta s, \quad t \in \mathbb{T} \tag{3.3}$$

for all $\lambda \in (0, 1)$, then (1.1) has a solution.

Proof. We define a linear and continuous operator $\mathcal{T} : \mathrm{C}(\mathbb{T}) \to \mathrm{C}(\mathbb{T})$ by

$$(\mathcal{T} y)(t) = \int_0^T g(t, s) y(s) \Delta s, \quad t \in \mathbb{T}$$

so that (3.3) is equivalent to the fixed point problem

$$y \in \lambda(\mathcal{T} \circ \mathcal{F})(y),$$

where $\mathcal{F} : \mathrm{C}(\mathbb{T}) \to \mathrm{CK}(\mathrm{C}(\mathbb{T}))$ is defined in (3.1). We also define

$$E = \mathrm{C}(\mathbb{T}), \quad U = \{u \in E : \ \|u\| < M\} .$$

Now we will apply Theorem 2 to the function $\mathcal{T} \circ \mathcal{F}$ by showing that

$$\mathcal{T} \circ \mathcal{F} : \overline{U} \to \mathrm{CK}(E) \quad \text{is upper semicontinuous and compact.} \tag{3.4}$$

Assume there exists $u \in \partial U$ and $\lambda \in (0, 1)$ with $u \in \lambda(\mathcal{T} \circ \mathcal{F})(u)$. Then $\|u\| = M$, and therefore the second possibility given in Theorem 2 is ruled out. Hence, if (3.4) is true, then the first possibility given in Theorem 2 holds so that $\mathcal{T} \circ \mathcal{F}$ has a fixed point in $\overline{U}$, which completes the proof. Hence it only remains to show (3.4). We first show that $\mathcal{T} \circ \mathcal{F} : \overline{U} \to \mathrm{CK}(E)$ is upper semicontinuous. Let $\{u_k\}_{k \in \mathbb{N}}, \{w_k\}_{k \in \mathbb{N}} \subset \mathbb{R}$ such that $u_k \to u_0$, $w_k \to w_0$ $(k \to \infty)$, and $w_k \in (\mathcal{T} \circ \mathcal{F})(u_k)$ for all $k \in \mathbb{N}$. Thus there exists $v_k \in \mathcal{F}(u_k)$ with $w_k = \mathcal{T} v_k$. Since $u_k \in \overline{U}$ for all $k \in \mathbb{N}$, (3.2) implies by [3, page 262] that there exists a compact set $\Omega \subset E$ with $\{v_k\}_{k \in \mathbb{N}} \subset \Omega$. Therefore there exists a convergent subsequence $\{v_{k_\nu}\}_{\nu \in \mathbb{N}}$ of $\{v_k\}_{k \in \mathbb{N}}$, say $v_{k_\nu} \to v_0$ as $\nu \to \infty$. Now

$$v_{k_\nu} \to v_0 \quad \text{and} \quad u_{k_\nu} \to u_0 \quad \text{as} \quad \nu \to \infty$$

and $v_{k_\nu}(t) \in F(t, u_{k_\nu}^\sigma(t))$ for all $t \in \mathbb{T}$. Thus, since $F(t, \cdot)$ is upper semicontinuous for all $t \in \mathbb{T}$, we may conclude $v_0(t) \in F(t, u_0^\sigma(t))$ for all $t \in \mathbb{T}$ and therefore $v_0 \in \mathcal{F}(u_0)$. Since $v_{k_\nu} \to v_0$ as $\nu \to \infty$ and $\mathcal{T} : E \to E$ is continuous, we see that $w_{k_\nu} = \mathcal{T} v_{k_\nu} \to \mathcal{T} v_0$ as $\nu \to \infty$, and hence

$$w_0 = \mathcal{T} v_0 \in (\mathcal{T} \circ \mathcal{F})(u_0).$$

Therefore $\mathcal{T} \circ \mathcal{F} : \overline{U} \to \mathrm{CK}(E)$ is upper semicontinuous. As it is also compact by the Arzelà–Ascoli theorem [3, Chapter 17], (3.4) holds and the proof is complete. ∎

Theorem 4. *Assume* (3.1) *and suppose that* $g : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ *is continuous. If there exists a nonnegative function* $h \in \mathrm{C}(\mathbb{T})$ *with*

$$|F(t, u)| \leq h(t) \quad \text{for all} \quad t \in \mathbb{T} \quad \text{and all} \quad u \in \mathbb{R},$$

then (1.1) *has a solution.*

Proof. If $\mathcal{T}$ and $\mathcal{F}$ are as in Theorem 3, then (1.1) is equivalent to the fixed point problem

$$y \in (\mathcal{T} \circ \mathcal{F})(y).$$

Now we can show as in Theorem 3 that $\mathcal{T} \circ \mathcal{F} : E \to \mathrm{CK}(E)$ is upper semicontinuous and compact, and hence the claim follows by using Theorem 1. ∎

4 Existence Results for Dynamic Inclusions

We first prove the following equivalence.

Theorem 5. *Assume* (3.1). *Define a function* $g : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ *by*

$$g(t, s) = \begin{cases} \frac{t\sigma(s)}{\sigma(T)} - t & \text{if} \quad t \leq s \\ \frac{t\sigma(s)}{\sigma(T)} - \sigma(s) & \text{if} \quad t \geq \sigma(s). \end{cases} \tag{4.1}$$

Then y *solves* (1.1) *iff* y *solves* (1.2).

Proof. First assume y solves (1.1). Then there exists a function $\tau \in \mathcal{F}(y)$ such that

$$
\begin{aligned}
y(t) &= \int_0^T g(t,s)\tau(s)\Delta s \\
&= \int_0^t g(t,s)\tau(s)\Delta s + \int_t^T g(t,s)\tau(s)\Delta s \\
&= \int_0^t \left(\frac{t\sigma(s)}{\sigma(T)} - \sigma(s) \right) \tau(s)\Delta s + \int_T^t \left(t - \frac{t\sigma(s)}{\sigma(T)} \right) \tau(s)\Delta s \\
&= \frac{t}{\sigma(T)} \int_0^t \sigma(s)\tau(s)\Delta s - \int_0^t \sigma(s)\tau(s)\Delta s + t \int_T^t \tau(s)\Delta s.
\end{aligned}
$$

We use the product rule (2.2) and the definition of the integral (2.3) to find

$$
y^\Delta(t) = \frac{1}{\sigma(T)} \int_0^t \sigma(s)\tau(s)\Delta s + \int_T^t \tau(s)\Delta s
$$

so that

$$
y^{\Delta\Delta}(t) = \tau(t) \in F(t, y^\sigma(t)) \quad \text{for all} \quad t \in \mathbb{T}^\kappa.
$$

Clearly, $y(0) = 0$ and

$$
\begin{aligned}
y(\sigma(T)) &= \int_0^T \sigma(s)\tau(s)\Delta s - \int_0^{\sigma(T)} \sigma(s)\tau(s)\Delta s + \sigma(T) \int_T^{\sigma(T)} \tau(s)\Delta s \\
&= \int_0^T \sigma(s)\tau(s)\Delta s - \int_0^{\sigma(T)} \sigma(s)\tau(s)\Delta s + \int_T^{\sigma(T)} \sigma(s)\tau(s)\Delta s \\
&= 0
\end{aligned}
$$

so that y solves (1.2). Conversely, assume now that y solves (1.2). Then

$$
\begin{aligned}
\int_0^T g(t,s)F(s,y^\sigma(s))\Delta s &\ni \int_0^T g(t,s)y^{\Delta\Delta}(s)\Delta s \\
&= \frac{t}{\sigma(T)} \int_0^T \sigma(s)y^{\Delta\Delta}(s)\Delta s - \int_0^t \sigma(s)y^{\Delta\Delta}(s)\Delta s + t \int_T^t y^{\Delta\Delta}(s)\Delta s \\
&= \frac{t}{\sigma(T)} \left(Ty^\Delta(T) - y(T) + y(0) \right) - \left(ty^\Delta(t) - y(t) + y(0) \right) + t \left(y^\Delta(t) - y^\Delta(T) \right) \\
&= \frac{tT}{\sigma(T)} y^\Delta(T) - \frac{t}{\sigma(T)} y(T) + y(t) - ty^\Delta(t) \\
&= -\frac{t\mu(T)}{\sigma(T)} y^\Delta(T) - \frac{t}{\sigma(T)} y(T) + y(t) \\
&= \frac{t}{\sigma(T)} \left(y(T) - y(\sigma(T)) \right) - \frac{t}{\sigma(T)} y(T) + y(t) \\
&= y(t)
\end{aligned}
$$

so that y solves (1.1). ∎

Using Theorem 5, we can now present our first existence result for second order dynamic inclusions.

Theorem 6. *Assume* (3.1). *Suppose there exists a continuous and nondecreasing function* $\Psi : [0, \infty) \to [0, \infty)$ *with* $\Psi(u) > 0$ *for* $u > 0$ *and a function* $q : \mathbb{T} \to [0, \infty)$ *such that* $|F(t, u)| \leq q(t)\Psi(|u|)$ *for all* $u \in \mathbb{R}$ *and* $t \in \mathbb{T}$. *If*

$$\sup_{c>0} \frac{c}{\Psi(c)} > Q_0 := \max_{t \in \mathbb{T}} \int_0^T |g(t, s)| q(s) \Delta s,$$

where g *is defined by* (4.1), *then* (1.2) *has a solution.*

Proof. Suppose $M > 0$ satisfies $M/\Psi(M) > Q_0$. Consider

$$y^{\Delta\Delta}(t) \in \lambda F(t, y^\sigma(t)), \quad t \in \mathbb{T}^\kappa, \quad y(0) = y(\sigma(T)) = 0$$

with $0 < \lambda < 1$. By Theorem 5, this is equivalent to (3.3). Let y be any solution of (3.3) for $0 < \lambda < 1$. Then

$$|y(t)| \leq \lambda \int_0^T |g(t, s)| q(s) \Psi(|y(s)|) \Delta s \leq \Psi(\|y\|) \int_0^T |g(t, s)| q(s) \Delta s \leq \Psi(\|y\|) Q_0$$

so that

$$\frac{\|y\|}{\Psi(\|y\|)} \leq Q_0.$$

If $\|y\| = M$, then $M/\Psi(M) \leq Q_0$, contradicting the first line of this proof. Hence the statement follows from Theorem 3. ∎

Our next existence result for second order dynamic inclusions uses upper and lower solutions defined as follows.

Definition 1. A function $\alpha \in C(\mathbb{T})$ is called a *lower solution* of (1.2) if

$$F(t, \alpha^\sigma(t)) \cap \left(-\infty, \alpha^{\Delta\Delta}(t)\right] \neq \emptyset, \quad t \in \mathbb{T}^\kappa, \quad \alpha(0) \leq 0, \quad \alpha(\sigma(T)) \leq 0.$$

A function $\beta \in C(\mathbb{T})$ is called an *upper solution* of (1.2) if

$$F(t, \beta^\sigma(t)) \cap \left[\beta^{\Delta\Delta}(t), \infty\right) \neq \emptyset, \quad t \in \mathbb{T}^\kappa, \quad \beta(0) \geq 0, \quad \beta(\sigma(T)) \geq 0.$$

Theorem 7. *Assume* (3.1). *Suppose for all* $r > 0$ *there exists a nonnegative function* $h_r \in C(\mathbb{T})$ *with* (3.2). *If there exist lower and upper solutions* α *and* β *of* (1.2) *with* $\alpha(t) \leq \beta(t)$ *for all* $t \in \mathbb{T}$, *then* (1.2) *has a solution* y *with* $\alpha(t) \leq y(t) \leq \beta(t)$ *for all* $t \in \mathbb{T}$.

Proof. We define

$$h(t, x) = \begin{cases} \alpha^\sigma(t) & \text{if } x < \alpha^\sigma(t) \\ \beta^\sigma(t) & \text{if } x > \beta^\sigma(t) \\ x & \text{otherwise,} \end{cases} \quad r(t, x) = \begin{cases} \varrho(x - \alpha^\sigma(t)) & \text{if } x < \alpha^\sigma(t) \\ \varrho(x - \beta^\sigma(t)) & \text{if } x > \beta^\sigma(t) \\ 0 & \text{otherwise,} \end{cases}$$

$$\varrho(x) = \begin{cases} x & \text{if } |x| \leq 1 \\ \frac{x}{|x|} & \text{if } |x| > 1, \end{cases} \quad \Gamma_+(t, x) = \begin{cases} \left(-\infty, \alpha^{\Delta\Delta}(t)\right] & \text{if } x < \alpha^\sigma(t) \\ \left[\beta^{\Delta\Delta}(t), \infty\right) & \text{if } x > \beta^\sigma(t) \\ \mathbb{R} & \text{otherwise,} \end{cases}$$

$$F_+(t, x) = F(t, h(t, x)) \cap \Gamma_+(t, x), \quad F_+^*(t, x) = F_+(t, x) + r(t, x).$$

We apply Theorem 4 to the function $F_+^* : \mathbb{T} \times \mathbb{R} \to \mathrm{CK}(\mathbb{R})$. Clearly, $\Gamma_+(t, \cdot)$ is upper semicountinuous for each $t \in \mathbb{T}$ and hence so is $F_+(t, \cdot)$ and therefore $F_+^*(t, \cdot)$. Now the modified problem

$$y^{\Delta\Delta}(t) \in F_+^*(t, y^\sigma(t)), \quad t \in \mathbb{T}^\kappa, \quad y(0) = y(\sigma(T)) = 0 \tag{4.2}$$

is equivalent by Theorem 5 to the problem

$$y(t) \in \int_0^T g(t, s) F_+^*(s, y^\sigma(s)) \Delta s, \quad t \in \mathbb{T}, \tag{4.3}$$

where g is given in (4.1). Since

$$|F_+^*(t, u)| \le |F_+(t, u)| + |r(t, u)| \le h_{\|\beta\|}(t) + 1,$$

Theorem 4 ensures that (4.3) has a solution $y \in C(\mathbb{T})$. Hence by the above equivalence of (4.3) and (4.2), we conclude that there exists a solution y of (4.2). It remains to show that $\alpha(t) \le y(t) \le \beta(t)$ holds for all $t \in \mathbb{T}$. Assume there exists $m \in \mathbb{T}$ with $y(m) > \beta(m)$. Define $u := y - \beta$ and let $\theta \in \mathbb{T}$ be such that $\max_{t \in \mathbb{T}} u(t) = u(\theta)$. Therefore $u(\theta) > 0$. By an argument given in [4] we may conclude that

$$u^{\Delta\Delta}(\rho(\theta)) \le 0 \quad \text{and} \quad \sigma(\rho(\theta)) = \theta. \tag{4.4}$$

But, since

$$\begin{aligned} y^{\Delta\Delta}(\rho(\theta)) &\in & F_+^*(\rho(\theta), y^\sigma(\rho(\theta))) \\ &=& F_+(\rho(\theta), y(\theta)) + r(\rho(\theta), y(\theta)) \\ &=& F_+(\rho(\theta), y(\theta)) + \varrho(y(\theta) - \beta(\theta)), \end{aligned}$$

there exists $w(\theta) \in \left[\beta^{\Delta\Delta}(\rho(\theta)), \infty\right)$ with

$$y^{\Delta\Delta}(\rho(\theta)) = w(\theta) + \varrho(y(\theta) - \beta(\theta)) \ge \beta^{\Delta\Delta}(\rho(\theta)) + \varrho(y(\theta) - \beta(\theta))$$

so that

$$u^{\Delta\Delta}(\rho(\theta)) = y^{\Delta\Delta}(\rho(\theta) - \beta(\theta)) \ge \varrho(y(\theta) - \beta(\theta)) > 0,$$

contradicting (4.4). Hence $y(t) \le \beta(t)$ for all $t \in \mathbb{T}$, and a similar argument shows $\alpha(t) \le y(t)$ for all $t \in \mathbb{T}$. The proof is complete. ∎

5 Time Scales with Differentiable Forward Jump

In this section we present an existence result for the second order dynamic inclusion problem (1.2) subject to the assumption that the time scale $\mathbb{T}$ has a differentiable forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$. That is, we assume throughout this final section that

$$\sigma \quad \text{is differentiable.} \tag{5.1}$$

We start with some auxiliary results that are needed in the proof of our existence result. These results about time scales satisfying (5.1) are also interesting in their own right.

Lemma 1. *Assume (5.1). If $y : \mathbb{T} \to \mathbb{R}$ is differentiable, then so is $y^\sigma : \mathbb{T}^\kappa \to \mathbb{R}$, and we have*

$$y^{\sigma\Delta} = \sigma^\Delta y^{\Delta\sigma}.$$

Proof. Using (2.1), we have

$$y^\sigma(t) = y(t) + \mu(t)y^\Delta(t) = y(t) - ty^\Delta(t) + \sigma(t)y^\Delta(t)$$

so that, by the product rule (2.2),

$$y^{\sigma\Delta}(t) = y^\Delta(t) - \sigma(t)y^{\Delta\Delta}(t) + \sigma(t)y^{\Delta\Delta}(t) + \sigma^\Delta(t)y^{\Delta\sigma}(t).$$

This completes the proof. ∎

Lemma 2. *Assume (5.1). If $y : \mathbb{T} \to \mathbb{R}$ is twice differentiable, then so is $y^2 : \mathbb{T} \to \mathbb{R}$, and we have*

$$\left(y^2\right)^{\Delta\Delta} = 2y^\sigma y^{\Delta\Delta} + \left(y^\Delta\right)^2 + \sigma^\Delta \left(y^{\Delta\sigma}\right)^2.$$

Proof. First, $(y^2)^\Delta = y^\Delta y + y^\Delta y^\sigma$ by the product rule (2.2). Applying the product rule again, we find

$$\left(y^2\right)^{\Delta\Delta} = y^{\Delta\Delta}y^\sigma + y^\Delta y^\Delta + y^{\Delta\Delta}y^\sigma + y^{\Delta\sigma}y^{\sigma\Delta}.$$

Now the claim follows by using Lemma 1. ∎

Theorem 8. *Assume (3.1) and (5.1). If there exist constants $L, K \geq 0$ with*

$$|F(t,u)| \leq \inf_{w \in F(t,u)} (Luw + K) \quad \text{for all} \quad (t,u) \in \mathbb{T} \times \mathbb{R}, \tag{5.2}$$

then any solution y of (1.2) satisfies

$$\|y\| \leq K \max_{t \in \mathbb{T}} \int_0^T |g(t,s)|\Delta s.$$

Proof. Suppose y solves (1.2). Then, by Theorem 5, y also solves (1.1), i.e., there exists $\tau \in \mathcal{F}(y)$ such that

$$y(t) = \int_0^T g(t,s)\tau(s)\Delta s \quad \text{for all} \quad t \in \mathbb{T}.$$

Since $\tau \in \mathcal{F}(y)$, we have for all $s \in \mathbb{T}$

$$|\tau(s)| \leq \inf_{w \in F(s,y^\sigma(s))} (Ly^\sigma(s)w + K) \leq Ly^\sigma(s)y^{\Delta\Delta}(s) + K \leq \frac{L}{2}\left(y^2\right)^{\Delta\Delta}(s) + K,$$

where we used Lemma 2 and the fact that σ is an increasing function. Thus

$$\begin{aligned}
|y(t)| &\leq \int_0^T |g(t,s)||\tau(s)|\Delta s \\
&\leq \int_0^T |g(t,s)| \left[\frac{L}{2}\left(y^2\right)^{\Delta\Delta}(s) + K\right]\Delta s \\
&= -\frac{L}{2}y^2(t) + K\int_0^T |g(t,s)|\Delta s \\
&\leq K\int_0^T |g(t,s)|\Delta s
\end{aligned}$$

for all $t \in \mathbb{T}$, where we have used

$$\int_0^T |g(t,s)| \left(y^2\right)^{\Delta\Delta}(s)\Delta s = -y^2(t). \tag{5.3}$$

It remains to prove formula (5.3). To this end, we use (4.1) to find

$$\int_0^T |g(t,s)| \left(y^2\right)^{\Delta}(s)\Delta s = \int_0^t \sigma(s) \left(y^2\right)^{\Delta\Delta}(s)\Delta s - \frac{t}{\sigma(T)} \int_0^T \sigma(s) \left(y^2\right)^{\Delta\Delta}(s)\Delta s$$

$$+ t \int_t^T \left(y^2\right)^{\Delta\Delta}(s)\Delta s$$

$$= t \left(y^2\right)^{\Delta}(t) - \int_0^t \left(y^2\right)^{\Delta}(s)\Delta s - \frac{t}{\sigma(T)} \left[T \left(y^2\right)^{\Delta}(T) - \int_0^T \left(y^2\right)^{\Delta}(s)\Delta s\right]$$

$$+ t \left[\left(y^2\right)^{\Delta}(T) - \left(y^2\right)^{\Delta}(t)\right]$$

$$= -y^2(T) + y^2(0) - \frac{t}{\sigma(T)} \left[T \left(y^2\right)^{\Delta}(T) - y^2(T) + y^2(0)\right] + t \left(y^2\right)^{\Delta}(T)$$

$$= -y^2(t) + t\frac{\mu(T) \left(y^2\right)^{\Delta}(T) + y^2(T)}{\sigma(T)}$$

$$= -y^2(t).$$

This completes the proof.   ∎

Theorem 9. *If* (3.1), (5.1), *and* (5.2) *hold, then* (1.2) *has a solution.*

Proof. We may multiply both sides of the inequality in (5.2) by $\lambda \in [0,1]$ to find

$$|\lambda F(t,u)| \leq \inf_{w \in \lambda F(t,u)} (Luw + K) \quad \text{for all} \quad (t,u) \in \mathbb{T} \times \mathbb{R}.$$

Therefore Theorem 8 is applicable to

$$y^{\Delta\Delta}(t) \in \lambda F(t, y^\sigma(t)), \quad t \in \mathbb{T}^\kappa, \quad y(0) = y(\sigma(T)) = 0, \tag{5.4}$$

and thus $\|y\| \leq R$ for all solutions y of (5.4), where R is defined by

$$R := K \max_{t \in \mathbb{T}} \int_0^T |g(t,s)|\Delta s.$$

Now choose U to be the set

$$U := \{y \in \mathrm{C}(\mathbb{T}): \|y\| < R + 1\}.$$

In view of the above argument, there cannot be any solution y of (5.4) with $\|y\| = R+1$. Hence, the relevant operator arguments in the previous theorems are applicable. We have the necessary compactness and upper semicontinuity and thus, by Theorem 2, the boundary value problem (1.2) must have a solution. This concludes the proof.   ∎

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