Twisted Volterra equation

Sergei D SILVESTROV

*Centre for Mathematical Sciences, Department of Mathematics, Lund Institute of
Technology, Lund University, Box 118, SE-221 00 Lund, Sweden*
  E-mail: sergei.silvestrov@math.lth.se

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Abstract

In this paper an extension of the q-deformed Volterra equation associated with linear
rescaling to the general non-linear rescaling is obtained.

1   Introduction

One of the most famous non-linear differential-difference equations is

$$\frac{dc_n}{dt} = c_n(c_{n+1} - c_{n-1}), \quad n \in \mathbb{Z}, \, c_n(t) > 0 \tag{1.1}$$

often referred to as the Volterra equation or Volterra model as a tribute to the Vito
Volterra whose pioneering works stimulated a broad use of models based on such type of
equations in biological and ecological problems. This equation also proved to be quite
universal having many applications in Physics. For example it appears in the study of
the fine structure of the spectrum of Langmuir oscillations in plasma, it can be viewed
as a difference analogue of the famous Kortweg-de Vries equation [6], it is also shown to
be related to Liouville equation, and after suitable Hamiltonian interpretation, to provide
a latice deformation of the Virasoro algebra [3] important in String theory. A quantum
version has been investigated in [1, 2].

A starting point for this article is the following q-deformation of the Volterra model

$$_{q^2}D_t R_n = q^2 \, _{q^2}T_t(R_n R_{n+1}) - R_n R_{n-1}, \quad n \in \mathbb{Z}, \tag{1.2}$$

where $_{q^2}D_t f(t) = \frac{f(q^2 t) - f(t)}{(q^2 - 1)t} = \frac{_{q^2}T_t f(t) - f(t)}{(q^2 - 1)t}$ is the Jackson q^2-difference operator, and
$_{q^2}T_t f(t) = f(q^2 t)$ is the operator of multiplicative rescaling by q^2 with respect to t. The
differential-difference Volterra model is recovered in the limit for $q = 1$. This non-linear q-
difference difference equation is related to similarity reduction for the q-deformed Painlevé-
I equation and to our knowledge has appeared first in the work of Frank Nijhoff [5] and
further studied in depth by Sara Lombardo [4].

Some special features of this equation are connected with the fact that the "time" vari-
able t is rescaled linearly that is via multiplication by a parameter q. Such linear rescaling

has rather simple dynamics. It is intriguing and from several respects important to have models involving non-linear time rescalings. In this article we deduce a generalization of the q-deformed Volterra equation to the situation of non-linear rescaling $\sigma(t)$ in t instead of multiplication by q^2. Let $\sigma(x)$ be a real-valued function as defined above. We will use the following operator ${}_\sigma D_t f(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}$ called the σ-twisted derivative of f. The name is justified by the observation that this operator obeys the so called σ-twisted Leibniz rule

$$ {}_\sigma D_t(fg)(t) = ({}_\sigma D_t f)(t)g(t) + f(\sigma(t))({}_\sigma D_t g)(t) \tag{1.3} $$

which is easily proved as follows

$$
\begin{aligned}
{}_\sigma D_t(fg)(t) &= \frac{f(\sigma(t))g(\sigma(t)) - f(t)g(t)}{\sigma(t) - t} = \\
&= \frac{(f(\sigma(t)) - f(t))g(t) + f(\sigma(t))(g(\sigma(t)) - g(t))}{\sigma(t) - t} = ({}_\sigma D_t f)(t)g(t) + f(\sigma(t))({}_\sigma D_t g)(t).
\end{aligned}
$$

This operator generalizes the Jackson q^2-difference operator ${}_{q^2} D_t : f(t) \mapsto \frac{f(q^2 t) - f(t)}{q^2 t - t}$ obtained in the special case of linear rescaling $\sigma(t) = q^2 t$.

2 The σ-twisted Volterra equation

In our derivation of the σ-twisted Volterra equation, we will start with the following system of equations

$$ \hat{\psi}_{n+1} + R_n \hat{\psi}_{n-1} = \lambda \hat{\psi}_n \tag{2.1} $$

and

$$ {}_\sigma D_t \hat{\psi}_n = Q_n \hat{\psi}_{n-2}. \tag{2.2} $$

where $R_n = R_n(t)$ and $Q_n = Q_n(t)$. Equation (2.1) corresponds to the standard auxiliary linear problem for the Volterra model, and (2.2) is an equation defining the time dependence of the spectral data [4]. Applying ${}_\sigma D_t$ to (2.1) we get

$$ {}_\sigma D_t \hat{\psi}_{n+1} + {}_\sigma D_t(R_n \hat{\psi}_{n-1}) = \lambda {}_\sigma D_t \hat{\psi}_n = \lambda Q_n \hat{\psi}_{n-2}, $$

where the last equality is due to (2.2). By using the σ-twisted Leibniz rule (1.3) in combination with (2.2), we find that

$$ Q_{n+1} \hat{\psi}_{n-1} + ({}_\sigma D_t R_n) \hat{\psi}_{n-1} + R_n(\sigma(t)) Q_{n-1} \hat{\psi}_{n-3} = \lambda Q_n \hat{\psi}_{n-2}. \tag{2.3} $$

Noting that $\lambda \hat{\psi}_{n-2} = \hat{\psi}_{n-1} + R_{n-2} \hat{\psi}_{n-3}$ by (2.1), and inserting this into (2.3), we get

$$ Q_{n+1} \hat{\psi}_{n-1} - Q_n \hat{\psi}_{n-1} + ({}_\sigma D_t R_n) \hat{\psi}_{n-1} + R_n(\sigma(t)) Q_{n-1} \hat{\psi}_{n-3} - Q_n R_{n-2} \hat{\psi}_{n-3} = 0 $$

which is the same as

$$ (Q_{n+1} - Q_n + {}_\sigma D_t R_n)) \hat{\psi}_{n-1} + (R_n(\sigma(t)) Q_{n-1} - Q_n R_{n-2}) \hat{\psi}_{n-3} = 0. \tag{2.4} $$

If coefficients are put to zero, then (2.4) holds for any $\{\psi_n\}_{n\in\mathbb{Z}}$. We have

$$_\sigma D_t R_n = Q_n - Q_{n+1} \tag{2.5}$$

and

$$R_n(\sigma(t))Q_{n-1} = Q_n R_{n-2}(t). \tag{2.6}$$

These equations reduce to (5) and (6) from [4] in the special case of the linear rescaling $\sigma(t) = q^2 t$. Now, since

$$_\sigma D_t R_n = \frac{R_n(\sigma(t)) - R_n(t)}{\sigma(t) - t} = Q_n - Q_{n+1}$$

we find that

$$R_n(\sigma(t)) = R_n + (\sigma(t) - t)(Q_n - Q_{n+1}).$$

Therefore, (2.6) can be written as

$$(R_n + (\sigma(t) - t)(Q_n - Q_{n+1}))Q_{n-1} = Q_n R_{n-2}.$$

This is equivalent to $(R_n + (t - \sigma(t))Q_{n+1} + (\sigma(t) - t)Q_n)Q_{n-1} = Q_n R_{n-2}$ which, after introducing $\tilde{R}_n = R_n + (t - \sigma(t))Q_{n+1}$, takes the form

$$(\tilde{R}_n + (\sigma(t) - t)Q_n)Q_{n-1} = Q_n(\tilde{R}_{n-2} - (t - \sigma(t))Q_{n-1}),$$

becoming, after cancellations,

$$\tilde{R}_n Q_{n-1} = Q_n \tilde{R}_{n-2}. \tag{2.7}$$

Repeated use of the recurrence (2.7) yields $\prod_{k=0}^{p} \tilde{R}_{n-k} Q_{n-p-1} = Q_n \prod_{k=2}^{p+2} \tilde{R}_{n-k}$. From this it can be deduced that $Q_n = -c(t)\tilde{R}_n \tilde{R}_{n-1}$ for some $c(t)$ independent of n. If we insert $R_n = \tilde{R}_n + (\sigma(t) - t)Q_{n+1}$ into (2.5), we get using σ-twisted Leibniz rule, that

$$_\sigma D_t(\tilde{R}_n + (\sigma(t) - t)Q_{n+1}) = {}_\sigma D_t \tilde{R}_n + {}_\sigma D_t(\sigma(t)Q_{n+1}) - {}_\sigma D_t(tQ_{n+1}) =$$

$$= {}_\sigma D_t \tilde{R}_n + ({}_\sigma D_t \sigma(t))Q_{n+1} + \sigma(\sigma(t))({}_\sigma D_t Q_{n+1}) - ({}_\sigma D_t(t))Q_{n+1} - \sigma(t)({}_\sigma D_t Q_{n+1}) =$$

$$= {}_\sigma D_t \tilde{R}_n + \left(\frac{\sigma(\sigma(t)) - \sigma(t)}{\sigma(t) - t}\right)Q_{n+1} + \sigma(\sigma(t))\left(\frac{Q_{n+1}(\sigma(t)) - Q_{n+1}(t)}{\sigma(t) - t}\right) -$$

$$- \left(\frac{\sigma(t) - t}{\sigma(t) - t}\right)Q_{n+1} - \sigma(t)\left(\frac{Q_{n+1}(\sigma(t)) - Q_{n+1}(t)}{\sigma(t) - t}\right) =$$

$$= {}_\sigma D_t \tilde{R}_n + \frac{\sigma(\sigma(t))Q_{n+1}(\sigma(t)) - \sigma(t)Q_{n+1}(\sigma(t))}{\sigma(t) - t} - Q_{n+1} =$$

$$= {}_\sigma D_t \tilde{R}_n + \frac{\sigma(\sigma(t)) - \sigma(t)}{\sigma(t) - t} {}_\sigma T_t Q_{n+1} - Q_{n+1} = Q_n - Q_{n+1} \tag{2.8}$$

where the last equality follows by (2.5), and $_\sigma T_t : f(t) \mapsto f(\sigma(t))$ is the composition or rescaling operator by transformation σ. The equation above can be written as

$$_\sigma D_t \tilde{R}_n = \frac{\sigma(t) - \sigma(\sigma(t))}{\sigma(t) - t} {}_\sigma T_t Q_{n+1} + Q_n.$$

Finally, since $Q_n = -c(t)\tilde{R}_n\tilde{R}_{n-1}$, we get

$$_\sigma D_t\tilde{R}_n = \frac{\sigma(\sigma(t)) - \sigma(t)}{\sigma(t) - t} \,_\sigma T_t(c(t)\tilde{R}_{n+1}\tilde{R}_n) - c(t)\tilde{R}_n\tilde{R}_{n-1}$$

which can be expressed as

$$_\sigma D_t\tilde{R}_n = (_\sigma D_t\sigma(t))_\sigma T_t(c(t)\tilde{R}_{n+1}\tilde{R}_n) - c(t)\tilde{R}_n\tilde{R}_{n-1}. \tag{2.9}$$

We call this equation the σ-twisted Volterra equation. When $\sigma(t) = q^2 t$ and $c(t) \equiv 1$, this equation becomes the q-deformed Volterra equation (1.2).

Integrability or solvability of this equation depends on the properties of the twisting map σ. However, some general constructions of classes of solutions work for basically any general σ. Of cause, under additional requirements on the allowed function spaces for solutions, a need arises for further investigation of the compatibility of general solutions with such extra requirements, often bringing dependence on further specific properties of σ into play. In this paper, however it is reasonable to refrain from going into this elaborate analysis, leaving its comprehensive account to future publications. Nevertheless, at least one simple way to construct solutions can be mentioned here. Given any pair $\tilde{R}_0(t)$, $\tilde{R}_1(t)$ of σ-invariant functions, i.e. functions that satisfy $\tilde{R}_0(\sigma(t)) = \tilde{R}_0(t)$ and $\tilde{R}_1(t)(\sigma(t)) = \tilde{R}_1(t)$, the sequence defined by

$$\tilde{R}_n(t) = \begin{cases} \left(\frac{c(t)}{c(\sigma(t))}\frac{1}{_\sigma D_t(\sigma(t))}\right)^k \tilde{R}_0(t), & \text{if } n = 2k \\ \left(\frac{c(t)}{c(\sigma(t))}\frac{1}{_\sigma D_t(\sigma(t))}\right)^k \tilde{R}_1(t), & \text{if } n = 2k+1 \end{cases}, \qquad n \in \mathbb{Z}$$

is a solution of (2.9) if $_\sigma T_t\left(\frac{c(t)}{c(\sigma(t))}\frac{1}{_\sigma D_t(\sigma(t))}\right) = \frac{c(t)}{c(\sigma(t))}\frac{1}{_\sigma D_t(\sigma(t))}$ or, when reformulated in terms of σ-twisted derivative, if $_\sigma D_t\left(\frac{c(t)}{c(\sigma(t))}\frac{1}{_\sigma D_t(\sigma(t))}\right) = 0$.

In [4], one-soliton solution is explicitly constructed and time-dependence of the spectral data is investigated for the q-deformed Volterra equation (1.2). There are strong indications that extension of these results to the general twisted Volterra equation (2.9) should be possible. But before a sufficiently complete account for this can be presented, further understanding of the interplay between properties of the twisting map σ, spectral data and symmetries is needed. Solution of this interesting problem should bring to light several new interesting phenomena especially for non-linear twist maps σ when compared to the linear twist case of q-deformed Volterra equation.

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