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Asymptotic symmetries of difference equations on a lattice

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Abstract

It is known that many equations of interest in Mathematical Physics display solutions which are only asymptotically invariant under transformations (e.g. scaling and/or translations) which are not symmetries of the considered equation. In this note we extend the approach to asymptotic symmetries for the analysis of PDEs, to the case of difference equations.

1 Introduction

When we consider a differential equation the analysis of symmetries leads, via a standard procedure [4, 19, 34, 36, 37], to the determination of invariant solutions.

This method has been extended to consider *conditional symmetries* [10, 17, 28, 35], i.e. transformations which do not leave the equation invariant, but such that some of its solutions are left invariant. This notion can be further generalized to consider *partial symmetries* where a subset of solutions, each of them not necessarily invariant, is mapped into itself by the transformation at hand [9] (see also [7] for a short review).

In a recent contribution [21] the notion of *asymptotic invariant solutions* with respect to standard symmetries and conditional symmetries for differential equations has been introduced. This theory easily extends also to partial symmetries. The approach of [21] can be seen as a development of [20], based in turn on the renormalization group approach to differential equations [2, 3, 5, 11, 23]. In this way we were able to explain the asymptotic scaling symmetry observed in numerical experiments analyzing anomalous diffusion and reaction-diffusion equations [32]. In this note, *asymptotic* is meant in the sense of "for large values of some dependent or independent variables".

Another interesting notion in this field is that of *approximate symmetries* [1, 18]. Here one considers transformations which leave invariant the lower order approximations to the solution of a system depending on a small parameter. This notion may also mean an approximate symmetry for a series expansion near a fixed point [8, 22].

All these approaches and results have been introduced in the case of differential equations. However, in many physical (or biological, chemical, etc.) models – and in numerical simulations of continuous phenomena – one is rather interested in difference equations on a lattice, and their symmetry properties [13, 16, 26, 27, 29, 30, 38]. The reader is referred to the bibliography contained in [38] for applications.

The purpose of the present note is to show that the approach of [21] also extends to equations defined on a lattice.

We will not try to be as general as possible, but rather consider, after a brief introduction to the generalities of equations on a lattice and their symmetries, a well defined equation. We hope that, in this way, the general method will result clearer, and it should be easy for the reader to apply it to other equations of interest. In fact, we will focus on reaction-diffusion equations [6, 12, 14], and in particular on the classical FKPP equation [15, 25, 33].

The only possible symmetries for discrete equations if we do not want to change essentially the underlying lattice are discrete translations, rotations of the lattice variables and continuous transformations of the dependent variable.

We are mainly interested here in equations whose solutions (as observed in numerical experiments) are asymptotically well described by a *travelling front*. We believe that the reader interested in more complex asymptotic symmetries will easily understand how to deal with them by comparing the present work and [21].

In Section 2 we present the generalities on equations defined on a lattice while in Section 3 we will consider symmetries for equations defined on a lattice. In Section 4 we introduce the FKPP equation both in the continuous and on a lattice and analyze it so as to deduce the correct asymptotic behavior by requiring that asymptotically the system has the correct required symmetries. Section 5 is devoted to a final discussion of the results and some concluding remarks.

2 Difference equations on a lattice.

For simplicity we will consider a lattice Λ in $\mathbf{R}^2 = (x,t)$, and a difference equation for a real function $u(x,t)$ on it. Higher dimensional lattices and matrix functions defined on them could be defined in the same way. We will use the notation of [29, 38]. Thus the points on the lattice, having coordinates $(x_{m,n}, t_{m,n})$, will be indexed by a couple of numbers $(m,n) \in \mathbf{Z}^2$, and to each site is associated a real variable $u_{m,n} \in \mathbf{R}$.

The lattice is intrinsically described by assigning the difference between neighboring points and will be of the form

$$\begin{aligned} x_{m+1,n} - x_{m,n} &= \xi_{m,n} \ , \quad & x_{m,n+1} - x_{m,n} &= \eta_{m,n} \\ t_{m+1,n} - t_{m,n} &= \tau_{m,n} \ , \quad & t_{m,n+1} - t_{m,n} &= \vartheta_{m,n} \ . \end{aligned} \tag{1}$$

These will be referred to as the *lattice equations*. The set of values $\xi_{m,n}$ defines a function $\xi : \Lambda \to \mathbf{R}$ via $\xi(x_{m,n}, t_{m,n}, u_{m,n}) = \xi_{m,n}$, and similarly for η, τ, ϑ. By the notation $u_{m,n}$ we mean $u(x_{m,n}, t_{m,n})$. The shifts $(m,n) \to (m+1,n)$ and $(m,n) \to (m,n+1)$ correspond to $(x \to x + \xi(x,t,u), t \to t + \tau(x,t,u))$ and to $(x \to x + \eta(x,t,u), t \to t + \vartheta(x,t,u))$ respectively. The presence of u allows for solution depending lattices.

We will then consider a difference equation for the variables $u_{m,n}$:

$$F_{m,n}[u] : \quad \mathcal{F}[x_{m,n}, t_{m,n}, \{u_{m+j,n+i}\}_{(i,j)\in\mathbf{Z}^2}] = 0. \tag{2}$$

Given some initial boundary conditions, this equation must allow us to get u in all points of the lattice (past or future).

As an example of difference equation we can consider the "discrete heat equation", obtained as the natural discretization (but only one of the possible ones) of $u_t = u_{xx}$,

$$(\delta t)^{-1}\,[u_{m,n+1} - u_{m,n}] \;=\; (\delta x)^{-2}\,[u_{m+1,n} - 2u_{m,n} + u_{m-1,n}]\;. \tag{3}$$

where $\xi_{m,n} = \delta x$, $\eta_{m,n} = 0$, $\tau_{m,n} = 0$ and $\vartheta_{m,n} = \delta t$. We can express the discretizations in terms of an operator Δ_k, a *difference operator of order* $j - i$. Acting with Δ_k on an arbitrary smooth function $f_k = F(z_k)$ we have

$$\Delta_k f_k = \frac{1}{\delta_k}\sum_{\ell=i}^{j} a_\ell\, f_{\ell+k}\;,\;\; \sum_{\ell=i}^{j} a_\ell = 0\;,\;\; \sum_{\ell=i}^{j} \ell\, a_\ell = 1\;, \tag{4}$$

where $\delta_k = z_{k+1} - z_k$. In eq. (3) the simplest discrete derivative on t, corresponding to $j = 1$ and $i = 0$, has been considered.

3 Symmetries

A transformation $S : (x,t,u(x,t)) \to (x',t',u'(x',t'))$ will be a symmetry of the lattice equation if it takes $\Lambda \subset \mathbf{R}^2$ into itself. It will be a symmetry of the difference equation defined on Λ if it is a symmetry of the lattice and leaves the equation $F_{m,n}[u]$ invariant. Most of the transformations for the lattice will be discrete symmetries. If we consider a uniform regular lattice Λ_R, the point (m,n) will have coordinates $x_{mn} = x_{0,0} + m\delta x$, $t_{mn} = t_{0,0} + n\delta t$ and the lattice equations are

$$\begin{aligned} x_{m+1,n} - x_{m,n} &= \delta x\;, \quad & x_{m,n+1} - x_{m,n} &= 0 \\ t_{m+1,n} - t_{m,n} &= 0\;, \quad & t_{m,n+1} - t_{m,n} &= \delta t\;. \end{aligned} \tag{5}$$

We will denote by p the ratio $p = \delta x/\delta t$ ($p \in \mathbf{R}$). On Λ_R the discrete symmetries are easy to determine. They are given by

$$\begin{cases} \hat{T}_x :\; (x,t) \to (x+\delta x, t) & \text{[shift in } x\text{, equivalent to } (m,n) \to (m+1,n)]; \\ \hat{T}_t :\; (x,t) \to (x, t+\delta t) & \text{[shift in } t\text{, equivalent to } (m,n) \to (m,n+1)]; \\ \hat{B}_x :\; (x,t) \to (-x,t) & \text{[inversion in } x\text{, equivalent to } (m,n) \to (-m,n)]; \\ \hat{B}_t :\; (x,t) \to (x,-t) & \text{[inversion in } t\text{, equivalent to } (m,n) \to (m,-n)]; \\ \hat{R} :\; (x,t) \to (-pt, x/p) & \text{[rotation by } \pi/2 \text{ with a scale factor,} \\ & \quad \text{equivalent to } (m,n) \to (-n,m)]. \end{cases}$$

Moreover, if we accept $S\Lambda_R \subset \Lambda_R$ rather than $S\Lambda_R = \Lambda_R$, we have the further trasformation

$$S_q \;:\; (x,t) \to (q_1 x, q_2 t)\;,\;\; (q_1,q_2) \in \mathbf{Z} \text{ (discrete scaling)}\;. \tag{6}$$

Apart from the discrete symmetries given by the transformation written above we can introduce Lie symmetries which are better described in term of an infinitesimal symmetry generator $\hat{X}$, which, taking into account that the lattice Λ_R is subject just to discrete transformations, will involve only transformations of the dependent variable $u_{m,n}$

$$\hat{X}_{m,n} = Q(x_{m,n}, t_{m,n}, \{u_{m+j,n+i}\}_{(i,j)\in \mathbf{Z}^2})\partial_{u_{m,n}}. \tag{7}$$

Formula (7) will be the infinitesimal generator of a symmetry for (2) if

$$\sum_{j=-1}^{1}\sum_{i=0}^{1} \hat{X}_{m+j,n+i}\mathcal{F}|_{\mathcal{F}=0} = 0 \tag{8}$$

Eq. (8) is equivalent to the request that the flow generated by eq. (2), when $u_{m,n} = u_{m,n}(\lambda)$, is compatible with

$$u_{m,n,\lambda} = Q(x_{m,n}, t_{m,n}, \{u_{m+j,n+i}\}_{(i,j)\in \mathbf{Z}^2}) \tag{9}$$

Let us notice that the symmetries generated by the infinitesimal generator (7), if $(i,j) \neq (0,0)$, are not Lie point symmetries but *generalized symmetries*.

Solutions invariant with respect to the symmetries of infinitesimal generator (7) are obtained by solving the difference equation (2) together with the invariance condition $Q(x_{m,n}, t_{m,n}, \{u_{m+j,n+i}\}_{(i,j)\in \mathbf{Z}^2}) = 0$. A particularly interesting class of function symmetries is when the function Q takes the form

$$\begin{aligned} Q &= \varphi(x_{m,n}, t_{m,n}, u_{m,n}) - \xi(x_{m,n}, t_{m,n}, u_{m,n})\Delta_m u_{m,n} - \\ &\quad -\tau(x_{m,n}, t_{m,n}, u_{m,n})\Delta_n u_{m,n} \end{aligned} \tag{10}$$

which in the continuous limit goes over to point symmetries.

When the transformation is not a symmetry, but there is an invariant solution, we say that we have a *conditional symmetry* for our problem. In this case to get the invariant solution we can add to the equation the condition $Q = 0$.

If we consider an equation $F_{m,n}[u]$ which does not explicitly depend on x and t, it is invariant under $\hat{T}_x$ and $\hat{T}_t$ transformation. In this case we can consider travelling wave solutions, depending on $\zeta = x - vt$. Defining $v = k(\delta x/\delta t)$ with $k \in \mathbf{Z}$, the travelling wave solutions can be obtained by imposing [27]

$$Q = (\hat{T}_x^{-k} - \hat{T}_t)u_{m,n}. \tag{11}$$

For $k = 1$ this can be read, similarly as in the continuous case where the condition is $u_x + (1/v)u_t = 0$, as $Q = \Delta_x^- u_{m,n} + \frac{1}{v}\Delta_t^+ u_{m,n}$ where Δ_x^- and Δ_t^+ are the 'down' and 'up' discrete derivatives defined as

$$\Delta_t^+ = \frac{T_t - 1}{\delta t} \qquad \Delta_x^- = \frac{1 - T_x^{-1}}{\delta x}. \tag{12}$$

As the regular lattice Λ_R is invariant under

$$\hat{M} = \hat{T}_x^k \hat{T}_t : (m, n, u_{m,n}) \to (m + k, n + 1, u_{m+k,n+1}),$$

we can pass to the variables $(\zeta = x - vt, \tau = t, w(\zeta, \tau) = u(x - vt, t))$. Obviously Q and $\hat{M}$ are related, as we have $Q = \hat{T}_t(\hat{M}^{-1} - 1)u_{m,n}$. So we can introduce two new indices

($\mu = m - kn, \nu = n$). In these new independent variables the difference equation $F_{m,n}[u]$ will transform into a new difference equation for the variable $w_{\mu,\nu}$, $\Phi_{\mu,\nu}[w]$.

The reduction of eq. (2) by the symmetry (11) implies that $w_{\mu,\nu+1} = w_{\mu,\nu}$; so, as $w_{\mu,\nu}$ is independent on ν, we can write $w_{\mu,\nu} = A_\mu$ and reduce $\Phi_{\mu,\nu}[w]$ to $\tilde{\Phi}_\mu[A]$. If this equation admits a solution, it represents a travelling wave solution.

Let us now consider a continuous transformation on A generated by the infinitesimal symmetry operator

$$\hat{X} = \Psi A_\mu \partial_{A_\mu}, \tag{13}$$

where Ψ is a constant. The infinitesimal transformation (13) corresponds to the transformation $\tilde{A}_\mu = A_\mu + \lambda \Psi A_\mu + O(\lambda^2)$. If the quantity

$$W_\mu \;:=\; \left(\hat{X}\tilde{\Phi}_\mu\,[A]\right) \;-\; \left(\tilde{\Phi}_\mu\,[\hat{X}A]\right)$$

is not zero but goes to zero for large μ, then $\hat{X}$ is an asymptotic symmetry of $\tilde{\Phi}_\mu[A]$.

Finally, if there exist $\hat{X}$-invariant asymptotic travelling wave solutions of the original equation $F_{m,n}[u]$, we will say that $\hat{X}$ is an *asymptotic conditional symmetry* of $F_{m,n}[u]$.

4 The discrete FKPP equation and its symmetries

In the rest of this note we will focus on the standard FKPP [15, 25, 33] reaction-diffusion equation

$$u_t \;=\; u_{xx} \;+\; u(1-u). \tag{14}$$

See e.g. [6, 12, 14] for a discussion of its properties. Here we are specially interested in its asymptotic solutions for large t. For sufficiently localized non-negative initial data,

$$u(x, t=0) = \begin{cases} \stackrel{x\to+\infty}{\longrightarrow} e^{-k_0(x+k_1)}, & (k_0, k_1) > 0, \quad \text{for } x > 0 \\ = 1, & \text{for } x < 0. \end{cases} \tag{15}$$

the asymptotic solutions correspond to a stable front travelling with constant speed and smoothly connecting the "fresh" (unstable) stationary state $u = 0$ ahead of it and the "exhaust" (stable) stationary state $u = 1$ behind it.

The difference equation representing its simplest discretization on a regular lattice Λ_R is given by

$$\begin{aligned} F_{m,n}[u]: \quad & \frac{u_{m,n+1} - u_{m,n}}{\delta t} \;= \\ & = \frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{\delta x^2} \;+\; u_{m,n}\,(1 - u_{m,n})\;. \end{aligned} \tag{16}$$

Let us stress that δx and δt are small but finite parameters, representing the lattice spacing in the x- and in the t-direction respectively.

As noted above, in analyzing the symmetries of eq. (16), we should restrict our attention to transformations $S : (x, t, u(x, t)) \to (x', t', u'(x', t'))$ which are also symmetries of the lattice equations (5).

Let us first consider transformations which do not act on the $u_{m,n}$. It is easy to see that shifts in x and t are symmetries of the equation (16). This is also the case for the inversion in x, but not for the inversion in t. Also, since x and t are intrinsically different here, the rotation R is not a symmetry of the equation. We can recover the time inversion by considering the map $B_t : (x, t, u(x, t)) \to (x, -t, u(x, -t))$, corresponding to $(m, n, u_{m,n}) \to (m, -n, u_{m,-n})$. This represents a time inversion and backward dynamics.

It is easy to see (by the same argument as for the continuous equation) that there is no way to have a scaling symmetry of the independent variables for eq. (16).

4.1 The discrete FKPP in a moving frame

Following the results presented in Section 3, a set of values $u_{m,n}$ will represent a travelling solution with speed $v = k\delta x/\delta t = kp$ (with $k \in \mathbf{Z}$) if, for any $j \in \mathbf{Z}$,

$$u_{m+kj,n+j} \;=\; u_{m,n} \; . \tag{17}$$

The equivalent of passing to a moving frame of reference will be passing to the independent variables

$$\zeta = x - vt \; ; \; \tau = t. \tag{18}$$

The inverse of the change (18) is given by $t = \tau$, $x = \zeta + v\tau$. It follows from

$$\begin{aligned} \zeta_{m,n} \;&=\; x_{m,n} - vt_{m,n} \;=\; m(\delta x) - \frac{k\delta x}{\delta t}\, n\,\delta t \;=\; (m - kn)\delta x \; , \\ \tau_{m,n} \;&=\; t_{m,n} \;=\; n\,\delta t \; . \end{aligned} \tag{19}$$

that the lattice equations, in terms of the new coordinates, are given by

$$\begin{aligned} \zeta_{m+1,n} - \zeta_{m,n} \;=\; \delta x \; , \quad & \zeta_{m,n+1} - \zeta_{m,n} \;=\; -v\,\delta t \;=\; -k\,\delta x \; ; \\ \tau_{m+1,n} - \tau_{m,n} \;=\; 0 \; , \quad & \tau_{m,n+1} - \tau_{m,n} \;=\; \delta t \; . \end{aligned} \tag{20}$$

The lattice point with coordinates $(\zeta = \mu\delta x, \tau = \nu\delta t)$, has coordinates $(x = m\delta x, t = n\delta t)$ in the original system, with $\mu = m - kn$, $\nu = n$. The inverse map is $m = \mu + k\nu$, $n = \nu$.

We will denote by $w_{\mu,\nu}$ the dependent variable associated to the point of coordinates $\zeta = \mu\delta x, \tau = \nu\delta t$. We have

$$w_{\mu,\nu} \;\equiv\; u_{\mu+k\nu,\nu}; \; u_{m,n} \;=\; w_{m-kn,n}. \tag{21}$$

We can now use (21) to express (16) in the new variable w. We get

$$\begin{aligned} \frac{w_{m-k(n+1),n+1} - w_{m-kn,n}}{\delta t} \;&=\; w_{m-kn,n}(1 - w_{m-kn,n}) + \\ &+\; \frac{w_{m+1-kn,n} - 2w_{m-kn,n} + w_{m-1-kn,n}}{\delta x^2}. \end{aligned} \tag{22}$$

Passing now to the indices (μ, ν), we can rewrite eq. (22) as

$$\Phi_{\mu,\nu}[w] : \; \frac{w_{\mu-k,\nu+1} - w_{\mu,\nu}}{\delta t} \;=\; \frac{w_{\mu+1,\nu} - 2w_{\mu,\nu} + w_{\mu-1,\nu}}{\delta x^2} \;+\; w_{\mu,\nu}\,(1 - w_{\mu,\nu}) \, . \tag{23}$$

Eq. (23) can be simplified by recalling that $\delta t = \frac{k}{v}\delta x$. So the equation $\Phi_{\mu,\nu}[w]$ reads

$$[w_{\mu+1,\nu} - 2w_{\mu,\nu} + w_{\mu-1,\nu}] - \frac{v}{k}\delta x\,[w_{\mu-k,\nu+1} - w_{\mu,\nu}] + \delta x^2 w_{\mu,\nu}\,(1 - w_{\mu,\nu})\ . \tag{24}$$

Let us recall that δx is a small but finite parameter, representing the lattice spacing in the x-direction.

If we reduce eq. (24) with respect to a proper combination of the translations, we get that the solution of (24) will depend only on the ζ (not on the τ) coordinate, i.e. $w_{\mu,\nu+j} = w_{\mu,\nu}$ for all $j \in \mathbf{Z}$. If $w_{\mu,\nu}$ does not depend on the τ variable, we can as well restrict to a fixed value of the ν index, and write simply $A_\mu := w_{\mu,\nu}$. Inserting this ansatz into (24) we get the *reduced difference equation*

$$\tilde{\Phi}_\mu :\quad [A_{\mu+1} - 2A_\mu + A_{\mu-1}] \;+\; \frac{v}{k}\delta x\,[A_\mu - A_{\mu-k}] \;+\; \delta x^2\, A_\mu\,(1 - A_\mu) \;=\; 0\ . \tag{25}$$

4.2 Travelling fronts

A solution to (25) is, by construction, a travelling wave solution to the difference FKPP equation (16). In the continuum case, we know this will actually be a travelling front. In the moving frame coordinates (ζ,τ), it will quickly tend to zero for positive ζ and to one for negative ζ. Moreover, it is stable. If we start with a sufficiently regular initial datum, the (non translation-invariant) solution will tend to this moving front as time goes by. Finally, the front solution is characterized by a sharp transition between the two stationary states $u = 0$ and $u = 1$. The region with small u is described by an exponential, $u \simeq \exp(-\alpha\zeta)$ (with $1/\alpha$ a characteristic width).

The equivalent of the exponential behaviour in our discrete setting is

$$A_{\mu+1} \;=\; (1 - \alpha\delta x)\,A_\mu. \tag{26}$$

In fact, as $\mu = \zeta/\delta x$, $\lim_{\delta x\to 0} A_\mu = A_0 \exp(-\alpha\zeta)$. Solving eq. (26) we get

$$A_\mu = (1 - \alpha\delta x)^\mu A_0. \tag{27}$$

Putting formula (27) in (25), with $k = 1$, we have

$$\delta x^2\left[\alpha^2 - (v + \delta x)\alpha + 1 - (1 - \alpha\delta x)^{\mu+1} A_0\right] = 0. \tag{28}$$

Eq. (28) is not solvable for $\mu \to \infty$ if $\alpha < 0$. The case $\alpha > 0$ gives us a monotonically decreasing solution for growing μ. For large μ, we can leave out the last term of eq. (28) and solve it for α:

$$\alpha^2 - (v + \delta x)\alpha + 1 = 0. \tag{29}$$

This equation admits two real solutions for $v + \delta x > 2$; studying the stability of the equation as in the continuous case, one finds that the velocity is $v = 2 - \delta x$. So the front, observed in the numerical integration of the discrete FKPP on Λ_R, would be just a little slower (of the order of δx) than the front, which is solution of the FKPP equation.

Note that we have just found an asymptotic symmetry. In fact the choice of a scaling invariant solution (26) is equivalent to set

$$Q = \alpha A_\mu + \Delta_\zeta A_\mu \tag{30}$$

The corresponding transformations, a scaling on A and a translation on ζ, are not a symmetry of the non-linear equation (25) as $W_\mu = \hat{X}\widehat{\Phi}_\mu[A] - \widehat{\Phi}_\mu[\hat{X}A] = -\alpha\delta x^3(1 - \alpha\delta x)A_{\mu+1}^2 \neq 0$. As for $\alpha < 0$ A_μ is a decreasing function of μ, we get that $\lim_{\mu\to\infty} W_\mu = 0$.

5 Discussion and Conclusions

In this note we have applied the method of asymptotic symmetries to a discrete version of the FPPK equation. In such a way we have shown that also in the discrete case the FKPP equation possesses an asymptotic scaling symmetry which gives the correct asymptotic behaviour of the system.

Long time ago Ibragimov and Fushchich [1, 18] introduced the notion of approximate symmetry (for differential equations); this is based on some small parameter ε appearing in the equation, and is a symmetry "up to terms of order ε". One could wonder what is the relation between approximate symmetries in this sense and our approach to the front solutions of the FKPP equation; note however that no explicit small parameter ε is present in the FKPP equation. A natural way to introduce such an ε would be to look for solutions whose amplitude is of order ε, i.e. $u_{n,m} = \varepsilon v_{n,m}$ (with $|v_{n,m}|$ of order one); proceeding in this way, however, the obtained scaling invariant solution will be small everywhere, not only asymptotically. Actually, in our approach the natural small parameter is inversely proportional to the group parameter generated by the asymptotic symmetry. Thus the small parameter comes into play only when we consider an additional object (the asymptotic symmetry). It appears, therefore, that our notion of asymptotic symmetry cannot be reconducted to the notion of approximate symmetries of Ibragimov and Fushchich [1, 18]; further work would be needed for a better understanding of the relation between these two notions.

Work is in progress to obtain the asymptotic solutions for the anomalous reaction diffusion equation. This equation is nonautonomous in the independent variables, so that no point symmetry is available. In such a case the use of a renormalization of the lattice, provided by eq. (6), might be important.

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