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Integrable flows and Bäcklund transformations on extended Stiefel varieties with application to the Euler top on the Lie group $SO(3)$

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Abstract

We show that the m-dimensional Euler–Manakov top on $so^*(m)$ can be represented as a Poisson reduction of an integrable Hamiltonian system on a symplectic extended Stiefel variety $\bar{\mathcal{V}}(k,m)$, and present its Lax representation with a rational parameter.

We also describe an integrable two-valued symplectic map $\mathcal{B}$ on the 4-dimensional variety $\mathcal{V}(2,3)$. The map admits two different reductions, namely, to the Lie group $SO(3)$ and to the coalgebra $so^*(3)$.

The first reduction provides a discretization of the motion of the classical Euler top in space and has a transparent geometric interpretation, which can be regarded as a discrete version of the celebrated Poinsot model of motion and which inherits some properties of another discrete system, the elliptic billiard.

The reduction of $\mathcal{B}$ to $so^*(3)$ gives a new explicit discretization of the Euler top in the angular momentum space, which preserves first integrals of the continuous system.

1 Introduction

In most publications the integrable m-dimensional Euler top is represented as a flow on the cotangent bundle $T^*SO(m)$ or on the coalgebra $so^*(m)$.

Recently, an alternative description of this problem as a system on a symplectic subvariety of the group product $SO(m) \times SO(m)$ was proposed in [4, 5].

A first discretization of the free m-dimensional top on $T^*SO(m)$ was constructed in [24, 21] by the method of factorization of matrix polynomials. This discretization is represented by a second order Lagrangian correspondence, which does not explicitly involve a time step, it is determined by initial data (a choice of two subsequent points on $SO(m)$).

On the other hand, in [23], Suris introduced a concept of an integrable discretisation of a finite-dimensional Hamiltonian system as a one parameter family of integrable Poisson maps parameterized by a time step ϵ, which differ from the identity map by $O(\epsilon)$, and whose Poisson structure and the integrals of motion differ at most by $O(\epsilon)$ from those of the continuous-time system.

In the special case where the discretization preserves exactly both the Poisson structure and the integrals of motion, one speak of an "exact discretization": one has a family of Bäcklund transformations, which map solutions into solutions and are interpolated by a hamiltonian flow generated by some function of the integrals of motion of the continuous system.

A class of implicitly defined Poisson maps $so^*(3) \to so^*(3)$ discretizing the classical Euler top in the space of the angular momentum was indicated in [6]. The maps preserve the energy and momentum integrals of the continuous problem and contain explicitly a time step parameter. It was shown that such a map preserves the standard Lie–Poisson structure on $so^*(3)$ if and only if its restriction onto complex invariant manifolds, open subsets of elliptic curves, is a shift, which is constant on each curve.

Recently, another integrable discretization of the top on $so^*(3)$, which is *explicit*, but does not preserve the integrals of the continuous problem was found in [14] by applying the Hirota method.

Contents of the paper. Our aim is twofold. First, in Section 2, we propose yet another description of the continuous n-dimensional Euler–Manakov top as a reduction of a Hamiltonian system on so called extended Stiefel variety $\bar{\mathcal{V}}(k,m)$, a symplectic submanifold of dimension $km - k^2/2$ in $\mathbb{R}^{km}$, where $2 \le k \le n$ is an even integer. We present a Lax representation of this system with a rational parameter, which, in a sense described below, is dual to Manakov's Lax pair found in [17].

The system possesses $k/2$ commuting symmetry fields $\mathcal{R}_l$ generated by Hamiltonians H_l. Its Marsden–Weinstein reduction with respect to the action of the fields gives rise to a Hamiltonian system on a rank k coadjoint orbit $\mathcal{S}_h^{(k)}$ in the coalgebra $so^*(m)$, whereas the original Poisson structure in $\bar{\mathcal{V}}(k,m)$ is a pull-back of the standard Lie–Poisson structure of $so^*(m)$ restricted onto the orbit. The reduced Hamiltonian system coincides with the Euler–Manakov system on $\mathcal{S}_h^{(k)}$. In case of the maximal rank k, the level variety $\{H_l = c_l\} \subset \bar{\mathcal{V}}(k,m)$ is the group $SO(m)$, and the restriction of the original system onto the group yields a flow describing the motion of the n-dimensional top in space.

Second, in Section 3, we present an intertwining relation (discrete Lax pair) generating a explicit λ^*-depended family of two-valued complex Bäcklund transformations $\mathcal{B}_\lambda^*$ of the variety $\mathcal{V}(2,3)$, which preserve the above Poisson structure and the first integrals of the continuous Hamiltonian system.

The restricton of $\mathcal{B}_\lambda^*$ onto the group $SO(3)$ provides a discretization of the motion of the classical Euler top in space and has a transparent geometric interpretation, which, in turn, can be regarded as a discrete version of the celebrated Poinsot model of motion and which inherits some properties of another discrete integrable system, the elliptic billiard (Figure 1).

On the other hand, the reduction of $\mathcal{B}_\lambda^*$ onto the coalgebra $so^*(3)$ gives a new explicit discretization of the classical Euler top, which also preserves its first integrals.

Like the Moser–Veselov correspondence, the both discretizations do not explicitly involve a time step and their continuous limits depend on the parameter λ^*.

2 Hamiltonian Systems on Extended Stiefel Varieties and Rank k Solutions of Frahm–Manakov top

Recall that the free motion of an m-dimensional rigid body is described by the Euler–Frahm equations ([11])

$$\dot{M} = [M, \Omega], \tag{2.1}$$

where $\Omega \in so(m)$ is the angular velocity, $M \in so^*(m)$ the angular momentum of the body in the moving frame. Following [19, 22], these equations are Hamiltonian with respect to the degenerate Lie–Poisson bracket on $so^*(m)$

$$\{M_{ij}, M_{kl}\}_{so(m)} = \delta_{il} M_{jk} - \delta_{il} M_{kj} + \delta_{kj} M_{jl} - \delta_{ik} M_{jl} \tag{2.2}$$

and $\Omega_{ij} = \partial H(M)/\partial M_{ij}$.

The restriction of $\{\cdot,\cdot\}_{so(m)}$ onto orbits of coadjoint action of $SO(m)$ in $so^*(m)$ is nondegenerate. A generic orbit $\mathcal{S}_h$ parameterized by $[m/2]$ independent Casimir functions of the bracket is thus a symplectic variety of dimension $m(m-1)/2-[m/2]$.

Equations (2.1) are known to be integrable provided M and Ω are related as $[M, a] = [\Omega, b]$, where a, b are constant commuting matrices, and all the eigenvalues of a and b are distinct. The integrability follows from the Lax representation with a rational spectral parameter found by Manakov in [17], or from a hyperelliptic Lax pair indicated in [8]. These Lax pairs provide a complete set of integrals of motion, whose involutivity can be proved by applying r-matrix theory.

For the concreteness, in the sequel we consider the case $a = \operatorname{diag}(a_1, \dots, a_m)$, $b = a^2$. Then $\Omega = AM + MA$, and equations (2.1) take the form

$$\dot{M} = [M, aM + Ma] \tag{2.3}$$

Apart from this "basic" system, there exists a whole hierarchy of "higher Manakov systems", which are defined by different relations between Ω and M, and which commute with (2.3).

Below we show that the restrictions of the Frahm–Manakov system on rank k orbits of coadjoint representation of $SO(m)$ in $so^*(m)$ are closely related to certain

Hamiltonian dynamical systems on extended Stiefel varieties. Recall that the standard Stiefel variety $\mathcal{V}(k,m)$ is the variety of ordered sets of k orthogonal vectors in $\mathbb{R}^m$ ($\mathbb{C}^m$) having fixed squares. It is a smooth variety of dimension $km-k(k+1)/2$ (see e.g., [7]).

Namely, as follows from (2.1), the angular momentum in space is a constant matrix. Hence, due to the Darboux theorem, in the case rank $M=k$ there exist k mutually orthogonal and *fixed in space* vectors $x^{(l)}, y^{(l)} \in \mathbb{R}^m$, $l=1,\dots,k/2$ such that $|x^{(l)}|^2 = |y^{(l)}|^2 = h_l$ and the momentum M can be represented in form

$$M = \sum_{l=1}^{k/2} x^{(l)} \wedge y^{(l)}, \quad \text{that is,} \quad M = \mathcal{X}^T\mathcal{Y} - \mathcal{Y}^T\mathcal{X}, \tag{2.4}$$
$$\mathcal{X}^T = (x^{(1)} \cdots x^{(k/2)}), \quad \mathcal{Y}^T = (y^{(1)} \cdots y^{(k/2)}).$$

Under the above conditions, the set of $k\times m$ matrices $\mathcal{Z} = (x^{(1)}\, y^{(1)} \cdots x^{(k/2)}\, y^{(k/2)})^T$ forms the *extended* Stiefel variety $\bar{\mathcal{V}}(k,m)$. In contrast to the standard Stiefel variety, for each index l, the absolute values $|x^{(l)}| = |y^{(l)}|$ are not fixed. Thus, $\bar{\mathcal{V}}(k,m)$ is of dimension $km - k^2/2$, and the $k\times m$ components of $\mathcal{Z}$ play the role of excessive coordinates on it.

Let
$$\omega = \operatorname{tr}(d\mathcal{X} \wedge d\mathcal{Y}^T) = \sum_{l=1}^{k}\sum_{i=1}^{m} dx_i^{(l)} \wedge d\,y_i^{(l)}$$

be the canonical symplectic structure on the space $\mathbb{R}^{km} = (x^{(1)}, y^{(1)}, \cdots, x^{(k/2)}, y^{(k/2)})$ and let $\bar\omega$ denote the restriction of 2-form ω onto $\bar{\mathcal{V}}(k,m) \subset \mathbb{R}^{km}$. The latter subvariety is defined by conditions

$$(x^{(l)}, x^{(l)}) - (y^{(l)}, y^{(l)}) = 0, \quad (x^{(l)}, x^{(s)}) = (y^{(l)}, y^{(s)}) = 0, \quad \mathcal{Y}\mathcal{X}^T = 0, \tag{2.5}$$
$$l, s = 1, \dots, k/2,$$

which consist of $k^2/2$ independent scalar equations $f_s(x,y) = 0$. The matrix of standard Poisson brackets of the constraint functions f_s in $\mathbb{R}^{km}$ is nondegenerate. It follows that 2-form $\bar\omega$ is also nondegenerate and the extended Stiefel variety is symplectic.

Since the vectors are fixed in space, in the frame attached to the top they satisfy the Poisson equations $\dot x^{(l)} = -\Omega x^{(l)}$, $\dot y^{(l)} = -\Omega y^{(l)}$, $\Omega \in so(m)$, which imply

$$\dot{\mathcal{X}} = \mathcal{X}\Omega, \quad \dot{\mathcal{Y}} = \mathcal{Y}\Omega. \tag{2.6}$$

As above, we put $\Omega = aM + Ma$, $a = \operatorname{diag}(a_1, \dots, a_m)$ and define a dynamical system on $\bar{\mathcal{V}}(k,m)$, which is generated by (2.6), (2.4):

$$\begin{aligned} \dot{\mathcal{X}} &= \mathcal{X}[a(\mathcal{X}^T\mathcal{Y} - \mathcal{Y}^T\mathcal{X}) + \mathcal{X}^T\mathcal{Y}a], \\ \dot{\mathcal{Y}} &= \mathcal{Y}[a(\mathcal{X}^T\mathcal{Y} - \mathcal{Y}^T\mathcal{X}) - \mathcal{Y}^T\mathcal{X}a]. \end{aligned} \tag{2.7}$$

Theorem 2.1. 1) *Under the substitution* (2.4) *solutions of the system* (2.7) *give rank k solutions of the Frahm–Manakov system (2.3) on $so^*(m)$.*

2) *Up to the action of the discrete group generated by reflections* $(\mathcal{X},\mathcal{Y})\to(-\mathcal{X},-\mathcal{Y})$, *the system* (2.7) *is described by* $k\times k$ *Lax pair with rational parameter* λ

$$\dot L(\lambda)=[\,L(\lambda),A(\lambda)\,],\qquad L,A\in \mathrm{sp}(k/2),\quad \lambda\in\mathbb{C}\,, \tag{2.8}$$

$$L=\begin{pmatrix}-\mathcal{X}(\lambda\mathbf{I}-a)^{-1}\mathcal{Y}^T & -\mathcal{X}(\lambda\mathbf{I}-a)^{-1}\mathcal{X}^T\\ \mathcal{Y}(\lambda\mathbf{I}-a)^{-1}\mathcal{Y}^T & \mathcal{Y}(\lambda\mathbf{I}-a)^{-1}\mathcal{X}^T\end{pmatrix}\equiv\sum_{i=1}^m\frac{\mathcal{N}_i}{\lambda-a_i}\,, \tag{2.9}$$

$$\mathcal{N}_i=\begin{pmatrix}\bar x_i\bar y_i^T & -\bar x_i\bar x_i^T\\ -\bar y_i\bar y_i^T & -\bar y_i\bar x_i^T\end{pmatrix},$$

$$A=\begin{pmatrix}\mathcal{X}(a+\lambda\mathbf{I})\mathcal{Y}^T & \mathcal{X}(a+\lambda\mathbf{I})\mathcal{X}^T\\ -\mathcal{Y}(a+\lambda\mathbf{I})\mathcal{Y}^T & -\mathcal{Y}(a+\lambda\mathbf{I})\mathcal{X}^T\end{pmatrix}, \tag{2.10}$$

where $\bar x_i=\left(x_i^{(1)},\dots,x_i^{(k/2)}\right)^T$ *(respectively* $\bar y_i=\left(x_i^{(1)},\dots,x_i^{(k/2)}\right)^T$*) is i-th column of* $\mathcal{X}$ *(respectively of* $\mathcal{Y}$*), and* $\mathbf{I}$ *is the unit* $m\times m$ *matrix.*

Proof. The first statement follows directly from the derivation of the system (2.7). Further, we calculate the derivative $\dot L(\lambda)$ by virtue of equations (2.7). In view of matrix relations $(\lambda\mathbf{I}-a)^{-1}a=\lambda(\lambda\mathbf{I}-a)^{-1}-\mathbf{I}$ and $\mathcal{Y}\mathcal{X}^T=\mathcal{X}\mathcal{Y}^T=0$, the derivative coincides with the commutator in (2.8). ■

Remark 2.1. Notice that the entries of matrices

$$\Phi(\lambda)L(\lambda),\quad \Phi(\lambda)A(\lambda),\quad \text{where}\quad \Phi(\lambda)=(\lambda-a_1)\cdots(\lambda-a_n)$$

are polynomials in λ, and, under the substitution (2.4), the coefficients of the characteristic polynomial $|\Phi(\lambda)L(\lambda)-w\mathbf{I}|$ can be expressed in terms of M_{ij} only as follows

$$|w\mathbf{I}-L(\lambda)|=w^k+\sum_{l=2}^k w^{k-l}\Psi^{l-1}(\lambda)\,\widetilde{\mathcal{I}}_l(\lambda,M)\,,\qquad l=2,4,\dots,k,$$

$$\widetilde{\mathcal{I}}_l(\lambda,M)=\sum_I^m\frac{\Phi(\lambda)}{(\lambda-a_{i_1})\cdots(\lambda-a_{i_l})}\,|M|_I^I\,, \tag{2.11}$$

where $|M|_I^I$ are diagonal minors of order l corresponding to multi-indices $I=\{i_1,\dots i_k\}\subset\{1,\dots,m\}$, $1\le i_1<\cdots<i_l\le m$. Notice that the leading coefficients $H_{l,m-l}(M)=\sum_I^m|M|_I^I$ form a complete set of Casimir functions on $so^*(m)$.

The $k\times k$ matrix $L(\lambda)$ in (2.9) belongs to a wide class of Lax operators of the form

$$Y+\sum_{i=1}^n\frac{G_i^TF_i}{\lambda-a_i},$$

where $Y\in gl(k)$ is a constant matrix and G_i,F_i are $k_i\times k$ matrices. Such Lax matrices can be regarded as images of moment maps to the loop algebra $\widetilde{gl}(k)$, and integrable systems generated by them have been studied in the series of papers

[1, 1, 2, 3] in connection with the duality to so called rank k perturbations of constant diagonal matrices of dimension $n \times n$ (following Moser [20]). In particular, the $k \times k$ Lax matrix (2.9) is dual to the $n \times n$ Lax matrix in the Manakov representation,

$$\mathcal{L}(\mu) = a + \frac{1}{\mu}(\mathcal{X}^T \mathcal{Y} - \mathcal{Y}^T \mathcal{X}) \equiv a + \frac{1}{\mu} M, \tag{2.12}$$

in the sense that under the relation (2.4) the spectral curves $|L(\lambda) - \mu \mathbf{I}| = 0$ and $|\mathcal{L}(\mu) - w\mathbf{I}| = 0$ are birationally equivalent and the parameter λ plays the role of the eigenvalue parameter for (2.12). The characteristic polynomials of the dual Lax matrices are related by the Weinstein–Aronjan formula (see [1]).

Remark 2.2. The matrix $A(\lambda)$ in (2.10) can be represented in form

$$A(\lambda) = [\, \lambda^{-m+2} \Phi(\lambda) L(\lambda)\,]_+ + (a_1 + \cdots + a_n) L_0, \qquad L_0 = \begin{pmatrix} 0 & \mathcal{X}\mathcal{X}^T \\ -\mathcal{Y}\mathcal{Y}^T & 0 \end{pmatrix}$$

where $[\ ]_+$ denotes the polynomial part in λ of the expression. Notice that the Lax equation $\dot{L} = [L, L_0]$ describes the vector flow

$$\dot{x^{(l)}} = (x^{(l)}, x^{(l)}) y^{(l)}, \quad \dot{y^{(l)}} = -(y^{(l)}, y^{(l)}) x^{(l)}, \qquad l = 1, \ldots, k/2. \tag{2.13}$$

For each index $\mathcal{R}_l$, equations (2.13) generate rotations $\mathcal{R}_l$ in 2-planes spanned by the vectors $x^{(l)}, y^{(l)}$, which leave the momentum M invariant.

Let $\overline{\{\cdot,\cdot\}}$ be the Poisson bracket on $\bar{\mathcal{V}}(k,m)$ obtained as the Dirac restriction of the standard bracket in $\mathbb{R}^{km}$. Symplectic properties of our system are descibed by

Proposition 2.2. *The dynamical system* (2.7) *on* $\bar{\mathcal{V}}(k,m)$ *is Hamiltonian with respect to* $\overline{\{\cdot,\cdot\}}$ *with the Hamilton function* $\bar{H}(\mathcal{X},\mathcal{Y}) = -\frac{1}{4}\mathrm{tr}(M^2(\mathcal{X},\mathcal{Y})A)$. *In the abundant coordinates* $\mathcal{X},\mathcal{Y}$ *it admits the canonical representation*

$$\dot{x}_i^{(l)} = \left.\frac{\partial \bar{H}}{\partial y_i^{(l)}}\right|_{\bar{\mathcal{V}}(k,m)}, \quad \dot{y}_i^{(l)} = -\left.\frac{\partial \bar{H}}{\partial x_i^{(l)}}\right|_{\bar{\mathcal{V}}(k,m)}, \tag{2.14}$$
$$i = 1, \ldots, n, \quad l = 1, \ldots, k/2.$$

Proof. The equivalence of equations (2.14) and (2.7) on $\bar{\mathcal{V}}(k,m)$ is verified by direct calculations. Next, according to the Dirac formalizm, the standard bracket and $\overline{\{\cdot,\cdot\}}$ are different by terms containing $\{f_s, \bar{H}\}$. The latter equal zero since the constraint functions f_s given by (2.5) are invariants of the flow generated by $\bar{H}(\mathcal{X},\mathcal{Y})$ on $\mathbb{R}^{km}$. Hence, equations (2.7) or (2.14) are Hamiltonian with respect to $\overline{\{\cdot,\cdot\}}$. ■

Rotations $\mathcal{R}_l$ given by (2.13) are generated by the Hamiltonians $H_l(x,y)$, the restrictions of the functions $\frac{1}{2}(x^{(l)}, x^{(l)})(y^{(l)}, y^{(l)})$ on $\bar{\mathcal{V}}(k,m)$. Clearly, these functions are first integrals of the system (2.7) and moreover they commute with H.

Let us fix the values of the Hamiltonians by putting

$$(x^{(l)}, x^{(l)}) = (y^{(l)}, y^{(l)}) = h_l, \quad h_l = \mathrm{const} \neq 0, \quad l = 1, \ldots, k/2.$$

These conditions define the customary Stiefel variety $\mathcal{V}(k,m)$. Under the substitution (2.4), the factor variety $\mathcal{V}(k,m)/\{\mathcal{R}_1,\dots,\mathcal{R}_{k/2}\}$ coincides with a rank k coadjoint orbit $\mathcal{S}_h^{(k)} \subset so^*(m)$ of dimension $k\left(m-\frac{k}{2}\right)-k$, which is parameterized by the constants $h_1,\dots,h_{k/2}$. Notice that $M^2 = h_1^2+\dots+h_{k/2}^2$.

Theorem 2.3. 1) *Under the map $\bar{\mathcal{V}}(k,m) \to \mathcal{S}_h^{(k)}$, the Lie–Poisson bracket on $\mathcal{S}_h^{(k)} \subset so^*(m)$ is the push-forward of the bracket $\overline{\{\cdot,\cdot\}}$.*

2) *The Poisson (Marsden–Weinstein) reduction of the system (2.7) obtained by fixing values of $H_l(x,y)$ and by factorization by $\mathcal{R}_l$, $l=1,\dots,k/2$ coincides with the restriction of the Frahm–Manakov system with Hamiltonian $H(M)=\frac{1}{2}\sum_{i\le j}(a_i+a_j)M_{ij}^2$ onto the orbit $\mathcal{S}_h^{(k)}$.*

Proof. 1). In view of (2.2), (2.4),

$$\{M_{ij}(\mathcal{X},\mathcal{Y}), M_{kl}(\mathcal{X},\mathcal{Y})\} = \{M_{ij}, M_{kl}\}_{so(n)}(\mathcal{X},\mathcal{Y}),$$

i.e., the canonical bracket on $\mathbb{R}^{km}$ is the pull-back of the bracket $\{\cdot,\cdot\}_{so(n)}$ on $\mathcal{S}_h^{(k)} \subset so^*(m)$. On the other hand, on $\bar{\mathcal{V}}(k,m)$,

$$\{M_{ij}(\mathcal{X},\mathcal{Y}), M_{kl}(\mathcal{X},\mathcal{Y})\} = \overline{\{M_{ij}(\mathcal{X},\mathcal{Y}), M_{kl}(\mathcal{X},\mathcal{Y})\}},$$

since for any i,j,s, $\{M_{ij}(\mathcal{X},\mathcal{Y}), f_s(\mathcal{X},\mathcal{Y})\}=0$. This proves item 1).

2). By item 1) and Proposition 2.2, the Poisson reduction of system (2.7) onto $\mathcal{S}_h^{(k)}$ is described by the Lie–Poisson bracket $\{\cdot,\cdot\}_{so(n)}$ and the Hamiltonian $H(M) = \bar{H}(\mathcal{X},\mathcal{Y}) = \sum_{i\le j}(a_i+a_j)M_{ij}^2$, i.e., it is the corresponding restriction of the Frahm–Manakov system. ■

The reduced system on the orbit $\mathcal{S}_h^{(k)}$ is integrable and its generic invariant manifolds are tori of dimension $\frac{1}{2}\dim\mathcal{S}_h^{(k)}$ (see, e.g., [19]). On the other hand, the preimage of a generic point $M\in\mathcal{S}_h^{(k)}$ in $\mathcal{V}(k,m)$ is a $k/2$-fold product of circles $S^1\times\dots\times S^1$ (in the complex case $\mathbb{C}^*\times\dots\times\mathbb{C}^*$). This implies that the original system on $\bar{\mathcal{V}}(k,m)$ has generic invariant tori of dimension $\frac{1}{2}\dim\mathcal{S}_h^{(k)}+k/2=(m-k/2)k/2$, i.e., a half of dimension of the symplectic manifold $\bar{\mathcal{V}}(k,m)$. Hence, the original system (2.7) is also integrable.

To get a global view on the above manifolds, we represent them in the following diagram, with the dimension indicated above, where arrows denote the corresponding relations (restrictions or factorizations).

$$\begin{array}{ccccccc}
\mathbb{R}^{km} & \xrightarrow{f_s=0} & \bar{\mathcal{V}}(k,m) & \xrightarrow{|x^{(l)}|^2=|y^{(l)}|^2=h_l} & \mathcal{V}(k,m) & \xrightarrow{\mathcal{R}} & \mathcal{S}_h^{(k)} \\
\boxed{k\times m} & & \boxed{k\left(m-\frac{k}{2}\right)} & & \boxed{k\left(m-\frac{k}{2}\right)-\frac{k}{2}} & & \boxed{k\left(m-\frac{k}{2}\right)-k}
\end{array}$$

Remark 2.3. In the case of maximal rank k ($k = m$ or $k = m-1$), when $\mathcal{S}_h^{(k)}$ is a generic coadjoint orbit $\mathcal{S}_h$, the Stiefel variety $\mathcal{V}(k,m)$ is isomorphic to the group $SO(m)$. Then the following commutative diagram holds

$$\begin{array}{ccc} \bar{\mathcal{V}}(k,m) & \xrightarrow{|x^{(l)}|^2=|y^{(l)}|^2=h_l} & SO(m) \\ \mathcal{R}\big\downarrow & & \mathcal{R}\big\downarrow \\ so^*(m) & \xrightarrow{H_{l,m-l}(M)=c_l} & \mathcal{S}_h\,, \end{array}$$

where the values $\{c_l\}$ of nonzero Casimir functions $H_{l,m-l}(M)$ correspond to the constants $\{h_l\}$. The mapping $SO(m) \xrightarrow{\mathcal{R}} \mathcal{S}_h$ can be regarded as a multi-dimensional analog of the Hopf fibration $SO(3) \xrightarrow{S^1} S^2$. The restriction of the system (2.7) onto $\mathcal{V}(k,m)$ yields an integrable flow on the group $SO(m)$ which describes the motion of the Frahm–Manakov top in space for the chosen angular momentum.

For $m = 3$ such a flow was considered in [15, 16] from the point of view of its hydrodynamical interpretation.

A generalization of the Chasles theorem. If the rank k is not maximal, then the components of $\mathcal{X}, \mathcal{Y}$ themselves are not sufficient to form a complete set of coordinates on $SO(m)$ and to determine the position of the top in space uniquely. However, in this case one can make use of the following geometric property described in [8]. Let us fix a part of constants of motion by putting in (2.11)

$$\widetilde{\mathcal{I}}_k(s,M) = c_0(s-c_1)\cdots(s-c_{m-k}), \qquad c_0, c_1, \ldots, c_{m-k} = \text{const} \tag{2.15}$$

and consider family of confocal cones in $\mathbb{R}^m = (X_1, \ldots, X_n)$

$$\bar{Q}(s) = \left\{ \frac{X_1^2}{s-a_1} + \cdots + \frac{X_n^2}{s-a_n} = 0 \right\}. \tag{2.16}$$

Let $\bar{\Lambda} \subset \mathbb{R}^m$ be a k-plane spanned by the orthogonal vectors $x^{(1)}, y^{(1)}, \ldots, x^{(k/2)}, y^{(k/2)}$.

Proposition 2.4. ([8]).

1). *Under the motion of the Frahm–Manakov top with constants (2.15) the k-plane $\bar{\Lambda}$ is tangent to the fixed cones $\bar{Q}(c_1), \ldots, \bar{Q}(c_{m-k})$.*

2). *Let $\phi^{(\alpha)}$ be a normal vector of the cone $\bar{Q}(c_\alpha)$ at a point of the contact line $\bar{\Lambda} \cap \bar{Q}(c_\alpha)$. Then the vectors $\phi^{(1)}, \ldots, \phi^{(m-k)}$ together with $x^{(1)}, y^{(1)}, \ldots, x^{(k/2)}, y^{(k/2)}$ form an orthogonal frame in $\mathbb{R}^m$ which is fixed in space.*

For fixed polynomial $\widetilde{\mathcal{I}}_k(s,M)$, the vectors $\phi^{(l)}$ can be calculated in terms of $x^{(s)}, y^{(s)}$ and, thereby, the position of the top in space is completely determined. Proposition 2.4 defines a single-valued map $\bar{\mathcal{V}}(k,m) \to SO(m)$ under which generic invariant tori of dimension $(m-k/2)k/2$ on $\bar{\mathcal{V}}(k,m)$ become tori of the same dimension on the group $SO(m)$.

Note that the above proposition generalizes the celebrated Chasles theorem on the propery of the tangent line to a geodesic on a quadric.

The rank 2 case. In the simplest case $k = 2$ the angular momentum can be represented in form

$$M = x \wedge y, \quad x = x^{(1)} = (x_1, \dots, x_m)^T, \quad y = y^{(1)} = (y_1, \dots, y_m)^T \tag{2.17}$$

and equations (2.7) describe a Hamiltonian system on the extended Stiefel variety $\bar{\mathcal{V}}(2,m) = \left\{ (x,y) \middle| \, |x| = |y|, \ (x,y) = 0 \right\}$,

$$\begin{aligned} \dot{x} &= -(y, ax)x + (x, ax)y + ay(x,x), \\ \dot{y} &= -(y, ay)x + (x, ay)y - ax(y,y) \end{aligned} \tag{2.18}$$

with the Hamiltonian

$$\bar{H} = \frac{1}{2}(x, ax)(y,y) - (ax, y)(x,y) + \frac{1}{2}(y, ay)(x,x) = \frac{1}{2}\sum_{i<j}^{m}(a_i + a_j)M_{ij}^2.$$

Equivalently, this system describes the evolution of fixed orthogonal vectors x, y in a frame attached to the m-dimensional body. The system admits the following 2×2 Lax pair arising from (2.8),

$$\begin{aligned} &\dot{L}(\lambda) = [\, L(\lambda), \mathcal{A}(\lambda)\,], \\ &L(\lambda) = \Phi(\lambda) \begin{pmatrix} -\sum_{i=1}^m \frac{x_i y_i}{\lambda - a_i} & -\sum_{i=1}^m \frac{x_i^2}{\lambda - a_i} \\ \sum_{i=1}^m \frac{y_i^2}{\lambda - a_i} & \sum_{i=1}^m \frac{x_i y_i}{\lambda - a_i} \end{pmatrix}, \\ &\mathcal{A}(\lambda) = \begin{pmatrix} -\sum_{i=1}^m (\lambda + a_i) x_i y_i & -\sum_{i=1}^m (\lambda + a_i) x_i^2 \\ \sum_{i=1}^m (\lambda + a_i) y_i^2 & \sum_{i=1}^m (\lambda + a_i) x_i y_i \end{pmatrix}, \\ &\Phi(\lambda) = (\lambda - a_1) \cdots (\lambda - a_m), \end{aligned} \tag{2.19}$$

The Lax representation (2.19) was first indicated in [3], where it was shown to be dual to an $n \times n$ Lax pair for the rank 2 case found by Moser in [20].

In view of relation (2.17), the characteristic polynomial $|L(\lambda) - \mu \mathbf{I}|$ for (2.19) can be written in form $\Phi(\lambda)\widetilde{\mathcal{I}}_2(\lambda, M) + \mu^2$, where $\widetilde{\mathcal{I}}_2(\lambda, M)$ is the family of quadratic integrals defined in (2.11),

$$\begin{aligned} \widetilde{\mathcal{I}}_2(\lambda, M) &= \sum_{i<j} \frac{\Phi(\lambda)}{(\lambda - a_i)(\lambda - a_j)} M_{ij}^2 \\ &= \lambda^{m-2} H_{2,m-2}(M) + \lambda^{m-3} H_{2,m-3}(M) + \cdots + H_{20}(M). \end{aligned} \tag{2.20}$$

Notice that $H_{2,m-2}(M) = \sum_{i<j}^m M_{ij}^2 = (y,y)(x,x)$ is a Casimir function of the standard Lie–Poisson bracket on $so^*(m)$. With respect to the Poisson bracket $\overline{\{\,,\,\}}$ on $\bar{\mathcal{V}}(2,m)$, this function generates permanent rotations of the top in the fixed 2-plane $\bar{\Lambda} = \operatorname{span}(y, x)$, which leave the components of M invariant.

Let us fix the constants of motion by putting

$$\widetilde{\mathcal{I}}_2(\lambda, M) = c_0(\lambda - c_1) \cdots (\lambda - c_{m-2}), \qquad c_0, c_1, \dots, c_{m-2} = \text{const}. \tag{2.21}$$

This defines hyperelliptic spectral curve in $\mathbb{C}^2 = (\lambda, \mu)$ of genus $g = m - 2$

$$\mathcal{C} = \{\mu^2 = -c_0\, \Phi(\lambda)\, (\lambda - c_1) \cdots (\lambda - c_{m-2})\}\,. \tag{2.22}$$

As noticed in [9], the real generic $(g+1)$-dimensional invariant tori of the system can be extended to open subsets of generalized Jacobian varieties $\mathrm{Jac}(\mathcal{C}, \infty_\pm)$, which are extensions of the customary g-dimensional Jacobian $\mathrm{Jac}(\mathcal{C})$ by $\mathbb{C}^*$ and which can be regarded as the factor of $\mathbb{C}^2$ by the lattice generated by $(2g+1)$ independent period vectors of g holomorphic differentials $\bar\omega_1 \ldots, \bar\omega_g$ and a meromorphic differential of the third kind $\Omega_{\infty_\pm}$ having a pair of simple poles at the infinite points $\infty_\pm$ on the curve $\mathcal{C}$.

The coefficients of the matrix polynomial $L(\lambda)$ are meromorphic functions on $\mathrm{Jac}(\mathcal{C}, \infty_\pm)$, whereas the components of the momentum M_{ij} and the normal vectors $\phi^{(\alpha)}$ are meromorphic on a covering of the Jacobian $\mathrm{Jac}(\mathcal{C})$ itself (the $\mathbb{C}^*$-extension is factored out by the action of $\mathcal{R} = SO(2)$).

In the classical case $m = 3$ the curves $\mathcal{C}$ become elliptic ones and generic invariant tori in $\bar{\mathcal{V}}(2,3)$ and in $SO(3)$ are 2-dimensional. Since now rank M =2 in the generic case, the above commutative diagram takes the form

$$\begin{array}{ccc} \bar{\mathcal{V}}(2,3) & \xrightarrow{|x|^2=|y|^2=h} & SO(3) \\ {\scriptstyle SO(2)}\big\downarrow & & {\scriptstyle SO(2)}\big\downarrow \\ so^*(3) & \xrightarrow{(M,M)=h^2} & {S^2}_h \end{array}$$

${S^2}_h$ being the coadjoint orbit (2-dimensional sphere) corresponding to the constant h.

3 Bäcklund transformation on $\bar{\mathcal{V}}(2,3)$, $SO(3)$ and discretization of the clasical Euler top

A first integrable discretization of the m-dimensional Euler–Manakov top was constructed in [24, 21] by the method of factorization of matrix polynomials. It was represented by the correspondence $(\Omega, M) \to (\widetilde\Omega, \widetilde M)$, $\Omega \in SO(m)\,, M \in so^*(m)$, which, in our notation reads

$$M = \Omega^T A - A\Omega, \quad \widetilde M = \Omega M \Omega^T. \tag{3.1}$$

Given $\widetilde M$, the new matrix $\widetilde\Omega$ is found from equation $\widetilde M = \widetilde\Omega^T A - A\widetilde\Omega$, whose solution is not unique.

In given section we describe a symplectic map $\bar{\mathcal{B}}_{\lambda *}\ : \bar{\mathcal{V}}(2,3) \to \bar{\mathcal{V}}(2,3)$, $\bar{\mathcal{B}}_{\lambda *}(x,y) = (\widetilde x, \widetilde y)$ governed by an arbitrary parameter $\lambda^* \in \mathbb{C}$, which preserves the first integrals of the continuous system (2.18) and whose restriction to each generic complex torus, generalized Jacobian $\mathrm{Jac}(\mathcal{C}, \infty_\pm)$, is given by shift by the 2-dimensional vector

$$S = \int_{E_-}^{E_+} (\bar\omega, \Omega_{\infty_\pm})^T, \qquad E_\pm = (\lambda^*, \pm\mu^*),$$

$E_\pm$ being involutive points on the elliptic spectral curve, the simplest case of (2.22),

$$\mathcal{C} = \{\mu^2 = -c_0\,(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)\,(\lambda - c_1)\}.$$

Note that S is a correctly defined vector in the generalized Jacobian: under a change of integration path on $\mathcal{C}$ it increases by a period vector of $\mathrm{Jac}(\mathcal{C}, \infty_\pm)$. We also emphasize that here λ^* is a constant parameter, whereas the conjugated coordinate μ^* depends on the equation of the curve.

Bäcklund transformation on $\bar{\mathcal{V}}(2,3)$. As shown in [10] by applying an addition theorem for a class of meromorphic functions on generalized hyperelliptic Jacobians, such a map admits intertwining relation (discrete Lax pair)

$$\widetilde{L}(\lambda)M(\lambda|\lambda^*) = M(\lambda|\lambda^*)L(\lambda), \tag{3.2}$$

$$M(\lambda|\lambda^*) = \begin{pmatrix} -\alpha(\lambda-\lambda^*)+\beta & 1 \\ -\beta^2 & -\alpha(\lambda-\lambda^*)-\beta \end{pmatrix},$$

$$\begin{aligned} \beta(\lambda^*) &= -\frac{\mu^* + L_{11}(\lambda^*)}{L_{12}(\lambda^*)} \equiv -\frac{\mu^* + a_1^* a_2^* a_3^* (x, (a-\lambda^*\mathbf{I})^{-1} y)}{a_1^* a_2^* a_3^* (x, (a-\lambda^*\mathbf{I})^{-1} x)}, \\ \alpha &= \left.\frac{\partial \beta(\lambda)}{\partial \lambda}\right|_{\lambda=\lambda^*}, \quad a_i^* = a_i - \lambda^*, \end{aligned} \tag{3.3}$$

where $L(\lambda)$ is defined in (2.19) and $\widetilde{L}(\lambda)$ depends on the new variables $\widetilde{x}, \widetilde{y}$ in the same way as $L(\lambda)$ depends on x, y. In view of (2.20) for $m = 3$,

$$\mu^* = \sqrt{\Phi(\lambda^*) \sum_{k=1}^{3} a_k^* (x_i y_j - x_j y_i)^2}\,, \qquad (i,j,k) = (1,2,3). \tag{3.4}$$

Now putting in (3.2) subsequently $\lambda = a_1, a_3, a_3$ and calculating the matrices $M(a_i)L(a_i|x,y)M^{-1}(a_i)$, we find

$$\begin{aligned} \widetilde{x}_i^2 &= \frac{(y_i + \beta x_i + \alpha(a_i - \lambda^*)x_i)^2}{\alpha^2 (a_i - \lambda^*)^2}, \\ \widetilde{y}_i^2 &= \frac{(\beta y_i + \beta^2 x_i - \alpha(a_i - \lambda^*) y_i)^2}{\alpha^2 (a_i - \lambda^*)^2}, \\ \widetilde{x}_i \widetilde{y}_i &= -\frac{(y_i + \beta x_i + \alpha(a_i - \lambda^*)x_i)\,(\beta y_i + \beta^2 x_i - \alpha(a_i - \lambda^*) y_i)}{\alpha^2 (a_i - \lambda^*)^2}. \end{aligned} \tag{3.5}$$

¿From here the new variables can be recovered up to the action of the group generated by reflections $(\widetilde{x}_i, \widetilde{y}_i) \to (-\widetilde{x}_i, -\widetilde{y}_i)$. Imposing the condition of the existence of a continuous limit (see below), we choose the following relations

$$\widetilde{x}_i - x_i = \frac{y_i + \beta x_i}{\alpha(a_i - \lambda^*)}, \quad \widetilde{y}_i - y_i = -\frac{\beta(y_i + \beta x_i)}{\alpha(a_i - \lambda^*)}, \qquad i = 1,2,3. \tag{3.6}$$

These expressions together with (3.3), (3.4) describe the map $\bar{\mathcal{B}}_{\lambda^*} : \bar{\mathcal{V}}(2,3) \to \bar{\mathcal{V}}(2,3)$ in an *explicit* form. Since a generic parameter λ^* corresponds to two values of μ^*, the map is generally two-valued.

Geometric model. The restriction of the map onto the group $SO(3)$ admits a transparent geometric interpretation, which can be regarded as a "discrete version" of the kinematic Poinsot model (see, e.g., [25]). Namely, let $|x| = |y| = 1$ and let

$$R = ||x\ y\ \ x \wedge y|| \in SO(3)$$

be rotation matrix defining a position of a rigid body in space. We attach to the body a cone $K_2 = \{(X, (a - \lambda^* \mathbf{I})^{-1} X) = 0\}$, which is fixed in the body frame (X_1, X_2, X_3), and assume that

$$0 < a_1 < a_2 < a_3, \quad a_1 < \lambda^* < a_2 \quad \text{or} \quad a_2 < \lambda^* < a_3. \tag{3.7}$$

Under these conditions the cone is real and regular. Let Π be 2-plane spanned by x, y, which is thus fixed in space and orthogonal to the momentum vector $M = x \wedge y$. Assume also that x, y are such that Π has a nonempty real intersection with the cone K_2 along lines L_1, L_2. One can show that under this condition the coordinates μ^* defined in (3.4) and the parameters α, β are real.

Theorem 3.1. *Let* $K_1 = \{(X, (a - h\mathbf{I})^{-1} X) = 0\}$, $h = \text{const}$ *be a unique cone attached to the body such that it is confocal to* K_2 *and tangent to the fixed plane* Π. *Then then new position of the body defined by the rotation matrix* $\widetilde{R} = ||\widetilde{x}\ \widetilde{y}\ \ \widetilde{x} \wedge \widetilde{y}||$ *and expressions (3.6) is obtained from the original position by rotating the cones* K_1, K_2 *about axis* L_1 *or* L_2 *until* K_1 *again touches* Π.

This geometric construction is illustrated on Figure 1. In the new position $\widetilde{R}$ determined by rotation about L_2, the cone K_2 intersects Π along L_2 and another line L_3. Then the next iteration is generated by rotation about L_2 or L_3. The two-valuedness of the map $\bar{\mathcal{B}}_{\lambda *}$ is now related to the possibility of rotation about two different axes in $\mathbb{R}^3$. It follows that N-th iteration of the map is only $(N+1)$-valued, not 2^N-valued. By fixing a sign of μ^* in (3.4), $\bar{\mathcal{B}}_{\lambda *}$ becomes single-valued and generates a sequence of points on $SO(3)$.

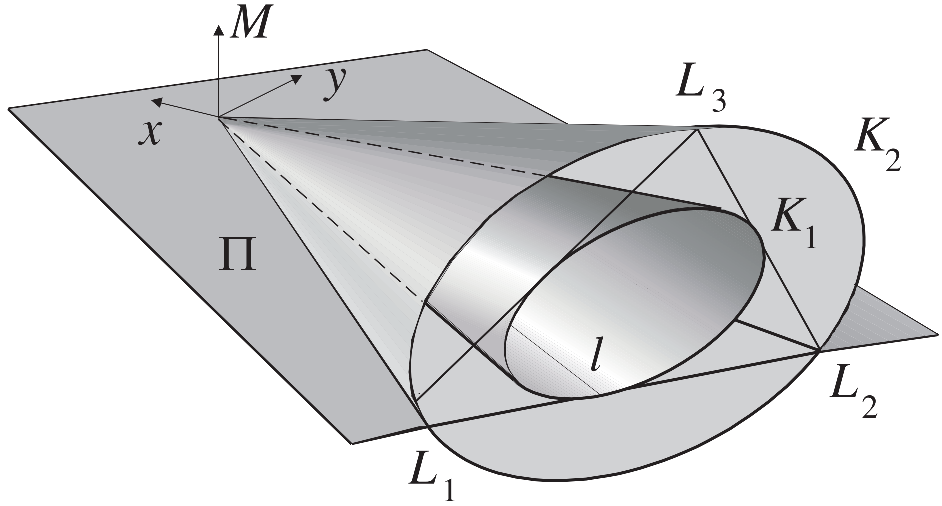

Figure 1

This geometric model was first proposed in [10] as a certain limit of a kinematical model of motion of 4-dimensional Frahm–Manakov top in space.

Remark 3.1. As follows from (3.6), the map $\bar{\mathcal{B}}_{\lambda*}$ admits particular solutions, for which the vector $M = x \wedge y$ remains to be an eigenvector of the inertia tensor $A = \operatorname{diag}(a_1, a_2, a_3)$, whereas x, y themselves rotate by a fixed angle in the plane Λ. Such solutions can be regarded as analogs of stationary permanent rotations of the classical Euler top about its principal intertia axes.

Continuous limit. Note that when x, y are chosen such that Π is (almost) tangent to the cone K_2 ($c \to \lambda^*$), K_1 and K_2 confluent and, according to the above model, the cone $K_1 = K_2$ is rolling without sliding over the fixed plane thus giving a continuous limit motion on the group $SO(3)$.

From the algebraic geometrical point of view, in the above limit the points E_-, E_+ on the spectral curve $\mathcal{C}$ come together to a branch point $E_0 = (c, 0)$ and the shift vector S on the generalized Jacobian tends to zero. Let ϵ be a small complex parameter. Setting $\mu^* = \epsilon$, $\lambda - \lambda^* = \mathrm{const} \cdot \epsilon^2$ in (3.3), we have the expansions

$$\beta = -\frac{(x, a^{-1}y)}{(x, a^{-1}x)} + O(\epsilon), \qquad \alpha = \frac{1}{\epsilon}\varkappa(x, a^{-1}x) + O(1), \qquad \varkappa = \left.\frac{\partial \mu^2(\lambda)}{\partial \lambda}\right|_{\lambda=\lambda^*},$$

where $\varkappa$ is a real nonzero constant. Now we set

$$\widetilde{x} = x + \dot{x}\epsilon + O(\epsilon^2), \quad \widetilde{y} = y + \dot{y}\epsilon + O(\epsilon^2).$$

Substituting the above expansions into (3.6), then matching the coefficients at ϵ in both sides and taking into account relation

$$(x, (a^*)^{-1}y)^2 = (x, (a^*)^{-1}x)(y, (a^*)^{-1}y) + O(\epsilon^2),$$

we obtain the following differential equations describing the limit flow on a subset of $\bar{\mathcal{V}}(2,3)$

$$\begin{aligned} \dot{x} &= \frac{\det a^*}{\varkappa}\Big[(x, (a^*)^{-1}y)(a^*)^{-1}x - (x, (a^*)^{-1}x)(a^*)^{-1}y\Big] \equiv \frac{1}{\varkappa} x \wedge a^*(x \wedge y), \\ \dot{y} &= \frac{\det a^*}{\varkappa}\Big[(y, (a^*)^{-1}y)(a^*)^{-1}x - (x, (a^*)^{-1}y)(a^*)^{-1}x\Big] \equiv \frac{1}{\varkappa} y \wedge a^*(x \wedge y), \\ a^* &= a - \lambda^* \mathbf{I}. \end{aligned} \tag{3.8}$$

These equations are Hamiltonian with the Hamilton function

$$H = \frac{\det a^*}{2\varkappa}\Big[(x, (a^*)^{-1}x)(y, (a^*)^{-1}y) - (x, (a^*)^{-1}y)^2\Big] \equiv \frac{1}{2\varkappa}\sum_{i=1}^{3}(a_i - \lambda^*)M_i^2.$$

Notice that this function equals zero on the limit continuous flow. The restriction of this flow on $\operatorname{Jac}(\mathcal{C}, \infty_\pm)$ is tangent to the curve $\mathcal{C} \subset \operatorname{Jac}(\mathcal{C}, \infty_\pm)$ at the point E_0.

The above asymptotic of α, β explains the specific choice of sign of $\widetilde{x}, \widetilde{y}$ made in the passage from relations (3.5) to the map (3.6).

Proof. (theorem 3.1). The condition for Π to be tangent to the cone $K_1 = \{(X, (a - h\mathbf{I})^{-1}X) = 0\}$ has the form

$$\sum_{k=1}^{3}(h - a_k)(x_i y_j - x_j y_i)^2 \equiv \mathcal{I}_2(h, M) = 0.$$

Comparing this with the family of integral (2.21) for $m = 3$, we conclude that $h = c_1$, which is constant under the map, hence the plane spanned by $\widetilde{x}, \widetilde{y}$ is again tangent to K_1.

Next, any translation in $SO(3)$ is represented as a finite rotation about an axis in $\mathbb{R}^3$. As follows from relations (3.6), $\widetilde{y} + \beta\widetilde{x} = y + \beta x$, hence the line along the vector $\ell = y + \beta x$ is invariant of the action of $\bar{\mathcal{B}}_{\lambda*}$ on $\mathbb{R}^3$ and therefore represents the axis of such a rotation. Finally, in view of (3.3), we have

$$\begin{aligned}(\ell, (a^*)^{-1}\ell) &= (y, (a^*)^{-1}y) - 2\frac{\mu^* + L_{11}(\lambda^*)}{L_{12}(\lambda^*)}(x, (a^*)^{-1}y) \\ &\quad + \frac{(\mu^*)^2 + 2\mu^* L_{11}(\lambda^*) + L_{11}^2(\lambda^*)}{L_{12}^2(\lambda^*)}(x, (a^*)^{-1}x) \\ &= \frac{1}{L_{12}(\lambda^*)}\left[(y, (a^*)^{-1}y)(x, (a^*)^{-1}x) - (x, (a^*)^{-1}y)^2 + (\mu^*)^2\right],\end{aligned}$$

which equals zero by virtue of (3.4). Hence $(\ell, (a^*)^{-1}\ell) = 0$, which imply that the vector ℓ lies on the cone K_2. This establishes the theorem. ■

Remark 3.2. When the attached cone K_2 does not have real intersection with $\Pi = \operatorname{span}(x, y)$, the coordinate μ^* is imaginary and, according to (3.6), (3.3), the new values $\widetilde{x}, \widetilde{y}$ are complex. As a result, under the reality conditions (3.7) the map $\bar{\mathcal{B}}_{\lambda*}$ is real only on the subset $\mathfrak{R} \subset \bar{\mathcal{V}}(2, 3)$ defined by unequality

$$\sum_{k=1}^{3}(\lambda^* - a_k)(x_i y_j - x_j y_i)^2 \leq 0$$

On the boundary of $\mathfrak{R}$, the map tends to the identical one.

Reduction to the coalgebra $so^*(3)$**.** Under the factorization by rotations of $\mathcal{R} = SO(2)$, the transformation $\bar{\mathcal{B}}_{\lambda*}$ induces a map $\mathcal{B}_{\lambda*} : so(3)^* \to so(3)^*$ such that

$$\widetilde{M} \equiv \mathcal{B}_{\lambda*}M(x, y) = \widetilde{x} \wedge \widetilde{y}.$$

The latter map is correctly defined, i.e., it does not depend on a concrete choice of vectors x, y giving the same M. It preserves the first integrals of the classical Euler top on $so^*(3)$ and its generic invariant manifolds are open subsets of 4-fold unramified coverings of the complex torus $\operatorname{Jac}(\mathcal{C}) = \mathcal{C}$. The restriction of $\mathcal{B}_{\lambda*}$ onto $\operatorname{Jac}(\mathcal{C})$ is given by shift by the holomorphic integral $e = \int_{E_-}^{E_+} \bar{\omega}$, which thus depends only on the constants c_0, c_1. According to a theorem in [6], this implies that the map $\mathcal{B}_{\lambda*}$ preserves the standard Lie–Poisson structure on $so^*(3)$.

Proposition 3.2. *Vectors $M, \widetilde{M}$ satisfy the following symmetric relations*

$$\widetilde{M} - M = \varkappa (\widetilde{M} + M) \wedge a(\widetilde{M} + M), \tag{3.9}$$

$$\varkappa = \sqrt{2(aM + a\widetilde{M}, aM + a\widetilde{M})}\, \frac{\sqrt{1 - (M, \widetilde{M})/c_0}}{\sqrt{1 + (M, \widetilde{M})/c_0}}, \tag{3.10}$$

$$(M, a^* M) = (\widetilde{M}, a^* \widetilde{M}) = -(M, a^* \widetilde{M}), \tag{3.11}$$

where, as above, $c_0 = (M, M) = (\widetilde{M}, \widetilde{M})$.

Relation (3.9) was previously obtained by another method in [6], as an implicit map describing a Poisson discretization of the Euler top in $so^*(3)$.

Proof. (proposition 3.2). In view of relations (3.6), we find

$$\widetilde{x} \wedge \widetilde{y} = \widetilde{M} = x \wedge y + \frac{1}{\alpha} (a^*)^{-1} \ell \wedge \ell, \tag{3.12}$$

where, as above, $\ell = y + \beta x$, $a^* = a - \lambda^* \mathbf{I}$. Note that vector $(a^*)^{-1} \ell$ is normal to the cone K_2 at a point of the intersection line L_2 or L_1. Hence, $\widetilde{M} - M$ is orthogonal to $(a^*)^{-1} \ell$ and ℓ. Next, since ℓ lies in the planes $\Pi, \widetilde{\Pi}$, this vector is orthogonal to $M, \widetilde{M}$. This, together with the equality $|M| = |\widetilde{M}|$ implies that the sum $\widetilde{M} + M$ is parallel to $(a^*)^{-1} \ell$ and $a^*(\widetilde{M} + M)$ is parallel to ℓ. As a result, (3.12) implies (3.9).

To find factor $\varkappa$, we first introduce angle ϕ between vectors $\widetilde{M}$ and M. Since $|M| = |\widetilde{M}|$, the vectors $\widetilde{M} - M$ and $\widetilde{M} + M$ are orthogonal, and we have

$$|\widetilde{M} - M| = \frac{1}{2} |\widetilde{M} + M| \tan \frac{\phi}{2} \equiv \frac{1}{2} |\widetilde{M} + M| \frac{\sqrt{1 - (M, \widetilde{M})/c_0}}{\sqrt{1 + (M, \widetilde{M})/c_0}}.$$

On the other hand, since $a^*(\widetilde{M} + M)$ is orthogonal to $\widetilde{M}, M$, from (3.9) and the properties of the vector product we deduce

$$|\widetilde{M} - M| = \varkappa |\widetilde{M} + M|\, |a^*(\widetilde{M} + M)|.$$

Comparing the right hand sides of the above two relations, we obtain (3.10).

The first equality in (3.11) holds because the map $\mathcal{B}_{\lambda *}$ preserves the first integrals of the Euler top. Next, since the vector $a^*(\widetilde{M} + M)$ lies on the cone K_2, we have $((\widetilde{M} + M), a^*(\widetilde{M} + M)) = 0$. Expanding this and using the first equality in (3.11) yields the second equality. ■

Remark 3.3. The fact that the difference $\widetilde{M} - M$ is orthogonal to $(a^*)^{-1} \ell$ and ℓ implies that *the angle between the normal vector* $(a^*)^{-1} \ell$ *and the plane* Π *equals the angle between* $(a^*)^{-1} \ell$ *and* $\widetilde{\Pi}$. This property can be regarded as a projective version of the Birkhoff condition of elastic impacts, hence the geometric construction of Theorem 3.1 illustrated in Figure 1 describes a projective analog of the plane elliptic billiard. (Note that no any plane section of the cones and of the sequence of Π gives such a plane billiard.)

To obtain the map $\mathcal{B}_{\lambda*} : so^*(3) \to so^*(3)$ in an explicit form, we use the fact that the vector ℓ satisfies the system of homogeneous equations

$$(\ell, (a^*)^{-1}\ell) = 0, \quad (\ell, M) = 0.$$

One of its solutions, $\bar\ell = (\bar\ell_1, \bar\ell_2, \bar\ell_3)^T$, normalized by the condition $\bar\ell_3 = 1$, has the form

$$\bar\ell_1 = -a_1^* \frac{M_1M_3 - \sqrt{D}\, a_2^* M_2}{a_1^* M_1^2 + a_2^* M_2^2}, \quad \bar\ell_2 = -a_2^* \frac{M_2M_3 + \sqrt{D}\, a_1^* M_1}{a_1^* M_1^2 + a_2^* M_2^2}, \quad \bar\ell_3 = 1 \tag{3.13}$$

$$D = \frac{a_1^* M_1^2 + a_2^* M_2^2 + a_3 M_3^2}{a_1^* a_2^* a_3^*}. \tag{3.14}$$

Substituting this instead of ℓ into (3.12) and symmeterising obtained expressions, we arrive at the following relations

$$\begin{aligned}
\widetilde{M}_1 - M_1 &= \chi(a_2 - a_3)\Big(\Delta_1\, M_2M_3 - \sqrt{D}\,(a_1^*)^2 M_1^3 (a_2^* M_2^2 - a_3^* M_3^2)\Big), \\
\widetilde{M}_2 - M_2 &= \chi(a_3 - a_1)\Big(\Delta_2\, M_1M_3 - \sqrt{D}\,(a_2^*)^2 M_2^3 (a_3^* M_3^2 - a_1^* M_1^2)\Big), \\
\widetilde{M}_3 - M_3 &= \chi(a_1 - a_2)\Big(\Delta_3 M_1M_2 - \sqrt{D}\,(a_3^*)^2 M_3^3 (a_1^* M_1^2 - a_2^* M_2^2)\Big),
\end{aligned} \tag{3.15}$$

where

$$\Delta_i = a_j^* a_k^* M_j^2 M_k^2 - (a_i^*)^2 M_i^4, \qquad (i,j,k) = (1,2,3)$$

and χ is a common factor.

Next, multiplying the both sides of (3.15) by $a_1^* M_1, a_2^* M_2, a_3^* M_3$ respectively, then summing and using the second equality in (3.11), we find the factor χ in form

$$\chi = -\frac{2(M, a^* M)}{\sum a_1(a_2 - a_3)\Big[\Delta_1 M_1M_2M_3 - \sqrt{D}(a_1^*)^2 M_1^4 (a_2^* M_2^2 - a_3^* M_3^2)\Big]}, \tag{3.16}$$

where the summation in the denominator ranges over the three terms obtained by the cyclic permutations of indices (1,2,3).

We summarize our results on $\mathcal{B}_{\lambda*}$.

Theorem 3.3. *The map $\mathcal{B}_{\lambda*} : so^*(3) \to so^*(3)$ given by (3.15), (3.16), (3.14) preserves the first integrals of the continuous Euler top, as well as the standard Lie–Poisson structure on $so^*(3)$. Under reality conditions (3.7), the map is real inside the domain*

$$\mathfrak{R} = \left\{ \sum_{k=1}^{3} (\lambda^* - a_k) M_k^2 \le 0 \right\}.$$

Its restriction onto $Jac(\mathcal{C}) = \mathcal{C}$ is represented by the shift by the holomorphic integral $e = \int_{E_-}^{E_+} \bar\omega$. On the boundary of $\mathfrak{R}$ the shift vector e tends to zero and the continuous limit of $\mathcal{B}_{\lambda*}$ coincides with the Euler equations $\dot M = [M, aM]$.

Remark 3.4. Since generic solutions $M_1(t), M_2(t), M_3(t)$ of the classical Euler top are proportional to the elliptic functions $sn(u), cn(u), dn(u)$ with argument u depending linearly on time t, the relations (3.15), (3.14), (3.16) can be regarded as a set of explicit addition formulae for these functions, in case when their moduli (parameters of the curve $\mathcal{C}$) are not fixed.

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