A Novel Approach to Improve the Frequency Resolution Based on Sparse Representation

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***Abstract*—How to improve the frequency resolution, especially in the low frequency bands, is an important problem for many applications. A novel approach based on sparse representation was proposed for this problem. In this work, we first discussed the design of the over-completed dictionary with the high frequency resolution, and then Matching Pursuit was advised to perform sparse decomposition using Matching Pursuit Tool Kit. Next, the frequency information was extracted from the code book of the results of Matching Pursuit. Finally, the experiments demonstrated the improvement of the frequency resolution for the proposed approach.**

Keywords-Fast Fourier Transform;Matching Pursuit; sparse represantion; frequency resolution; MPTK;

I. INTRODUCTION

In the fields of the signal processing, many applications need to get the accurate frequency information. For example , we need to extract the frequency information from the original music signals to estimate the pitch. Certainly, the finer the frequency resolution is, the better result we can get. However, the frequency resolution is always finite in our present methods of the signal processing.

Our traditional method of spectral analysis is Fast Fourier Transform(FFT). According to the theory of FFT, for the N point signal, with the sample rate f_s , the frequency resolution is f_s / N . If we want to improve the frequency resolution, we can adopt two methods: one is to diminish f_s, the other is to increase N . However , in practice , the two approaches don't work, because the two parameters always can not be changed due to some reasons. How to improve the frequency resolution in case of the two parameters unchanging? In article [1], an algorithm was proposed by Yang based on modulated FFT. The algorithm shifted the frequency of the real signals by modulation in time domain, and then performed traditional FFT. It avoided the defect of the half redundancy of the results of FFT and doubled the frequency resolution .

The Short Time Fourier Transform(STFT) processes the signal based on FFT. It has constant frequency resolution all the time. With the adaptive time-frequency window, the Wavelet Transform(WT) can overcome the disadvantages of the constant resolution. In the low frequency bands, it can produce high frequency resolution and low time resolution. In the high frequency bands, it can produce low frequency resolution and high time resolution. Lu [2] proposed a method to estimate spectrum based on Morlet wavelet, taking advantage of the WT's good characteristics, and obtained a higher accuracy of estimation than that of FFT, under the condition of the short signal. But in this way, we need to adjust the factors α and β according to the central frequency. It increased the complexity, and could not get satisfied results.

In the fields of the signal processing, sparse representation, a new approach to analyze the signal, has become popular. The traditional methods of the signal processing such as FFT based on the complete basis have limited expressiveness, while decomposition based on over-complete dictionary (redundant dictionary) can represent the signals more sparsely and more accurately to some extent, because it can adaptively choose appropriate atoms to decompose the signal according to the intrinsic characteristics of the signal itself. In this work, we proposed a novel approach to improve the frequency resolution based on the sparse representation.

The rest of this article is organized as follows. We start by introducing sparse decomposition of the signals using Matching pursuit(MP)[3]. Then, we explain the approach to improve the frequency resolution, in which we first need to design a dictionary with the higher frequency resolution, and then perform MP using Matching Pursuit Tool Kit (MPTK)[4], and finally extract the frequency information from the results of the MPTK. Finally, We show our experiments and analysis, and summarize in the end of the paper .

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II. SPARSE RPRESENTATION

Sparse representation has become more and more important tool to analyze the signals. In this technique, the goal is to find the representation of a signal x as a weighted sum of elements from an over-complete dictionary. That is

$$x = \sum_{r \in T} \lambda_r g_r \quad (1)$$

where the dictionary $D = \{g_r \mid r \in T\}$ is redundant, with parameterized atom g_r. Many possible representations of x exist in this redundant dictionary. Several methods have been suggested to find the "optimal" representation of the form of (1), including MP, Basis Pursuit(BP)[5], Orthogonal Matching Pursuit(OMP)[6] , and so on. The definition of "optimal" is application dependent. We adopted MP to decompose the signal in this paper.

Put forward by [3], MP, with the flexibility of the redundant dictionary and the adaptability to signals, has tremendous vitality, and the papers and the applications has been emerging in an endless stream. MP is an iterative "greedy" algorithm. At every iteration, the algorithm selects the atom who is the most similar to the residual signal, and regards this atom as one of the components of the sparse representation. For improving the frequency resolution, we just need design a dictionary D with the high frequency resolution atoms g_r. The atoms selected after some iterations contain the frequency and the corresponding amplitude information of the analyzed signal. So, if we can create the redundant dictionary in which the atoms are characterized by the higher frequency resolution, we can get the finer resolution. In theory , Any resolution can be reached through designing the good enough dictionary.

We suppose that x is the signal to be decomposed, and D is the redundant dictionary. We define that the similarity measure function C is the inner product of g_r and x, i.e. $C(g_r, x) = <g_r, x>$. After m iterations, we will get a code book B.

$$B = \{b_n \mid b_n = (c_n, g_{r_n}), c_n = C(x_n, g_{r_n}), n = 1..m\} \quad (2)$$

The basic steps of MP is as follows.

1) Initialization: $n = 0; x_n = x; B = \phi;$

2) Compute the inner products $C(g_r, x_n)$ *for all* $g_r \in D$.

3) Find the maximum amongst all inner products:

$$r_n = \arg\max_{g_r \in D}(C(g_r, x_n)) \quad (3)$$

and add $b_n = (c_n, g_{r_n})$ *to code book* B.

4) Compute the residual signal as in (4) *, if precision reached, then stop, otherwise* $n = n+1$ *,and iterate to step 2).*

$$x_{n+1} = x_n - C(g_{r_n}, x_n) g_{r_n} \quad (4)$$

III. IMPROVE THE FREQUENCY RESOLUTION BASED ON MATCHING PURSUIT

We present our approach to improve the frequency resolution based on MP in this section. Firstly, we discuss the design of the dictionary, and then we introduce the MPTK to perform the MP using the dictionary designed. Finally, we analyze the book of the MP results to obtain frequency information. Provided the finer dictionary with the high frequency resolution, we can get the finer information of the analyzed signal.

A. Designing the Dictionary

In MP, choosing the suitable atoms for the redundant dictionary, namely, the design of over-complete dictionary of sparse representation, is a crucial step, in which the atoms should be similar to the original signal as far as possible.

Mallat and Zhang proposed a typical redundant dictionary in [3].

$$g_r(t) = \frac{K_r}{\sqrt{s}} g\left(\frac{t-u}{s}\right)\cos(\xi t + \phi) \quad (5)$$

Where $r = (s, \xi, u, \phi)$, $s \in R^+, \xi, u \in R, \phi \in (-\pi, \pi)$, and window g is Gaussian function, with K_r such that $\| g_r \| = 1$ .The window function adjusts the time and frequency resolution of the atoms by the scale factor s, so the adaptability is enhanced. When s is fixed, MP is similar to STFT, which is conducive to analyze the frequency components of the steady state signal; When ξ is fixed, MP is like WT, which can describe the singularity of the signal finely. Therefore, from the analysis of the characteristics of $g_r(t)$, MP can integrate the advantages of WT and STFT, and can represent the signals more precisely than WT and STFT. This dictionary presents good characteristics for time-frequency analysis, and many researchers employed this dictionary to process the signal. If we discretize the parameter ξ more finely than the frequency variable of FFT, we can get higher frequency resolution certainly. Obviously, this is a feasible way to improve the frequency resolution.

Next, let us consider a more simple dictionary for frequency resolution. We review the dictionary of FFT first. The dictionary of FFT is a collection of sinusoidal waveform indexed by $r = (w, v)$, where $w = [0, 2\pi)$ is an angular frequency variable and $v = \{0,1\}$ indicates phase type: sine or cosine. In detail,

$$g(\omega, 0) = \cos(\omega t) \;\; g(\omega, 1) = \sin(\omega t).$$

For the standard Fourier dictionary, we let r run through the set of all cosines with Fourier frequencies

$$\omega_k = 2\pi \mathrm{k/N} \;\; \mathrm{k} = 0, ..., N/2$$

And all sines with Fourier frequencies

$$\omega_k \;\; \mathrm{k} = 1, ..., N/2\text{-}1$$

This dictionary consists of N atoms; it is in fact a orthogonal basis. If we divide the frequencies into more than N components, and produce a dictionary with more than N atoms, we can obtain the higher frequency resolution too. For example, if we define cosines with frequencies

$$\omega_k = \pi \mathrm{k/N} \;\; \mathrm{k} = 0, ..., N$$

And sines with frequencies

$$\omega_k \;\; \mathrm{k} = 1, ..., N\text{-}1$$

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The dictionary will contain $2N$ atoms with higher frequency resolution. However, the dictionary is no more orthogonal basis, and it is over-complete. Meanwhile, the decomposition algorithm will not be FFT. Instead, we should employ sparse representation .

B. Performing MP

In the all algorithms of sparse representation, MP is the fastest one at present. But if we perform the MP algorithm in the usual way, the computation complexity will still appear high. We adopted the fast MP based on MPTK to decompose the signals. the algorithm divide the dictionary into a number of blocks with different scales, and each block contains the atoms with other parameters, such as dilations and frequencies. When updating the inner product (see the 2nd step of MP), we do not need to update all the inner products of all the atoms, only of those of the atoms who have the shared part with the selected atom, which has the maximum inner product in the previous iteration. When finding the maximum amongst all the inner products (see the 3rd step of MP), we do not need to search all the inner products of all the atoms, and only need to compare the current maximum with the inner products updated in the previous step. When searching, we scan the tree instead of the linear table. Eventually, the computation complexity decreases from $O(N^2)$ to $O(N \log N)$, where N is the length of the signal.

MPTK consists of packages for many dictionary, such as Gabor dictionary above, Harmonic dictionary[7] proposed by Gribonval, Any-wave dictionary, and so on. We can select one dictionary for our practical problem.

C. Extracting the frequency information from the MP book

After performing MPTK, how can we extract the frequency information from the results? The answer is code book B as depicted in (2). The code book contains the information of the atoms after n iterations. The book of the MPTK results corresponds to the code book B above, which has different structure according to the dictionary used.

If we select Gabor dictionary for MP, the atoms of the book contain the normalized frequency parameters named $freq$ and the corresponding amplitude parameters named amp . The values of the amplitude parameters diminish with the number of iteration increasing because the energy of the residual signal always reduces. The frequencies and the corresponding amplitudes of the analyzed signal are disclosed by $freq$ and amp . The actual frequencies need to be converted as equation below .

$$f = fs * atom.freq \qquad (6)$$

For Any-wave dictionary, the atoms of the book contain the index of the selected atom in the dictionary, we can judge which atoms were selected from the parameter $anywaveIdx$ of the atom, and then calculate the frequencies through the parameter. Like Gabor dictionary, the book also provides the corresponding amplitudes from the parameters amp .

IV. EXPERIMENTS

To demonstrate the procedure of our approach, we designed a simple over-complete Fourier dictionary, and performed MP using MPTK Any-wave package to extract the frequencies of the signal. We compared the results with those of FFT to verify the effect of our idea.

A. The Signal to Be Analyzed

The signal (Figure 1) was created by adding two sine functions with the frequency $f_1 = 1000Hz$ $f_2 = 1020Hz$ respectively. The sampling rate is $f_s = 5120Hz$, and the length of the signal is $N = 128$.

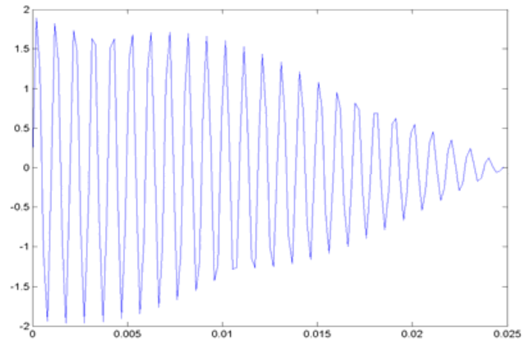

Figure 1. the signal to be analyzed

B. The Results of FFT

According to the theory of FFT, the frequency resolution is $\frac{f_s}{N} = 40Hz$, and the two frequency $f_1 = 1000Hz$ $f_2 = 1020Hz$ can not be distinguished. The result of FFT is showed by Figure 2, in which it displays the amplitudes around the frequencies analyzed only, from 950Hz to 1050Hz. It has three amplitudes , and the amplitude of the frequency $f_2 = 1020Hz$ certainly can not be seen.

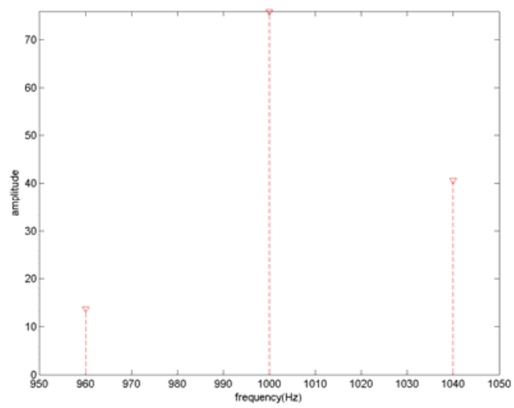

Figure 2. the result of FFT

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C. *The Results of MP*

The dictionary of MP in our experiments was generated from the Fourier basis. Suppose $r=(w)$,only one parameter, and the atoms g_r simply contain cosines .

$$g_r(t)=\cos(wt) \tag{7}$$

Where $w\in[0,\pi)$, For the angle frequency, we defined

$$\omega_k=\pi \mathrm{k}/(2^*\mathrm{N}) \quad \mathrm{k}=0,1...,2N-1 \tag{8}$$

So, the dictionary contained double atoms when compared with FFT. Apparently, it was over-complete . We used the MPTK Any-wave package to perform the MP algorithm. From the book of the MPTK Any-wave, we extracted the parameter *anywaveIdx* , and calculated the frequency

$$f=\frac{anywaveIdx}{2*N}*\frac{fs}{2} \tag{9}$$

Figure 3 shows the frequencies and the corresponding amplitudes generated by MPTK after 5 iterations. We can see that the two frequencies were distinguished explicitly. In fact, the true frequency resolution is 10 Hz , three times finer than FFT.

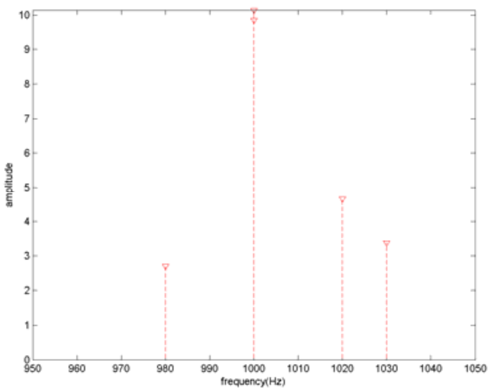

Figure 3. the result of MP

Experiments above demonstrated the correctness of our approach based on sparse representation.

V. CONCLUSION

In this paper, we introduced a novel approach to improve the frequency resolution based on sparse representation. Through deliberately designing the over-complete dictionary with the high frequency resolution, the frequencies which can not be identified by FFT, can be distinguished in this way. MPTK, as a good tool, can be adopted for performing MP.

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