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Searching for CAC-maps

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Abstract

For two-dimensional lattice equations the standard definition of integrability is that it should be possible to extend the map consistently to three dimensions, i.e., that it is "consistent around a cube" (CAC). Recently Adler, Bobenko and Suris conducted a search based on this principle, together with the additional assumptions of symmetry and "the tetrahedron property". We present here results of a search for CAC lattice equations assuming also the same symmetry properties, but not the tetrahedron property.

1 Introduction

In this paper we consider integrable difference equations defined on a 2-dimensional rectangular lattice (see Figure 1, which also introduces the notation for shifted positions). In general the map is given in terms of a multi-linear equation relating the four corner values of a plaquette

$$
\begin{aligned}
&K\, x x_{[1]} x_{[2]} x_{[12]} + L_1\, x x_{[1]} x_{[2]} + L_2\, x x_{[1]} x_{[12]} + L_3\, x x_{[2]} x_{[12]} + L_4\, x_{[1]} x_{[2]} x_{[12]} \\
&+ P_1\, x x_{[1]} + P_2\, x_{[1]} x_{[2]} + P_3\, x_{[2]} x_{[12]} + P_4\, x_{[12]} x + P_5\, x x_{[2]} + P_6\, x_{[1]} x_{[12]} \\
&+ Q_1\, x + Q_2\, x_{[1]} + Q_3\, x_{[2]} + Q_4\, x_{[12]} + U \\
&\qquad \equiv Q_{12}(x, x_{[1]}, x_{[2]}, x_{[12]}; p_1, p_2) = 0.
\end{aligned}
\tag{1.1}
$$

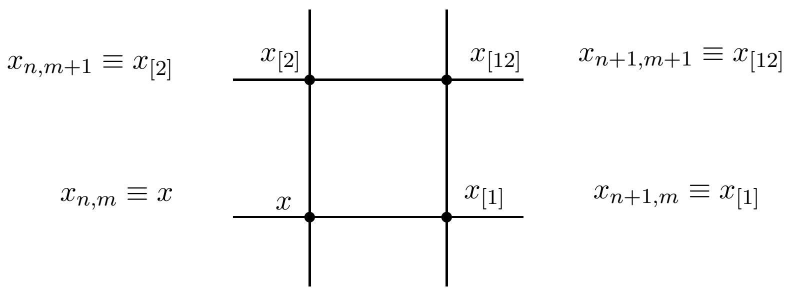

Figure 1. The lattice map is defined on each elementary square of the infinite lattice.

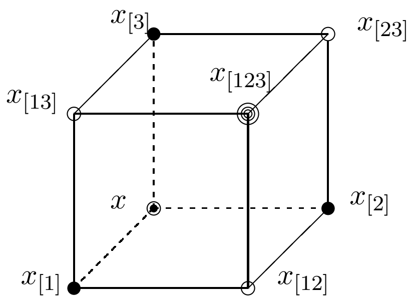

Figure 2. Given the values at the black circles, one should get a unique value for $x_{[123]}$, even though there are three possible way to compute it.

Here the coefficients K, L_ν, P_ν, Q_ν, U may depend on the two spectral parameters p_1, p_2. (A parameter is called *spectral*, if it is associated with a spatial direction.) If any 3 of the corner values are given then the fourth one can be obtained as a rational expression of the other three. One can therefore propagate any one-dimensional staircase-like initial value to cover the whole plane[1].

How should integrability be defined for such maps? The "Consistency Around the Cube" -property (CAC) is defined as follows: Adjoin a third direction (therefore assuming $x = x_{n,m,k}$) and use the same map (but with different spectral parameters) also in planes corresponding to indices 1,3 and 2,3, furthermore, on the parallel shifted planes one uses identical maps. If we assume that the values x, $x_{[1]}$, $x_{[2]}$, $x_{[3]}$ at black circles in Figure 2 are given, then the values at open circles are uniquely determined using the relevant map, but the value at $x_{[123]}$ can be computed in 3 different ways, and they must give the same result. That is, after computing $x_{[12]}$ from $Q_{12}(x, x_{[1]}, x_{[2]}, x_{[12]}; p_1, p_2) = 0$, $x_{[23]}$ from $Q_{23}(x, x_{[2]}, x_{[3]}, x_{[23]}; p_2, p_3) = 0$, and $x_{[31]}$ from $Q_{31}(x, x_{[3]}, x_{[1]}, x_{[31]}; p_3, p_1) = 0$, then $x_{[123]}$ computed from

$$
\begin{aligned}
&Q_{12}(x_{[3]}, x_{[31]}, x_{[23]}, x_{[123]}; p_1, p_2) = 0, &&\text{or}\\
&Q_{23}(x_{[1]}, x_{[12]}, x_{[31]}, x_{[123]}; p_2, p_3) = 0, &&\text{or}\\
&Q_{31}(x_{[2]}, x_{[23]}, x_{[12]}, x_{[123]}; p_3, p_1) = 0,
\end{aligned}
$$

should be the same. The functions Q_{ij} could be in principle be different, but are usually assumed to be identical.

The general notion that integrability is associated with a consistent extension to higher dimension can probably be traced back at least to Bianchi. More recently such observations were made in connection with the sine-Gordon system in [2], which contains the very diagram of Figure 2. In that context the diagram represented the observed property of some functions derived by Moutard transformations. The next step of abstraction was to consider CAC as a property of abstract maps defined on a lattice. For a generalized cross-ratio equation this was shown to hold in [3] and for the lattice MKdV [4]. In [5] it was shown for several other systems, and at the same time it was noted in [6] that CAC actually provides a Lax pair for the lattice equation. Any property strongly associated with integrability can be used to search for more integrable systems (in the past this has been done, e.g., with symmetries, conserved quantities, and the existence of multisoliton solutions with elastic scattering). In [7] such a search was performed based on the CAC property, with one extra assumption of *tetrahedron property*: By construction the $x_{[123]}$ computed above can only depend on x, $x_{[1]}$, $x_{[2]}$, $x_{[3]}$, and it is said that the system satisfies the tetrahedron property if $x_{[123]}$ actually does *not* depend on x.

The purpose of the present paper is to check the importance of this additional tetrahedron property: We will search for models having the same symmetry properties as were used in [7] but we do not assume the tetrahedron property. It has already been observed that there are CAC-systems without the tetrahedron property [8] but that system did not have the same symmetry properties as the models classified in [7]. Of course it would be more interesting to find all CAC-systems without *any* assumptions about symmetries, but such a general problem quickly leads to enormous systems of algebraic equations for the coefficients, and unfortunately they cannot be solved with presently available tools.

2 Searching for integrable lattices

2.1 Previous work

In [7] the consistency equations were solved in the case where the map Q was the same on all planes, and under the two additional assumptions of symmetry

$$\begin{aligned} &Q(x, x_{[1]}, x_{[2]}, x_{[12]}; p_1, p_2) = \varepsilon Q(x, x_{[2]}, x_{[1]}, x_{[12]}; p_2, p_1) \\ &= \sigma Q(x_{[1]}, x, x_{[12]}, x_{[2]}; p_1, p_2), \quad \varepsilon, \sigma = \pm 1, \end{aligned} \tag{2.1}$$

and tetrahedron property. With these assumptions the authors were able to get a full classification resulting with 9 integrable maps, they are quoted below as (A1-2), (H1-3) and (Q1-4). These lattice maps are not independent, but rather there is a fundamental one (Q4) from which all others can be obtained by various reductions.

The tetrahedron assumption is indeed satisfied by most of the well known models, but one can nevertheless ask whether it is a fundamental or essential property and whether there are any nontrivial models that do not satisfy it. It should be immediately observed that the more or less trivial models

$$x_{[1]}x_{[2]} - x x_{[12]} = 0, \text{and } y - y_{[1]} - y_{[2]} + y_{[12]} = 0, \tag{2.2}$$

($x = e^y$) do have CAC property but *not* the tetrahedron property, in fact one quickly finds that

$$x_{[123]} = x_{[1]}x_{[2]}x_{[3]}x^{-2}, \text{ and } y_{[123]} = y_{[1]} + y_{[2]} + y_{[3]} - 2y, \tag{2.3}$$

respectively. A nontrivial extension of this was given in [8].

2.2 The present setting

In this paper we take the symmetry assumption of (2.1) with $\varepsilon, \sigma = 1$, that is

$$\begin{aligned} Q(x, x_1, x_2, &x_{12}; r, s) := \\ &k(r, s)\, x x_1 x_2 x_{12} + l(r, s)\,(x x_1 x_2 + x x_1 x_{12} + x x_2 x_{12} + x_1 x_2 x_{12}) \\ &+ p_1(r, s)\,(x x_1 + x_{12} x_2) + p_2(r, s)\,(x x_{12} + x_1 x_2) + p_5(r, s)\,(x x_2 + x_1 x_{12}) \\ &+ q_1(r, s)\,(x + x_1 + x_{12} + x_2) + u(r, s) = 0. \end{aligned} \tag{2.4}$$

and require that the system satisfies the CAC-condition discussed above. We do not assume the "tetrahedron property". For notational convenience we call $r = p_1$, $s = p_2$, $t = p_3$.

We organized our search by the number of nonzero functions in the set of seven functions $k, l, p_1, p_2, p_5, q, u$. The same map was used on all planes of the CAC-cube.

2.3 Invariances

Lattice maps are in general invariant under rational linear (Moebius) transformations with constant coefficients. At the beginning we do not know which coefficients in Q depend on the spectral parameters and how, and therefore the full Moebius analysis can only be done after the results have been obtained. Furthermore only those transformations should be used which preserve the general form of the ansatz (2.4). The transformations that will be used most are

- Translation T: $x_* \to x_* + \lambda$
- Inversion I: $x_* \to 1/x_*$.
- Scaling: $x_* \to c\, x_*$.

One can sometimes also use transformations that only operate on a sublattice, that is on x_{nm} when $n+m$ is either even or odd. However, such transformations may conflict with the assumption that the same map is used on parallel planes. To be more precise, an alternate point transformation σ is only allowed if $Q(x, \sigma(x_1), \sigma(x_2), x_{12}; r, s) \propto Q(\sigma(x), x_1, x_2, \sigma(x_{12}); r, s)$. The following cases are often useful

- Inversion on alternate points Ia: $x \to x$, $x_1 \to 1/x_1$, $x_2 \to 1/x_2$, $x_{12} \to x_{12}$.
- Sign change on alternate points Sa: $x \to x$, $x_1 \to -x_1$, $x_2 \to -x_2$, $x_{12} \to x_{12}$.

Two examples of the above are: Sa takes (A1) of [7] to (Q1), while Ia takes (A2) to the $\delta = 0$ special case of (Q3). Note, however, that (Q3) with $\delta \neq 0$ does not have the symmetry property required for applying the sublattice transformation.

2.4 A typical condition

In the analysis of the ensuing equations one often arrives to the equation

$$A(r)(b(s) - c(t)) + B(s)(c(t) - a(r)) + C(t)(a(r) - b(s)) = 0, \tag{2.5}$$

in other words

$$\begin{vmatrix} 1 & a(r) & A(r) \\ 1 & b(s) & B(s) \\ 1 & c(t) & C(t) \end{vmatrix} = 0. \tag{2.6}$$

The solution of this is

$$\mu A(r) + \nu a(r) + \lambda = 0,\ \mu B(s) + \nu b(s) + \lambda = 0,\ \mu C(t) + \nu c(t) + \lambda = 0, \tag{2.7}$$

for some μ, ν, λ not all zero. There are three different cases:

1. if μ, ν are nonzero. Then we can take $A(r) = \alpha a(r) + \beta$, $B(s) = \alpha b(s) + \beta$, $C(t) = \alpha c(t) + \beta$, where α is nonzero. If the functions $a(r), b(s), c(t)$ are free they are often renamed as r, s, t.

2. If $\mu = 0$ then we must have ν, λ nonzero and then $a(r) = b(s) = c(t) = -\lambda/\nu =: \alpha$, a constant. If the functions $A(r), B(s), C(t)$ are free they are often renamed as r, s, t.

3. If $\nu = 0$ then we must have μ, λ nonzero and then $A(r) = B(s) = C(t) = -\lambda/\mu =: \alpha$, a constant. If the functions $a(r), b(s), c(t)$ are free they are often renamed as r, s, t.

3 Results

The computations were done following the definition of the CAC-property: An expression was computed for $x_{[123]}$ in three ways and the results subtracted, coefficients of x, $x_{[1]}$, $x_{[2]}$, $x_{[3]}$ were then extracted and the equations solved, step by step. All computations were done using REDUCE [9].

In presenting our results below we use the notation: N.Mx, where N=number of nonzero functions in Q; M=the number of the case, as classified by the choice of nonzero functions; x=t: the map satisfies tetrahedron property, x=n: it does not satisfy. The label is in italics if it can be transformed to something that has appeared earlier in the list, through the basic transformations listed in Section 2.3, in which case the transformation is given.

Since r, s, t below are free parameters, the substitution particular values (such as 1) to them is not considered as a new solution.

Many maps are connected by singular limits, but here such models are considered new.

1 nonzero function

1.1n $Q = x + x_1 + x_2 + x_{12}$

1.2n $Q = xx_{12} + x_1x_2$

1.3n $Q = xx_1x_2 + xx_1x_{12} + xx_2x_{12} + x_1x_2x_{12}$. Transforms to 1.1.n by I.

2 nonzero functions

2.1n $Q = (x + x_1 + x_2 + x_{12}) + 1$, Transforms to 1.1n by T.

2.2t $Q = s(xx_1 + x_{12}x_2) - t(xx_2 + x_1x_{12})$. (H3) with $\delta = 0$.

2.3n $Q = (xx_1x_2 + xx_1x_{12} + xx_2x_{12} + x_1x_2x_{12}) + (x + x_1 + x_{12} + x_2)$. Singular scaling limits lead to 1.1n or 1.3n

2.4n $Q = xx_1x_2x_{12} + 1$. Transforms to 1.2n by Ia.

2.5n $Q = xx_1x_2x_{12} + (xx_1x_2 + xx_1x_{12} + xx_2x_{12} + x_1x_2x_{12})$. Transforms to 2.1n by I.

3 nonzero functions

3.1n $Q=(xx_{12}+x_1x_2)+c(x+x_1+x_2+x_{12})+2c^2$, Transforms by T to 1.2n.

3.2n $Q=xx_1x_2x_{12}+d(xx_1x_2+xx_1x_{12}+xx_2x_{12}+x_1x_2x_{12})+2d^2(xx_{12}+x_1x_2)$, Transforms to 3.1n by I and scaling.

3.3t $Q=r(x_1x_{12}+xx_2)-s(x_{12}x_2+xx_1)+\beta(r/s-s/r)$, or by $s\to 1/s$, $r\to 1/r$
$Q=s(x_1x_{12}+xx_2)-r(x_{12}x_2+xx_1)+\delta(r^2-s^2)$. (H3)
The special case $\delta=0$ is 2.2t. The singular limit $r=1+\epsilon r', s=1+\epsilon s', \beta=\beta'/(2\epsilon)$, $\epsilon\to 0$ yields 3.3t':

3.3t' $Q=(x_1x_{12}+xx_2)-(x_{12}x_2+xx_1)+\delta(r-s)$. (H1)
alternatively $Q=(x-x_{12})(x_2-x_1)+\delta(r-s)$.

3.4 Obtained form 3.3 by transform I.

3.5t $Q=\sinh(s)(xx_1+x_{12}x_2)-\sinh(r)(xx_2+x_1x_{12})+(\cosh(r)-\cosh(s))(xx_{12}+x_1x_2)$. (Q3) with $\delta=0$. The singular limit $r,s\to\infty$ with redefinition $e^r=r'$ leads to 3.5t':

3.5t' $Q=s(xx_1+x_{12}x_2)-r(xx_2+x_1x_{12})+(r-s)(xx_{12}+x_1x_2)$. (Q1) or (A1) with $\delta=0$.

4 nonzero functions

4.1t1 $Q=(x-x_{12})(x_1-x_2)+(r-s)(x+x_1+x_2+x_{12})+(r^2-s^2)$. (H2)

4.1t2 $Q=s(x_1x_{12}+xx_2)-r(x_{12}x_2+xx_1)+c(r-s)(x+x_1+x_2+x_{12})+2c^2(s-r)+\delta(r^2-s^2)$, reduces by translation to 3.3t.

4.2 Obtained from 4.1 by I

4.3t $Q=\sinh(s)(xx_1+x_{12}x_2)-\sinh(r)(xx_2+x_1x_{12})+(\cosh(r)-\cosh(s))(xx_{12}+x_1x_2)+\delta(\cosh(r)-\cosh(s))/(\sinh(r)\sinh(s))$. (Q3).
Parameterization $\sinh(x)=2X/(1-X^2)$, $\cosh(x)=(1+X^2)/(1-X^2)$ is often used, and 4.2t' is then obtained by the singular limit $X\to 1+\epsilon X'$, $\delta=\delta/\epsilon^2$.

4.3t' $Q=r(x-x_2)(x_1-x_{12})-s(x-x_1)(x_2-x_{12})+\delta rs(r-s)$, (Q1) or (A1).

4.4 Obtained from 4.1 by I

4.5t $Q=(\sinh(r)-\sinh(s))(xx_1x_2x_{12}+1)-\cosh(r)(xx_1+x_2x_{12})+\cosh(s)(xx_2+x_1x_{12})$. (A2) with parameterization $e^r=\frac{\alpha+1}{\alpha-1}$ etc. in comparison with [7]. The singular limit $r,s\to\infty$ with redefinition $e^r=r'$ leads to 4.3t':

4.5t' $Q=(r-s)(xx_1x_2x_{12}+1)-r(xx_1+x_2x_{12})+s(xx_2+x_1x_{12})$. Transformation $x\to\frac{1-x}{1+x}$ takes this to (A1) with $\delta=0$. Also obtained from 3.5t' by Ia.

4.6n $Q=c(xx_1x_2x_{12}+1)+d(xx_1x_2+xx_1x_{12}+xx_2x_{12}+x_1x_2x_{12}+x+x_1+x_{12}+x_2)$
Limits $c=0$ and $d=0$ yield 2.3n and 2.4n, respectively. If $cd\neq 0$ Moebius transformation $x\to\frac{x\kappa-c}{xc+\kappa}$, $\kappa=\sqrt{c^2+4d^2}-2d$ takes this to 2.3n

5 nonzero functions

5.1t The generic case
$(xx_{12}+x_1x_2)(rs-\Delta)(r-s)+(xx_1+x_2x_{12})(s^2-\Delta)r-(xx_2+x_1x_{12})(r^2-\Delta)s$
$-(r-s)(x+x_1+x_2+x_{12})$
$+[\delta(rs-\Delta)rs+(r^2-rs+s^2-\Delta)](r-s)/(r^2-\Delta)/(s^2-\Delta)$
can be translated to a 4.3, $x_*\to x_*+1/\Delta$. However, if $\Delta=0$ we get the following new case at this level (with redefinition $r\to 1/r, s\to 1/s$):

5.1t $Q=r(x-x_2)(x_1-x_{12})-s(x-x_1)(x_2-x_{12})+rs(r-s)(x+x_1+x_2+x_{12})-rs(r-s)(r^2-rs+s^2+\delta)$. Translation $x\to x+\delta/4$ gives (Q2). If $\delta\neq 0$ then the singular limit $x_*\to x'_*/\epsilon$, $\delta\to\delta'/\epsilon^2$, $\epsilon\to 0$ yields (Q1) in 4.3t'.

5.3n $Q=(xx_1x_2+xx_1x_{12}+xx_2x_{12}+x_1x_2x_{12})+(xx_1+x_{12}x_2)+(xx_2+x_1x_{12})+(xx_{12}+x_1x_2)-1$ Reduces by translation to 2.3n.

5.5n $Q=(xx_1x_2+xx_1x_{12}+xx_2x_{12}+x_1x_2x_{12})+2(xx_1+x_{12}x_2)+2(xx_2+x_1x_{12})+2(xx_{12}+x_1x_2)+2(x+x_1+x_2+x_{12})$ Reduces by translation to 2.3n.

5.6t1

$$\begin{aligned} Q &= (h(r)f(s)-h(s)f(r))[(xx_1x_{12}x_2+1)f(r)f(s)-(xx_{12}+x_1x_2)] \\ &\quad +(f(r)^2f(s)^2-1)[(xx_1+x_{12}x_2)f(r)-(xx_2+x_1x_{12})f(s)], \end{aligned} \tag{3.1}$$

where $h^2=f^4+\delta f^2+1$. [One can parameterize this with Jacobi elliptic functions: $f(x)=k^{\frac{1}{4}}\,\mathrm{sn}(x,k)$, $h(x)=\mathrm{sn}'(x,k)$, $\delta=-(k+1)/\sqrt{k}$.] The map (3.1) is Moebius equivalent to (Q4) [10]. The singular limit $f(x)=x/\epsilon$, $h(x)=-(x/\epsilon)^2+\dots$, $\epsilon\to 0$ leads to 4.5t'.

5.6n2 $Q=(xx_1x_{12}x_2+1)+\tau_1(xx_{12}+x_1x_2)+\tau_2(xx_1+x_{12}x_2+xx_2+x_1x_{12})$, where $\tau_i=\pm 1$. If $\tau_1\tau_2=-1$ this is transformable to 1.2n, if $\tau_1\tau_2=1$ to 2.3n.

6 or 7 nonzero functions

At the moment a complete solution for these cases are still too hard to find.

4 Discussion

We have made a direct search of lattice models with CAC property and having the same symmetry properties as in [7], but without assuming the tetrahedron property. It turns out that all the results with spectral parameters also satisfy the tetrahedron property. One interesting result is (3.1): it is actually Moebius equivalent to (Q4) [10], but since it is much simpler in form (as it contain only even terms) it should be taken as the canonical form of (Q4).

As for non-tetrahedral (NT) models, three independent maps were found, the well known 1.1n, 1.2n and the cubic-linear 2.3n. None of them contain spectral parameters. However, one NT model with nontrivial spectral parameter dependence is known [8] (see also [11]), and the above three maps can be considered as special cases of it, but that map has different symmetry properties than the ones assumed here.

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