

Asymptotic behavior of discrete holomorphic maps z^c and $\log(z)$

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Abstract

It is shown that discrete analogs of z^c and $\log(z)$, defined via particular “integrable” circle patterns, have the same asymptotic behavior as their smooth counterparts. These discrete maps are described in terms of special solutions of discrete Painlevé-II equations, asymptotics of these solutions providing the behaviour of discrete z^c and $\log(z)$ at infinity.

1 Introduction

In this paper we discuss a very interesting and rich object: circle patterns mimicking the holomorphic maps z^c and $\log(z)$. Discrete z^2 and $\log(z)$ were guessed by Schramm and Kenyon (see [25]) as examples from the class of circle patterns with the combinatorics of the square grid introduced by Schramm in [24]. This class opens a new page in the classical theory of circle packings, enjoying new golden age after Thurston’s idea [26] about approximating the Riemann mapping by circle packings: it turned out that these circle patterns are governed by discrete integrable equation (the stationary Hirota equation), thus providing one with the whole machinery of the integrable system theory ([11]).

The striking analogy between circle patterns and holomorphic maps resulted in the development of discrete analytic function theory (for a good survey see [13]). Classical circle packings comprised of disjoint open disks were later generalized to circle patterns where the disks may overlap. Discrete versions of uniformization theorem, maximum principle, Schwarz’s lemma and rigidity properties and Dirichlet principle were established ([19],[16],[24]).

Different underlying combinatorics were considered: Schramm introduced square grid circle patterns, generalized by Bobenko and Hoffmann to hexagonal patterns with constant intersection angles in [8], hexagonal circle patterns with constant multi-ratios were studied by Bobenko, Hoffman and Suris in [7].

The difficult question of convergence was settled by Rodin and Sullivan [22] for general circle packings, He and Schramm [15] showed that the convergence is C^∞ for hexagonal packings, the uniform convergence for square grid circle patterns was established by Schramm [24].

On the other hand not very many examples are known: for circle packings with the hexagonal combinatorics the only explicitly described examples are Doyle spirals, which are discrete analogues of exponential maps [12], and conformally symmetric packings, which are analogues of a quotient of Airy functions [5]. For patterns with overlapping circles more examples are constructed: discrete versions of $\exp(z)$, $\operatorname{erf}(z)$ ([24]), z^c , $\log(z)$ ([2]) are constructed for patterns with underlying combinatorics of the square grid; z^c , $\log(z)$ are also described for hexagonal patterns with both multi-ratio ([7]) and constant angle ([8]) properties.

Discrete z^c is not only a very interesting example in discrete conformal geometry. It has mysterious relationships to other fields. It is constructed via some discrete isomonodromic problem and is governed by discrete Painlevé II equation ([2],[20]), thus giving geometrical interpretation thereof. Its linearization defines Green's function on critical graphs (see [9]) found in [18] in the frames of the theory of Dirac operator. Moreover, it seems to be a rather important tool for investigation of more general circle patterns and discrete minimal surfaces (see [4] for a brief survey and [6] for more details).

1.1 Circle patterns and discrete conformal maps

To visualize the analogy between Schramm's circle patterns and conformal maps, consider regular patterns composed of unit circles and suppose that the radii are being deformed so as to preserve the orthogonality of neighboring circles and the tangency of half-neighboring ones. Discrete maps taking intersection points of the unit circles of the standard regular patterns to the respective points of the deformed patterns mimic classical holomorphic functions, the deformed radii being analogous to $|f'(z)|$ (see Fig. 1).

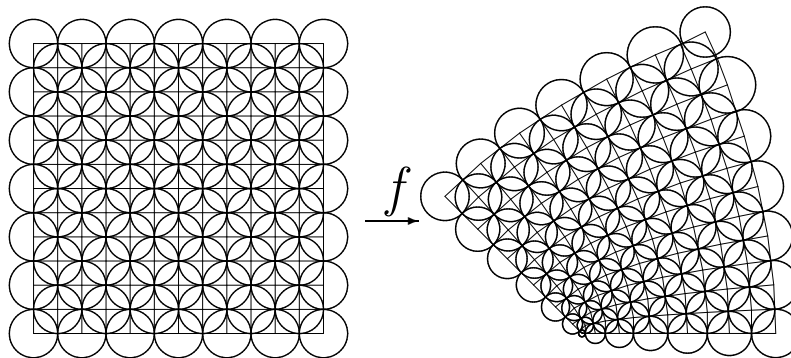


Figure 1. Schramm's circle patterns as discrete conformal map. Shown is the discrete version of the holomorphic mapping $z^{3/2}$.

It is easy to show that the lattice comprised of the centers of circles of Schramm's patterns and their intersection points is a special discrete conformal mapping (see Definition 1 below). The latter were introduced in [10] in the frames of discrete integrable geometry, originally without any relation to circle patterns.

Definition 1. A map $f : \mathbb{Z}^2 \rightarrow \mathbb{R}^2 = \mathbb{C}$ is called a discrete conformal map if all its elementary quadrilaterals are conformal squares, i.e., their cross-ratios are equal to -1:

$$q(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1}) := \frac{(f_{n,m} - f_{n+1,m})(f_{n+1,m+1} - f_{n,m+1})}{(f_{n+1,m} - f_{n+1,m+1})(f_{n,m+1} - f_{n,m})} = -1. \quad (1.1)$$

This definition is motivated by the following properties:

- 1) it is Möbius invariant,
- 2) a smooth map $f : D \subset \mathbb{C} \rightarrow \mathbb{C}$ is conformal (holomorphic or antiholomorphic) if and only if

$$\lim_{\epsilon \rightarrow 0} q(f(x, y), f(x + \epsilon, y), f(x + \epsilon, y + \epsilon), f(x, y + \epsilon)) = -1$$

for all $(x, y) \in D$.

Equation (1.1) can be supplemented with the following nonautonomous constraint:

$$cf_{n,m} = 2n \frac{(f_{n+1,m} - f_{n,m})(f_{n,m} - f_{n-1,m})}{(f_{n+1,m} - f_{n-1,m})} + 2m \frac{(f_{n,m+1} - f_{n,m})(f_{n,m} - f_{n,m-1})}{(f_{n,m+1} - f_{n,m-1})}. \quad (1.2)$$

This constraint, as well as its compatibility with (1.1), is derived from some monodromy problem (see [2] for detail). Let us assume $0 < c < 2$ and denote $\mathbb{Z}_+^2 = \{(n, m) \in \mathbb{Z}^2 : n, m \geq 0\}$. Motivated by the asymptotics of the constraint (1.2) at $n, m \rightarrow \infty$ and the properties

$$z^c(\mathbb{R}_+) \in \mathbb{R}_+, \quad z^c(i\mathbb{R}_+) \in e^{c\pi i/2}\mathbb{R}_+$$

of the holomorphic mapping z^c we use the following definition ([11]) of the “discrete” z^c .

Definition 2. The discrete conformal map $Z^c : \mathbb{Z}_+^2 \rightarrow \mathbb{C}$, $0 < c < 2$ is the solution of (1.1),(1.2) with the initial conditions

$$Z^c(0, 0) = 0, \quad Z^c(1, 0) = 1, \quad Z^c(0, 1) = e^{c\pi i/2}. \quad (1.3)$$

Obviously, $Z^c(n, 0) \in \mathbb{R}_+$ and $Z^c(0, m) \in e^{c\pi i/2}(\mathbb{R}_+)$ for any $n, m \in \mathbb{N}$.

A limit of rescaled Z^c as c approaches 2 gives discrete Z^2 . Discrete Log is defined as dual to Z^2 (see subsection 1.2 for detail).

Given initial data $f_{0,0} = 0$, $f_{1,0} = 1$, $f_{0,1} = e^{i\alpha}$ with $\alpha \in \mathbb{R}$, constraint (1.2) allows one to compute $f_{n,0}$ and $f_{0,m}$ for all $n, m \geq 1$. Now using equation (1.1) one can successively compute $f_{n,m}$ for any $n, m \in \mathbb{N}$. It turned out that all edges at the vertex $f_{n,m}$ with $n + m = 0 \pmod{2}$ are of the same length and therefore define a circle $C(z)$ of the radius

$$R(z) = |f_{n,m} - f_{n+1,m}| = |f_{n,m} - f_{n,m+1}| = |f_{n,m} - f_{n-1,m}| = |f_{n,m} - f_{n,m-1}|, \quad (1.4)$$

with the center at $f_{n,m}$, where we use complex labels $z = N + iM$ for the sublattice $\{n, m : n + m = 0 \pmod{2}\}$ with

$$N = (n - m)/2, \quad M = (n + m)/2.$$

Moreover, all angles between the neighboring edges at the vertex $f_{n,m}$ with $n + m = 1 \pmod{2}$ are equal to $\pi/2$. Therefore all circles $C(z)$ form circle patterns of Schramm type (see [24]), i.e. the circles of neighboring quadrilaterals intersect orthogonally and the circles of half-neighboring quadrilaterals with common vertex are tangent.

In [1] was proved:

$$\lim_{n \rightarrow \infty} Z_{n,m}^c = \infty, \quad \lim_{m \rightarrow \infty} Z_{n,m}^c = \infty.$$

In the present paper we establish the more accurate result

$$R(N_0 + iM) \simeq K(c)M^{c-1} \quad \text{as } M \rightarrow \infty$$

and corresponding asymptotics for Z^c . Thus asymptotic behavior of discrete z^c and $\log(z)$ is exactly that of their smooth counterparts as $R(z)$ is a discrete analog of $|f'(z)|$ for the corresponding smooth counterpart $f(z)$.

The main tool to prove this is a system of discrete Painlevé type equations for $R(z)$. For smooth Painlevé equations similar asymptotic problems have been studied in the frames of the isomonodromic deformation method [17]. Some discrete Painlevé equations were studied in that framework in [14]). The geometric origin of our equations permits us to find our asymptotics by bare-handed approach, studying linearized equations.

1.2 The discrete maps Z^2 and Log. Duality

Definition 2 was given for $0 < c < 2$. For $c < 0$ or $c > 2$, the radius function $R(1+i) = c/(2-c)$ of the corresponding circle patterns (found as a solution to equations for $R(z)$) becomes negative and some elementary quadrilaterals around $f_{0,0}$ intersect. But for $c = 2$, one can renormalize the initial values of f so that the corresponding map remains an immersion. Let us consider Z^c , with $0 < c < 2$, and make the following renormalization for the corresponding radii: $R \rightarrow \frac{2-c}{c}R$. Then as $c \rightarrow 2 - 0$ we have

$$R(0) = \frac{2-c}{c} \rightarrow +0, \quad R(1+i) = 1, \quad R(i) = \frac{2-c}{c} \tan \frac{c\pi}{4} \rightarrow \frac{2}{\pi}.$$

Definition 3. ([2]) $Z^2 : \mathbb{Z}_+^2 \rightarrow \mathbb{R}^2 = \mathbb{C}$ is the solution of (1.1), (1.2) with $c = 2$ and the initial conditions

$$Z^2(0,0) = Z^2(1,0) = Z^2(0,1) = 0, \quad Z^2(2,0) = 1, \quad Z^2(0,2) = -1, \quad Z^2(1,1) = i\frac{2}{\pi}.$$

In this definition, equations (1.1),(1.2) are regularized through multiplication by their denominators. Note that for the radii on the border one has $R(N+iN) = N$.

If $R(z)$ is a solution to (2.1) and therefore defines some immersed circle patterns, then $\tilde{R}(z) = \frac{1}{R(z)}$ also solves (2.1). This reflects the fact that for any discrete conformal map f there is *dual discrete conformal map* f^* defined by (see [11])

$$f_{n+1,m}^* - f_{n,m}^* = -\frac{1}{f_{n+1,m} - f_{n,m}}, \quad f_{n,m+1}^* - f_{n,m}^* = \frac{1}{f_{n,m+1} - f_{n,m}}. \quad (1.5)$$

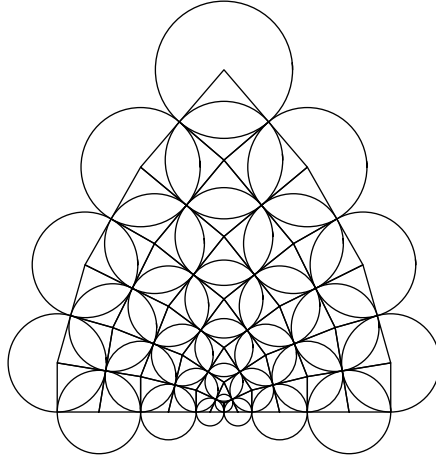


Figure 2. Discrete Z^2 .

The smooth limit of the duality (1.5) is

$$(f^*)' = -\frac{1}{f'}.$$

The dual of $f(z) = z^2$ is, up to a constant, $f^*(z) = \log z$. Motivated by this observation, we define the discrete logarithm as the discrete map dual to Z^2 , i.e. the map corresponding to the circle patterns with radii

$$R_{\text{Log}}(z) = \frac{1}{R_{Z^2}(z)},$$

where R_{Z^2} are the radii of the circles for Z^2 . Here one has $R_{\text{Log}}(0) = \infty$, i.e. the corresponding circle is a straight line. The corresponding constraint (1.2) can be also derived as a limit. Indeed, consider the map $g = \frac{2-c}{c}Z^c - \frac{2-c}{c}$. This map satisfies (1.1) and the constraint

$$cg_{n,m} + 2 - c = 2n \frac{(g_{n+1,m} - g_{n,m})(g_{n,m} - g_{n-1,m})}{(g_{n+1,m} - g_{n-1,m})} + 2m \frac{(g_{n,m+1} - g_{n,m})(g_{n,m} - g_{n,m-1})}{(g_{n,m+1} - g_{n,m-1})}.$$

Keeping in mind the limit procedure used to determine Z^2 , it is natural to define the discrete analogue of $\log z$ as the limit of g as $c \rightarrow +0$. The corresponding constraint becomes

$$1 = n \frac{(g_{n+1,m} - g_{n,m})(g_{n,m} - g_{n-1,m})}{(g_{n+1,m} - g_{n-1,m})} + m \frac{(g_{n,m+1} - g_{n,m})(g_{n,m} - g_{n,m-1})}{(g_{n,m+1} - g_{n,m-1})}. \quad (1.6)$$

Definition 4. ([2]) Log is the map $\text{Log} : \mathbb{Z}_+^2 \rightarrow \mathbb{R}^2 = \bar{\mathbb{C}}$ satisfying (1.1) and (1.6) with the initial conditions

$$\begin{aligned} \text{Log}(0,0) &= \infty, \quad \text{Log}(1,0) = 0, \quad \text{Log}(0,1) = i\pi, \\ \text{Log}(2,0) &= 1, \quad \text{Log}(0,2) = 1 + i\pi, \quad \text{Log}(1,1) = i\frac{\pi}{2}. \end{aligned}$$

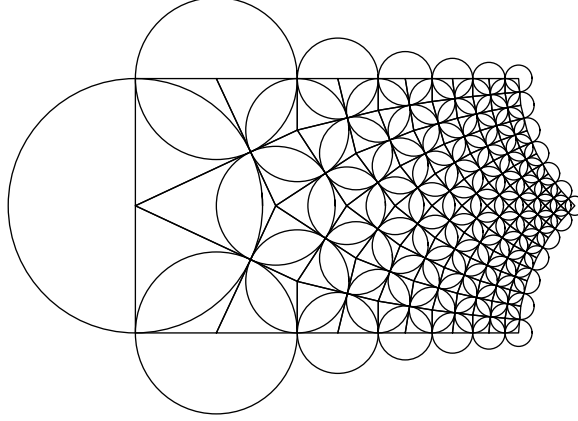


Figure 3. Discrete Log.

The circle patterns corresponding to the discrete conformal mappings Z^2 and Log were conjectured by O. Schramm and R. Kenyon (see [25]), in [2] it was proved that they are immersed.

2 Asymptotics of discrete z^c and $\log(z)$

As radius function for Schramm circle patterns $R(z)$ satisfies

$$R(z)^2 = \frac{\left(\frac{1}{R(z+1)} + \frac{1}{R(z+i)} + \frac{1}{R(z-1)} + \frac{1}{R(z-i)}\right) R(z+1)R(z+i)R(z-1)R(z-i)}{R(z+1) + R(z+i) + R(z-1) + R(z-i)} \quad (2.1)$$

Moreover, constraint (1.2) implies the following equations:

$$R(z)R(z+1)(-2M-c) + R(z+1)R(z+1+i)(2(N+1)-c) + R(z+1+i)R(z+i)(2(M+1)-c) + R(z+i)R(z)(-2N-c) = 0, \quad (2.2)$$

$$(N+M)(R(z)^2 - R(z+1)R(z-i))(R(z+i) + R(z+1)) + (M-N)(R(z)^2 - R(z+i)R(z+1))(R(z+1) + R(z-i)) = 0, \quad (2.3)$$

$$(N+M)(R(z)^2 - R(z+i)R(z-1))(R(z-1) + R(z-i)) + (M-N)(R(z)^2 - R(z-1)R(z-i))(R(z+i) + R(z-1)) = 0, \quad (2.4)$$

$$(N+M)(R(z)^2 - R(z+i)R(z-1))(R(z+1) + R(z+i)) + (N-M)(R(z)^2 - R(z+1)R(z+i))(R(z+i) + R(z-1)) = 0, \quad (2.5)$$

which are compatible with (2.1). Initial condition for Z^c is rewritten as

$$R(0) = 1, \quad R(i) = \tan \frac{c\pi}{4}. \quad (2.6)$$

Proposition 1. ([1]) Radius function $R(z)$ of Z^c satisfies

$$R(z) > 0, \quad (c-1)(R(z)^2 - R(z-i)R(z+1)) \geq 0 \quad (2.7)$$

in interior vertices of the quadrant

$$\mathbb{V} = \{z = N + iM : N, M \in \mathbb{Z}^2, M \geq |N|\}. \quad (2.8)$$

To treat z^c and $\log(z)$ on equal footing we agree that the case $c = 0$ in equations corresponds to discrete $\log(z)$. For the edges of unit squares with the vertices in \mathbb{V} defined by (2.8) we introduce X and Y via the radius ratios:

$$\frac{1 + X_{N,M}}{1 - X_{N,M}} = \frac{R_{N+1,M}}{R_{N,M}}, \quad \frac{1 + Y_{N,M}}{1 - Y_{N,M}} = \frac{R_{N,M}}{R_{N,M-1}}. \quad (2.9)$$

In these variables equations (2.3) and (2.2) read as:

$$(M - N) \frac{X_{N,M} + Y_{N,M+1}}{1 - X_{N,M}Y_{N,M+1}} + (M + N) \frac{X_{N,M} - Y_{N,M}}{1 + X_{N,M}Y_{N,M}} = 0, \quad (2.10)$$

$$(M - N) \frac{Y_{N,M+1} - X_{N,M}}{1 - X_{N,M}Y_{N,M+1}} + (M + N + 1) \frac{X_{N,M+1} + Y_{N,M+1}}{1 + X_{N,M+1}Y_{N,M+1}} = c - 1. \quad (2.11)$$

Moreover, X and Y satisfy

$$\frac{X_{N,M} + Y_{N,M+1}}{1 - X_{N,M}Y_{N,M+1}} = \frac{X_{N-1,M} + Y_{N,M}}{1 - X_{N-1,M}Y_{N,M}}, \quad (2.12)$$

$$\frac{X_{N,M} + Y_{N+1,M+1}}{1 + X_{N,M}Y_{N+1,M+1}} = \frac{X_{N,M+1} + Y_{N,M+1}}{1 + X_{N,M+1}Y_{N,M+1}}, \quad (2.13)$$

where (2.12) is equivalent to (2.1) and (2.13) is the compatibility condition.

Conditions (2.7), which hold for discrete z^c and $\log(z)$, turn out to be so restrictive for the corresponding solutions $X_{N,M}, Y_{N,M}$ of (2.10), (2.11), that they allow one to compute their asymptotic behavior. For definiteness we consider the case $c > 1$.

Lemma 1. For the solution $X_{N,M}, Y_{N,M}$ of (2.10), (2.11) corresponding to z^c in \mathbb{V} with $c > 1$ holds true:

$$-\frac{c-1}{M-N} \leq X_{N,M} \leq \frac{c-1}{M+N} \quad \text{and} \quad 0 \leq Y_{N,M+1} \leq \frac{c-1}{M+N} + \frac{2(c-1)}{M-N}. \quad (2.14)$$

Proof. Note that for the studied solutions $R \geq 0$ and therefore $-1 \leq X_{N,M} \leq 1$ and $-1 \leq Y_{N,M} \leq 1$. Let us denote for brevity $X_{N,M}$ by X , $X_{N,M+1}$ by \bar{X} , $Y_{N,M}$ by Y , $Y_{N,M+1}$ by \bar{Y} , and $R_{N,M}$ by R_1 , $R_{N+1,M}$ by R_2 , $R_{N+1,M+1}$ by R_3 , $R_{N,M+1}$ by R_4 , $R_{N,M-1}$ by R_5 . Then (2.7) together with (2.3), (2.4), (2.5) imply

$$R_1^2 \leq R_2R_4, \quad R_1^2 \geq R_2R_5, \quad R_3^2 \geq R_2R_4, \quad R_4^2 \geq R_1R_3, \quad R_2^2 \leq R_1R_3. \quad (2.15)$$

Rewriting the first inequality as $\frac{R_1}{R_2} \leq \frac{R_4}{R_1}$ and taking into account $1 + X \geq 0$ we have

$$X + \bar{Y} \geq 0. \quad (2.16)$$

Similarly the second inequality of (2.15) infers $Y - X \geq 0$ or after shifting

$$\bar{Y} \geq \bar{X}. \quad (2.17)$$

Combining the first and the third inequalities of (2.15) one gets $R_3 \geq R_1$ or

$$\bar{X} + \bar{Y} \geq 0, \quad (2.18)$$

which is equivalent to

$$X + \bar{Y}_{N+1, M+1} \geq 0 \quad (2.19)$$

due to (2.13). Similarly the fourth and the fifth imply $R_4 \geq R_2$ or

$$\bar{Y} \geq X. \quad (2.20)$$

Comparing (2.18) with (2.17) one gets $\bar{Y} \geq 0$. Rewriting (2.2) as

$$\frac{R_3}{R_1} = \frac{(2N + c)R_4 + (2M + c)R_2}{(2(M + 1) - c)R_4 + (2(N + 1) - c)R_2}$$

and taking into account $\frac{R_3}{R_1} \geq 1$ and $c \leq 2$ we can estimate $\frac{R_2}{R_4}$ in \mathbb{V} :

$$\frac{R_2}{R_4} \geq \frac{1 - \frac{c-1}{M-N}}{1 + \frac{c-1}{M-N}}$$

which reads as

$$\bar{Y} - X \leq (1 - \bar{Y}X)\epsilon_{N,M}, \quad \epsilon_{N,M} = \frac{c-1}{M-N}.$$

As $(1 - \bar{Y}X) \leq 2$ we have

$$\bar{Y} - X \leq 2\epsilon_{N,M}. \quad (2.21)$$

Similarly we can solve (2.2) with respect to $\frac{R_4}{R_2}$:

$$\frac{R_4}{R_2} = \frac{(2M + c)R_1 + (-2(N + 1) + c)R_3}{(-2N - c)R_1 + (2(M + 1) - c)R_3}.$$

Note that for $M > N$ holds $(-2N - c)R_1 + (2(M + 1) - c)R_3 \geq 0$ as $R_3 \geq R_1$, and $c \leq 2$. This together with $R_4 \geq R_2$ and $(1 - \bar{Y}\bar{X}) \leq 2$ gives after some calculations

$$\bar{X} + \bar{Y} \leq 2\delta_{N,M}, \quad (2.22)$$

with $\delta_{N,M} = \frac{c-1}{M+N+1}$. Now inequalities (2.16),(2.20),(2.21) yield

$$-\epsilon_{N,M} \leq X$$

and (2.17),(2.18),(2.22) imply

$$\bar{X} \leq \delta_{N,M},$$

which gives the first inequality of (2.14) after shifting backwards from $M + 1$ to M . Using (2.21) we easily get the second one. ■

The estimations obtained allow one to find asymptotic behavior for the radius-function in M -direction.

Theorem 1. *For the solution $R_{N,M}$ of (2.3),(2.2) corresponding to discrete Z^c and Log in \mathbb{V} holds true:*

$$R_{N_0,M} \simeq K(c)M^{c-1} \quad \text{as } M \rightarrow \infty, \quad (2.23)$$

with constant $K(c)$ independent of N_0 .

Proof. Because of the duality

$$R_{z^{2-c}} = \frac{1}{R_{z^c}}$$

it is enough to consider the case $c > 1$. Let us introduce $n = M - N_0$, $x_n = X_{N_0,M}$, $y_n = Y_{N_0,M}$. Then for large n Lemma 1 allows one to rewrite equations (2.10),(2.11) as

$$\begin{aligned} x_{n+1} &= 5x_n - 2y_n + \frac{c-1}{n} + U_{N_0}(n), \\ y_{n+1} &= -2x_n + y_n + V_{N_0}(n), \end{aligned} \quad (2.24)$$

where $U_{N_0}(n)$ and $V_{N_0}(n)$ are defined by discrete Z^c and satisfy

$$U_{N_0}(n) < C_1 \frac{1}{n^2}, \quad V_{N_0}(n) < C_2 \frac{1}{n^2}, \quad (2.25)$$

for natural n and some constants C_1, C_2 (depending on N_0). Thus the solution (x_n, y_n) , corresponding to discrete Z^c is a special solution of linear non-homogeneous system (2.24) having the order $\frac{1}{n}$ for large n . The eigenvalues of the system matrix are positive numbers $\lambda_1 = 3 - 2\sqrt{2} < 1$ and $\lambda_2 = 3 + 2\sqrt{2} > 1$. In the diagonal form system (2.24) takes the form:

$$\varphi_{n+1} = A\varphi_n + s_n + r_n \quad (2.26)$$

with

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \varphi_n = \begin{pmatrix} 1 & 1 \\ 1 + \sqrt{2} & 1 - \sqrt{2} \end{pmatrix}^{-1} \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \quad s_n = \begin{pmatrix} 2 - \sqrt{2} \\ 2 + \sqrt{2} \end{pmatrix} \frac{c-1}{4n},$$

and $|r_n| \leq \frac{G}{n^2}$ for some G . Looking for the solution in the form

$$\varphi_n = A^n c_n$$

we have the following recurrent formula for c_n :

$$c_{n+1} = A^{-n}(s_n + r_n) + c_n.$$

Integrating one gets

$$c_{n+1} = c_1 + A^{-2}(s_1 + r_1) + A^{-3}(s_2 + r_2) + \dots + A^{-1-n}(s_n + r_n). \quad (2.27)$$

Estimation of components of $\varphi_n = (a_n, b_n)^T$ implies

$$a_n = \frac{(2 - \sqrt{2})(c-1)}{4(1-\lambda_1)} \frac{1}{n} + O\left(\frac{1}{n^2}\right), \quad (2.28)$$

$$b_n = \lambda_2^n \left(b_1 + \frac{(2 + \sqrt{2})(c-1)}{4\lambda_2} \ln \frac{\lambda_2}{\lambda_2 - 1} + F_3(\lambda_2) \right) - \frac{(2 + \sqrt{2})(c-1)}{4(\lambda_2 - 1)} \frac{1}{n} + O\left(\frac{1}{n^2}\right). \quad (2.29)$$

(This estimating is purely technical and is relegated to Appendix.) As $b_n \rightarrow 0$ with $n \rightarrow \infty$ and $\lambda_2 > 1$ one deduces that the coefficient by λ_2^n vanishes. For original variables x_n, y_n the found asymptotics (2.28),(2.29) have especially simple form:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \frac{c-1}{2n} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O\left(\frac{1}{n^2}\right). \quad (2.30)$$

Asymptotic (2.30), the second equation (2.9) and

$$R_{N_0, N_0+n} = R_{N_0, N_0} \prod_{k=1}^n \frac{2k + (c-1)}{2k - (c-1)}$$

imply (2.23). The independence of $K(c)$ on N_0 easily follows from the first equation (2.9) and $x_n \rightarrow 0$. ■

3 Discussion and concluding remarks

Equation (1.2) allows one to “integrate” asymptotics (2.23) to get

$$Z^c(n_0 + n, m_0 + n) \simeq e^{c\pi i/4} K(c) n^c \quad \text{as } n \rightarrow \infty. \quad (3.1)$$

Thus the circles of Z^c not only cover the whole infinite sector with the angle $c\pi/2$ but the circle centers and intersection points mimic smooth map $z \rightarrow z^c$ also asymptotically. Moreover, $R(z)$, being analogous to $|f'(z)|$ of the corresponding smooth map, has the “right” asymptotics as well.

Discrete map Z^c is defined via constraint (1.2) which is isomonodromy condition for some linear equation. This seems to be rather far-fetched approach. It would be more naturally to define Z^c via circle patterns in a pure geometrical way: square grid Z^c with $0 < c < 2$ is an embedded infinite square grid circle patterns, the circles $C(z)$ being labeled by

$$\mathbb{V} = \{z = N + iM : N, M \in \mathbf{Z}^2, M \geq |N|\},$$

satisfying the following conditions:

- 1) the circles cover the infinite sector with the angle $c\pi/2$,
- 2) the centers of the border circles $C(N + iN)$ and $C(-N + iN)$ lie on the borders of this sector.

Conjecture 1. *Up to re-scaling there is unique square grid Z^c .*

A naive method to look for so defined discrete Z^c is to start with some equidistant $f_{n,0} \in \mathbb{R}$, $f_{0,m} \in i^c \mathbb{R}$:

$$|f_{2n,0} - f_{2n+1,0}| = |f_{2n,0} - f_{2n-1,0}| = |f_{0,2n} - f_{0,2n+1}| = |f_{0,2n} - f_{0,2n-1}|$$

and then compute $f_{n,m}$ for any $n, m > 0$ using equation (1.1). Such $f_{n,m}$ determines some circle patterns. But so determined map has a not very nice behavior (see an example in Fig. 4). For $c = 2/k$, $k \in \mathbb{N}$ the proof of Conjecture 1 easily follows from the rigidity

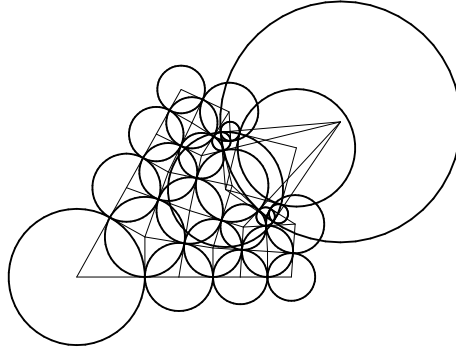


Figure 4. Non-immersed discrete conformal map

results obtained in [16].

Infinite embedded circle patterns define some (infinite) convex ideal polyhedron in \mathbb{H}^3 and the group generated by inversions in its faces. The known results (see for example [23]) imply the conjecture claim for rational c , but seem to be inapplicable for irrational as the group generated but the circles and the sector borders is not discrete any more. Unfortunately, the rigidity results for finite polyhedra [21] does not seem to be carried over to infinite case by induction.

One can also consider solutions of (1.1) subjected to (1.2) where n and m are not integer. It turned out that there exist initial data, so that the corresponding solution define immersed circle patterns. (The equations for radii are the same and therefore compatible. Existence of positive solution is provable by the arguments for the corresponding discrete Painlevé equation as in [2].) It is natural to call such circle patterns discrete map $z \rightarrow (z + z_0)^c$. In this case it can not be defined pure geometrically as the centers of border circles are not collinear. Therefore alternative approach to define $(Z + Z_0)^c$ would be the asymptotic behavior of the corresponding $R(z)$.

One can discard the restriction $0 < c < 2$ (as well as some circles $C(z)$ with small z) to define discrete Z^K for natural $K > 2$ (see [2]). All the asymptotic results obtained so far can be carried over on this case as well as on hexagonal Z^c with constant intersection angles defined in [8] since the governing equations are essentially the same ([3]).

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4 Appendix

For components of $\varphi_n = (a_n, b_n)^T$ formula (2.27) implies

$$\begin{aligned} a_n &= \lambda_1^n \left(a_1 + \frac{(2-\sqrt{2})(c-1)}{4} \left(\frac{1}{\lambda_1^2} + \frac{1}{2\lambda_1^3} + \dots + \frac{1}{(n-1)\lambda_1^n} \right) + \left(\frac{G_1}{\lambda_1^2} + \frac{G_2}{2^2\lambda_1^3} + \dots + \frac{G_{n-1}}{(n-1)^2\lambda_1^n} \right) \right) = \\ &= a_1\lambda_1^n + \frac{(2-\sqrt{2})(c-1)}{4} \left(\frac{1}{n-1} + \frac{\lambda_1}{(n-2)} + \dots + \lambda_1^{n-2} \right) + \left(\frac{G_1}{\lambda_1^2} + \frac{G_2}{2^2\lambda_1^3} + \dots + \frac{G_{n-1}}{(n-1)^2\lambda_1^n} \right) \end{aligned}$$

with some limited sequence G_n : $|G_n| \leq G$. The first sum, corresponding to s_n , is estimated as follows:

$$\begin{aligned} &\frac{1}{n-1} \left(1 + \frac{n-1}{n-2}\lambda_1 + \frac{n-1}{n-3}\lambda_1^2 \dots + (n-1)\lambda_1^{n-2} \right) = \\ &= \frac{1}{n-1} \left(1 + \lambda_1 + \lambda_1^2 \dots + \lambda_1^{n-2} \right) + \frac{\lambda_1}{n-1} \left(\frac{1}{n-2} + \frac{2}{n-3}\lambda_1 \dots + (n-2)\lambda_1^{n-3} \right) = \\ &= \frac{1}{n-1} \frac{1-\lambda_1^{n-1}}{1-\lambda_1} + \frac{\lambda_1}{(n-1)(n-2)} \left(1 + \frac{2(n-2)}{n-3}\lambda_1 + \frac{3(n-2)}{n-4}\lambda_1^2 \dots + (n-2)^2\lambda_1^{n-3} \right) = \\ &= \frac{1}{n-1} \frac{1-\lambda_1^{n-1}}{1-\lambda_1} + \frac{\lambda_1}{(n-1)(n-2)} F_1(n, \lambda_1) \end{aligned}$$

where

$$F_1(n, \lambda_1) < (1 + 2 \times 3\lambda_1 + 3 \times 4\lambda_1^2 + \dots) = \frac{2}{(1-\lambda_1)^3} - 1,$$

as $|\lambda_1| < 1$ and $(n-2)/(n-k) < k$ for $k \leq n-1$. The second sum is estimated by

$$\begin{aligned} &\frac{G}{(n-1)^2} \left(1 + \frac{(n-1)^2}{(n-2)^2}\lambda_1 + \frac{(n-1)^2}{(n-3)^2}\lambda_1^2 + \dots + \frac{(n-1)^2}{(n-k)^2}\lambda_1^{k-1} + \dots + (n-1)^2\lambda_1^{n-2} \right) = \\ &= \frac{G}{(n-1)^2} (1 + \lambda_1 + \lambda_1^2 + \dots + \lambda_1^{n-2} + \frac{(n-1)^2 - (n-2)^2}{(n-2)^2}\lambda_1 + \frac{(n-1)^2 - (n-3)^2}{(n-3)^2}\lambda_1^2 + \dots + \\ &+ \frac{(n-1)^2 - (n-k)^2}{(n-k)^2}\lambda_1^{k-1} + \dots + ((n-1)^2 - 1)\lambda_1^{n-2}) = \frac{G}{(n-1)^2} \left(\frac{1-\lambda_1^{n-1}}{1-\lambda_1} + \frac{(2(n-2)+1)}{(n-2)^2}\lambda_1 + \right. \\ &+ \dots + \frac{(2(n-k)(k-1)+(k-1)^2)}{(n-k)^2}\lambda_1^{k-1} + \dots + (2(n-2) + (n-2)^2)\lambda_1^{n-2} \left. \right) \leq \\ &\leq \frac{G}{(n-1)^2} \left(\frac{1}{1-\lambda_1} + 2\lambda_1(1 + 2\lambda_1 + 3\lambda_1^2 + \dots) + \lambda_1(1 + 2^2\lambda_1 + 3^2\lambda_1^2 + \dots) \right) = \frac{G}{(n-1)^2} F_2(\lambda_1). \end{aligned}$$

Summing up we infer estimation (2.28) for a_n . For the second component b_n one has

$$b_n = b_1\lambda_2^n + \frac{(2+\sqrt{2})(c-1)}{4}\lambda_2^n \left(\frac{1}{\lambda_2^2} + \frac{1}{2\lambda_2^3} + \dots + \frac{1}{(n-1)\lambda_2^n} \right) + \lambda_2^n \left(\frac{H_1}{\lambda_2^2} + \frac{H_2}{2^2\lambda_2^3} + \dots + \frac{H_{n-1}}{(n-1)^2\lambda_2^n} \right)$$

with some limited sequence H_n : $|H_n| \leq H$. The first sum in the previous formula is estimated as

$$\begin{aligned} &\lambda_2^{n-1} \int_0^{\frac{1}{\lambda_2}} (1 + x + x^2 \dots + x^{n-2}) dx = \lambda_2^{n-1} \int_0^{\frac{1}{\lambda_2}} \frac{1-x^{n-1}}{1-x} dx = \\ &= \lambda_2^{n-1} \left(\int_0^{\frac{1}{\lambda_2}} \frac{1}{1-x} dx - \int_0^{\frac{1}{\lambda_2}} (x^{n-1} + x^n + \dots) dx \right) = \\ &= \lambda_2^{n-1} \left(-\ln(1-x) \Big|_0^{\frac{1}{\lambda_2}} - \frac{1}{\lambda_2^n} \left(\frac{1}{n} + \frac{1}{(n+1)\lambda_2} + \frac{1}{(n+2)\lambda_2^2} \dots \right) \right) = \\ &= \lambda_2^{n-1} \ln \frac{\lambda_2}{\lambda_2-1} - \frac{1}{n\lambda_2} \left(1 + \frac{n}{(n+1)\lambda_2} + \frac{n}{(n+2)\lambda_2^2} \dots \right) = \\ &= \lambda_2^{n-1} \ln \frac{\lambda_2}{\lambda_2-1} - \frac{1}{n\lambda_2} \left(\left(1 + \frac{1}{\lambda_2} + \frac{1}{\lambda_2^2} + \dots \right) - \frac{1}{\lambda_2} \left(\frac{1}{(n+1)} + \frac{2}{(n+2)\lambda_2} + \dots + \frac{k}{(n+k)\lambda_2^{k-1}} + \dots \right) \right) = \\ &= \lambda_2^n \frac{1}{\lambda_2} \ln \frac{\lambda_2}{\lambda_2-1} - \frac{1}{n(\lambda_2-1)} + \frac{S_n(\lambda_2)}{n(n+1)\lambda_2^2}, \end{aligned}$$

Where

$$S_n(\lambda_2) = \left(1 + \frac{2(n+1)}{(n+2)\lambda_2} + \dots + \frac{k(n+1)}{(n+k)\lambda_2^{k-1}} + \dots \right) < \left(1 + \frac{2}{\lambda_2} + \dots + \frac{k}{\lambda_2^{k-1}} + \dots \right) = \frac{\lambda_2^2}{(\lambda_2-1)^2}.$$

For the second sum in the formula for b_n one has

$$\frac{1}{\lambda_2^2} \left(H_1 + \frac{H_2}{2^2\lambda_2} + \dots + \frac{H_{n-1}}{(n-1)^2\lambda_2^{n-2}} \right) = F_3(\lambda_2) - \frac{1}{\lambda_2^2} \left(\frac{H_n}{n^2\lambda_2^{n-1}} + \frac{H_{n+1}}{(n+1)^2\lambda_2^n} + \dots + \frac{H_{n+k}}{(n+k)^2\lambda_2^{n+k-1}} + \dots \right)$$

where

$$F_3(\lambda_2) = \frac{1}{\lambda_2^2} \left(H_1 + \frac{H_2}{2^2\lambda_2} + \dots + \frac{H_{k-1}}{(k-1)^2\lambda_2^{k-2}} + \dots \right)$$

and the second sum is estimated from above as

$$\frac{H}{\lambda_2^{n+1}n^2} \left(1 + \frac{n^2}{(n+1)^2\lambda_2} + \frac{n^2}{(n+2)^2\lambda_2^2} + \dots + \frac{n^2}{(n+k)^2\lambda_2^k} + \dots \right) < \frac{H}{\lambda_2^{n+1}n^2} \left(1 + \frac{1}{\lambda_2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_2^3} + \dots \right),$$

which implies (2.29).

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