Journal of Nonlinear Mathematical Physics Volume 12, Supplement 2 (2005), 95–136 SIDE VI

Bispectrality for deformed Calogero–Moser–Sutherland systems

Misha FEIGIN

Department of Mathematics, South Kensington campus, Imperial College London, London SW7 2AZ, UK
E-mail: m.feigin@imperial.ac.uk

This article is a part of the special issue titled "Symmetries and Integrability of Difference Equations (SIDE VI)"

Abstract

We prove bispectral duality for the generalized Calogero–Moser–Sutherland systems related to configurations $\mathcal{A}_{n,2}(m)$, $\mathcal{C}_n(l,m)$. The trigonometric axiomatics of the Baker–Akhiezer function is modified, the dual difference operators of rational Macdonald type and the Baker–Akhiezer functions related to both series are constructed.

1 Introduction

The original bispectral problem as it appeared in the paper by Duistermaat and Grunbaum [14] was devoted to investigations of the Sturm–Liouville operators such that they admit a family of eigenfunctions satisfying some differential equation in the spectral parameter. Part of the corresponding potentials, namely the rational KdV potentials, were described as those which can be obtained from 0 by applying the Darboux transformations. The corresponding Sturm–Liouville operators admit non-trivial commuting differential operators. In paper [14] the conditions in terms of local Laurent expansions for a potential to be a rational KdV potentials were also analyzed. These conditions generalized simple locus conditions from [1].

An example which may be looked at as a generalization of this picture to the many-dimensional case is given by the Calogero–Moser operator ([4], [22], [28], [23])

$$L = \Delta - \sum_{\alpha\in A} \frac{m_\alpha(m_\alpha+1)(\alpha,\alpha)}{(\alpha,x)^2}, \tag{1.1}$$

where A is a root system. When the parameters m_α are integer (and invariant) operator (1.1) can be included into a large supercomplete commutative ring of differential operators as it was discovered by Chalykh and Veselov in [11]. The key object of this construction is the multidimensional Baker–Akhiezer function $\psi(k,x) = \psi(k_1,\dots,k_n,x_1,\dots,x_n)$. This function is defined on a certain many-dimensional rational spectral variety, it is an eigenfunction for all operators from the commutative ring. The function ψ satisfies the same differential equations in the spectral variables k, as was shown by Chalykh, Styrkas, and Veselov [31], thus the bispectrality holds. The Baker–Akhiezer functions defined on the Riemann surfaces were introduced by Krichever for studying one variable rings of commuting differential operators and non-linear integrable equations [19] (see also [3]).

The generalization of the one-dimensional locus conditions from [14] to the multi-dimensional case led Chalykh, Veselov, and the author to an interesting class of Schroedinger operators of type (1.1) where A can be a Coxeter system and also more general locus configuration [10]. These operators can be included into the supercomplete rings of commuting differential operators, they admit the Baker–Akhiezer functions which also satisfy the differential equations in the spectral parameters.

Despite large number of locus configurations in the two-dimensional case the known examples in higher dimensions are quite exceptional and discrete. Besides operators (1.1) related to the Coxeter systems two other series of deformations $A_{n,1}(m)$, $\mathcal{C}_n(l,m)$ were found in [30], [9], [10], and one more configuration $A_{n,2}(m)$ appeared later in Chalykh–Veselov [12]. Configuration $A_{n,1}(m)$ becomes the root system A_n when m=1. Configuration $\mathcal{C}_n(l,m)$ specializes to the root system C_n at $l = m$. Configuration $A_{n,2}(m)$ is a complex extension of the root system A_{n-2}. When $n = 2$ the parameter m can be arbitrary complex and the corresponding operator coincides with the degeneration of the Hietarinta operator [16] (see also [10], [8]).

In this paper we analyze bispectrality and the Baker–Akhiezer functions for the trigonometric versions of the operators (1.1) for the configurations $\mathcal{C}_n(l,m)$ and $A_{n,2}(m)$, whereas the root systems and configuration $A_{n,1}(m)$ were considered by Chalykh [6]. Earlier in paper [9] the intertwining operators for the Schroedinger operators with trigonometric potentials related to the configurations $\mathcal{C}_2(m,l)$, $A_{n,1}(m)$ were constructed.

A construction with the multi-dimensional trigonometric Baker–Akhiezer function was also introduced by Chalykh and Veselov in [11]. Such a function is a certain eigenfunction for the generalized Calogero–Moser–Sutherland operator

$$L = \Delta - \sum_{\alpha\in A} \frac{m_\alpha(m_\alpha+1)(\alpha,\alpha)}{\sinh^2(\alpha,x)}. \tag{1.2}$$

It was shown in [11] that the Baker–Akhiezer function exists when A is a root system and the multiplicities m_α are integer and invariant. Then L is included into a supercomplete ring of commuting differential operators.

In the trigonometric case the dual operators happen to be the difference operators. These operators are also discretizations of the Calogero–Moser Hamiltonians, they were introduced by Ruijsenaars for the problem related to A_n root system [24] (see [26] for the classical version). The bispectral duality of the Calogero–Moser–Sutherland and Ruijsenaars systems was conjectured by Ruijsenaars in [25]. For an arbitrary reduced root system the difference operators were introduced by Macdonald [21]. The duality on the level of Macdonald polynomials was conjectured by Macdonald and proved first by Koornwinder [17] (see chapter VI of [20]) for the A_n case. For an arbitrary reduced root system the proof was obtained by Cherednik [13]. For the case of the BC_n system Macdonald polynomials were introduced by Koornwinder [18], their duality property was established in [29], [27].

In terms of the Baker–Akhiezer functions the bispectral duality for (1.2) related to any root system was established by Chalykh in [6]. Also it was done for the system $A_{n,1}$ thus the corresponding deformation of the rational Ruijsenaars–Macdonald operator appeared in [6].

The method of establishing the dual equations as well as of constructing the Baker–Akhiezer functions was introduced by Chalykh in [6], and it is as follows. The Baker–Akhiezer function should satisfy some shifting conditions as a function of the spectral variables k. One considers the space of functions satisfying these conditions and a certain difference operator in k such that the application of this operator leaves the space invariant. Then taking a proper initial function from the space and iterating the application of the operator we arrive at the Baker–Akhiezer function, besides that on the next step we get zero thus the dual equation appears. This method was first applied in the rational case [5] (see also [10]), it works also in the trigonometric difference case [7]. The corresponding formula for the Baker–Akhiezer functions in the rational case was found earlier by Berest [2] under assumption of existence.

In this paper we follow the described strategy to construct the Baker–Akhiezer functions and to establish the bispectrality for the configurations $\mathcal{C}_n(l,m)$, $A_{n,2}(m)$. On the way we introduce the generalizations of rational Macdonald operators related to these deformations. An interesting feature of the configuration $A_{n,2}(m)$ is that for the corresponding operator (1.2) there is no Baker–Akhiezer function in the original axiomatics [11]. Thus we modify conditions in variables k which should be imposed on the Baker–Akhiezer function in order to cover this case as well. The corresponding modification of rational Chalykh–Veselov axiomatics for the Baker–Akhiezer functions from [11] was carried out in [10]. We should mention that in our considerations we restrict ourselves to the simpler case when a configuration A has no parallel vectors although a deformation of BC_n system leading to algebraically integrable operators appeared in [8], so it is natural to expect the bispectral property for the degeneration of this model as well.

The structure of this paper is the following. In section 2 we give the modified axiomatics for the trigonometric Baker–Akhiezer function and review the Chalykh–Veselov construction [11] adopting it to the new settings. In section 3 we recall how the bispectrality allows construction of commuting operators in the spectral variables if we know commuting operators in x ([2], [14], [6]). Then we prove that the Baker–Akhiezer functions for the root systems and for the deformation $A_{n,1}(m)$ also satisfy modified axiomatics. In section 4 we consider configuration $\mathcal{C}_n(l,m)$. We introduce a deformed rational Macdonald operator for this case and we construct the Baker–Akhiezer function. Then we prove the bispectral property, and the family of commuting difference operators appears. In section 5 the analogous results are proved for the $A_{n,2}(m)$ configuration. In the last section we discuss necessary conditions for a configuration of vectors with multiplicities to admit the Baker–Akhiezer function. They reveal clear geometrical restrictions on the configurations. The presentation closely follows [15].

2 Baker–Akhiezer function and commuting differential operators

Let A be a finite set of non-collinear vectors $\alpha \in \mathbb{C}^n$, let every vector α have a multiplicity $m_\alpha \in \mathbb{N}$. Meaning by m this multiplicity function we will denote such configurations as $\mathcal{A} = (A, m)$. By the *Baker–Akhiezer function* $\psi(k,x)$ we will mean a function of two sets of variables $k, x \in \mathbb{C}^n$ of the form

$$\psi(k,x) = \Big(\prod_{\alpha\in A}(k,\alpha)^{m_\alpha} + \text{lower order polynomial in } k\Big)e^{(k,x)}, \tag{2.1}$$

$(k,x) = k_1x_1 + \ldots + k_nx_n$, which satisfies special properties. We introduce $-A$ to be the system of vectors $\{-\alpha|\alpha \in A\}$ with the multiplicities $m_{-\alpha} = m_\alpha$. Inside $A\cup -A$ we choose a positive subsystem A_+ consisting of those vectors which belong to some half-space inside $\mathbb{R}^{2n} \approx \mathbb{C}^n$. The half-space should be in a generic position such that for any $\alpha \in A$ either $\alpha \in A_+$ or $-\alpha \in A_+$. We say that a vector $\alpha \in A_+$ is *an edge vector* if α is not a linear combination of other vectors from A_+ with positive real coefficients.

In this paper we will assume that the set A of vectors α is such that all the vectors belong to some lattice of rank n in the space $\mathbb{C}^n$. Though constructions and most of the proofs work without this assumption in all known examples such a lattice does exist, also assumption on the lattice makes definition of the edge vectors and subsystems A_+ more invariant. Namely, we now have an n-dimensional real vector space V containing the system A which is spanned by a basis in the lattice. Positive subsystems $A_+ \subset (A \cup -A)$ are those which consist of the vectors belonging to a generic half-space in the real linear space V. We will also assume that A does not contain isotropic vectors $\alpha : (\alpha, \alpha) = 0$, as we will see such vectors do not contribute to the potential.

Definition 1. A function $\psi(k,x)$ of the form (2.1) is called the *Baker–Akhiezer function* for a configuration $\mathcal{A} = (A,m)$ (BA function) if for any choice of positive subsystem A_+ and for any choice of an edge vector α the following identities hold

$$\frac{\psi(k+s\alpha,x)}{\prod_{\substack{\beta\in A_+\\ \beta\neq\alpha}}\prod_{i=1}^{m_\beta}(k+i\beta+s\alpha,\beta)} \equiv \frac{\psi(k-s\alpha,x)}{\prod_{\substack{\beta\in A_+\\ \beta\neq\alpha}}\prod_{i=1}^{m_\beta}(k+i\beta-s\alpha,\beta)} \tag{2.2}$$

at $(k,\alpha) = 0$, $s = 1,\ldots,m_\alpha$.

Remark 1. For a given vector $\alpha \in A$ there are normally few choices of the subsystems A_+ such that the vector α is an edge vector. Therefore the existence of the Baker–Akhiezer function for a system A forces, in particular, the following compatibility conditions. Let A_+^1, A_+^2 be two choices of positive subsystems in A such that α is an edge vector. Then the following identity must hold:

$$\frac{\prod_{\substack{\beta\in A_+^1\\ \beta\neq\alpha}}\prod_{i=1}^{m_\beta}(k+i\beta+s\alpha,\beta)}{\prod_{\substack{\beta\in A_+^1\\ \beta\neq\alpha}}\prod_{i=1}^{m_\beta}(k+i\beta-s\alpha,\beta)} \equiv \frac{\prod_{\substack{\beta\in A_+^2\\ \beta\neq\alpha}}\prod_{i=1}^{m_\beta}(k+i\beta+s\alpha,\beta)}{\prod_{\substack{\beta\in A_+^2\\ \beta\neq\alpha}}\prod_{i=1}^{m_\beta}(k+i\beta-s\alpha,\beta)}$$

at $(k,\alpha) = 0$ for $s = 1,\ldots,m_\alpha$.

Introducing the functions $\psi_\alpha^{A_+}$ depending on the choices of positive subsystem A_+ and an edge vector α by formulas

$$\psi_\alpha^{A_+} = \frac{\psi(k,x)}{\prod_{\substack{\beta\in A_+\\ \beta\neq\alpha}}\prod_{i=1}^{m_\beta}(k+i\beta,\beta)}$$

conditions (2.2) take the following form

$$\psi_\alpha^{A_+}(k+s\alpha) \equiv \psi_\alpha^{A_+}(k-s\alpha), \qquad \text{if} \quad (\alpha,k)\equiv 0,\ s=1,\ldots,m_\alpha. \tag{2.2$'$}$$

Also it will be convenient for us to use the following equivalent form of equations (2.2)

$$\left(\delta_\alpha \frac{1}{(k,\alpha)}\right)^{s-1} \delta_\alpha \psi_\alpha^{A_+} \equiv 0, \qquad \text{at } (k,\alpha)=0,\ s=1,\ldots,m_\alpha. \tag{2.2$''$}$$

Here δ_α is an operator acting by the rule $\delta_\alpha f(k) = f(k+\alpha) - f(k-\alpha)$. It is obvious that conditions (2.2$'$) and (2.2$''$) are identical for $m_\alpha = 1$. One can also check that they are equivalent in general. Conditions (2.2) form an overdetermined system of equations for the coefficients of a polynomial in (2.1). It takes place the following statement.

Proposition 1. (*c.f.*[*11*]) *If a Baker–Akhiezer function exists then it is unique.*

Proof. Assume there are two functions $\varphi_1 = P_1(k,x)e^{(k,x)}$, $\varphi_2 = P_2(k,x)e^{(k,x)}$, which satisfy equations (2.2), and assume the highest terms of the polynomials $P_i(k,x)$ are both $\prod_{\alpha\in A}(k,\alpha)^{m_\alpha}$. Consider the difference $\varphi_1 - \varphi_2 = (P_1 - P_2)e^{(k,x)}$. This function also satisfies conditions (2.2) but the degree of the polynomial $P_1 - P_2$ is less than $\sum_{\alpha\in A} m_\alpha$. Thus the proof of the proposition reduces to the following statement.

Lemma 1. (*c.f.*[*11*]) *Let $\psi(k,x) = P(k,x)e^{(k,x)}$ satisfy conditions* (2.2) *with $P(k,x)$ being a polynomial in k with the highest term $P_0(k,x)$. Then $P_0(k,x)$ is divisible by $\prod_{\alpha\in A}(k,\alpha)^{m_\alpha}$.*

Proof. Consider condition (2.2$''$) for some subsystem A_+ and an edge vector α. We have

$$\psi_\alpha^{A_+} = \frac{P(k,x)}{Q(k)} e^{(k,x)},$$

where

$$Q(k) = \prod_{\substack{\beta\in A_+\\ \beta\neq\alpha}} \prod_{i=1}^{m_\beta} (k+i\beta,\beta).$$

We denote by $Q_0(k)$ the highest term of $Q(k)$ and consider conditions (2.2$''$) with $s=1$. We have

$$\frac{P(k+\alpha,x)}{Q(k+\alpha)} e^{(\alpha,x)} e^{(k,x)} - \frac{P(k-\alpha,x)}{Q(k-\alpha)} e^{-(\alpha,x)} e^{(k,x)} = \\ = e^{(k,x)} \frac{P_0(k,x)\,(e^{(\alpha,x)} - e^{(-\alpha,x)})\,Q_0(k) + \text{ lower terms}}{Q(k+\alpha)Q(k-\alpha)} \quad \vdots\, (k,\alpha).$$

As $Q_0(k) = \prod_{\substack{\beta\in A_+\\ \beta\neq\alpha}} (k,\beta)^{m_\beta}$ is not divisible by (k,α) we conclude that $P_0(k,x)$ should be divisible by (k,α).

Now we rewrite the obtained relation in the form

$$\delta_\alpha \psi_\alpha^{A_+} = (k,\alpha) \frac{\widetilde{P}(k,x)}{\widetilde{Q}(k)} e^{(k,x)} (e^{(\alpha,x)} - e^{(-\alpha,x)}),$$

where $\widetilde{P}$, $\widetilde{Q}$ are some polynomials in k with the highest terms $\widetilde{P}_0 = \frac{P_0}{(k,\alpha)}Q_0$, and $\widetilde{Q}_0 = \prod_{\substack{\beta\in A_+\\ \beta\neq\alpha}}(k,\beta)^{2m_\beta}$, so $\widetilde{Q}_0$ is again not divisible by (k,α). Considering conditions (2.2″) with $s=2$ we analogously conclude that $\widetilde{P}_0 \vdots (k,\alpha)$, that is $P_0 \vdots (k,\alpha)^2$. Continuing in this way up to $s = m_\alpha$ we obtain $P_0 \vdots (k,\alpha)^{m_\alpha}$. Since any vector $\alpha\in A$ is an edge vector for the proper choice of a subsystem A_+, system (2.2″) contains equations for all $\alpha \in A$. Therefore $P_0 \vdots \prod_{\alpha\in A}(k,\alpha)^{m_\alpha}$, and lemma is proven. ■

■

The existence of the Baker–Akhiezer function is possible for very special configurations $\mathcal{A}$ only. In this case $\psi(k,x)$ becomes a joint eigenfunction of a rich commutative ring of differential operators. Namely to any configuration $\mathcal{A}=(A,m)$ we relate the ring $R_\mathcal{A}$ of polynomials $p(k)$ which for any $\alpha\in A$ satisfy the conditions

$$p(k+s\alpha)\equiv p(k-s\alpha) \quad \text{at } (k,\alpha)=0,$$

where $s=1,\ldots,m_\alpha$.

Theorem 1. (*c.f.*[*11*]) *Assume configuration $\mathcal{A}$ admits the Baker–Akhiezer function. Then for any $p(k)\in R_\mathcal{A}$ there exists a differential operator $L_p(x,\partial_x)$ such that*

$$L_p(x,\partial_x)\,\psi(k,x) = p(k)\,\psi(k,x).$$

For any $p,q\in R_\mathcal{A}$ one has the commutativity $L_pL_q = L_qL_p$.

Proof. Consider function $\psi_1(k,x) = p(k)\,\psi(k,x) - p(\partial_x)\psi(k,x)$. Then the function ψ_1 satisfies conditions (2.2) and it has the form $\psi_1 = Q_1(k,x)e^{(k,x)}$ with $\deg Q_1 \leqslant \sum m_\alpha + \deg p - 1$. By lemma 1 the highest term of the polynomial Q_1 has the form $Q_1^0 = \prod(k,\alpha)^{m_\alpha} r(x,k)$. We define now $\psi_2(k,x) = \psi_1(k,x) - r(x,\partial/\partial x)\,\psi(k,x)$. We have $\psi_2(k,x) = Q_2(k,x)e^{(k,x)}$, where Q_2 is some polynomial of degree $\deg Q_2 \leqslant \sum m_\alpha + \deg p - 2$, and ψ_2 satisfies conditions (2.2). Therefore we can again apply lemma 1 and inductively we construct operator $L_p = p(\partial_x) + r(x,\partial_x) + \ldots$

The commutativity $[L_p, L_q] = 0$ follows from the condition that if an operator $L(x,\partial_x)$ satisfies $L(x,\partial_x)\psi(k,x) = 0$ for a function ψ of the form (2.1), then $L\equiv 0$. The theorem is proven. ■

We note that for any configuration $\mathcal{A}$ the ring $R_\mathcal{A}$ contains the polynomial $k^2 = k_1^2 + \ldots + k_n^2$. Indeed, $(k\pm s\alpha)^2 = (k\pm s\alpha, k\pm s\alpha) = (k,k)\pm 2s(\alpha,k) + s^2(\alpha,\alpha)$, and if $(\alpha,k)=0$ we have $(k+s\alpha)^2 = (k-s\alpha)^2$. The corresponding differential operator is the Schroedinger operator.

Proposition 2. (*c.f.*[*11*]) *In the settings of theorem 1 to the polynomial $p(k)=k^2$ it corresponds the operator*

$$L_{k^2} = \Delta - \sum_{\alpha\in A}\frac{m_\alpha(m_\alpha+1)(\alpha,\alpha)}{\sinh^2(\alpha,x)}.$$

Proof. Let

$$\psi(k,x)=P(k,x)e^{(k,x)}=(\prod_{\alpha\in A}(k,\alpha)^{m_\alpha}+P_1+\text{lower order terms})e^{(k,x)},$$

where P_1 is a polynomial of degree $\sum m_\alpha-1$. To obtain L_{k^2} we apply recurrent procedure described in the proof of theorem 1. We have

$$\psi_1(k,x)=k^2\psi(k,x)-\Delta\psi(k,x)= \\ =\Big(-2\sum_{i=1}^n k_i\frac{\partial}{\partial x_i}P-\Delta P\Big)e^{(k,x)}=\Big(-2\sum_{i=1}^n k_i\frac{\partial P_1}{\partial x_i}+R\Big)e^{(k,x)},$$

where R is some polynomial in k, $\deg R<\sum m_\alpha$. According to lemma 1

$$-2\sum_{i=1}^n k_i\frac{\partial P_1}{\partial x_i}=u(x)\prod_{\alpha\in A}(k,\alpha)^{m_\alpha}$$

for some function $u(x)$. Also from lemma 1 it follows that $\psi_1(k,x)-u(x)\psi(k,x)=0$. Thus

$$L_{k^2}=\Delta+u=\Delta-\frac{2}{\prod_{\alpha\in A}(k,\alpha)^{m_\alpha}}\sum_{i=1}^n k_i\frac{\partial P_1}{\partial x_i}.$$

And the proof of the proposition is reduced to the following lemma.

Lemma 2. (*c.f.*[*11*]) *Assume that a system $\mathcal{A}$ admits the Baker–Akhiezer function*

$$\psi(k,x)=P(k,x)e^{(k,x)}=\Big(\prod_{\alpha\in A}(k,\alpha)^{m_\alpha}+P_1+\dots\Big)e^{(k,x)},$$

where $P_1=P_1(k,x)$ are terms of order $\sum m_\alpha-1$ in the polynomial P. Then

$$P_1=-\Big(\prod_{\alpha\in A}(k,\alpha)^{m_\alpha}\Big)\sum_{\alpha\in A}\frac{m_\alpha(m_\alpha+1)}{2}\frac{(\alpha,\alpha)}{(\alpha,k)}\coth(\alpha,x).$$

Proof. We choose a subsystem A_+ and consider conditions (2.2″) for an arbitrary edge vector α. We want to show that P_1 is divisible by $(k,\alpha)^{m_\alpha-1}$ and to find $P_1/(k,\alpha)^{m_\alpha-1}$. For $s=1$ condition (2.2″) can be rewritten in the following way

$$\frac{1}{Q_1(k)}\Big\{T_1(k)(k,\alpha)^{m_\alpha}\prod_{\substack{\beta\in A\\ \beta\neq\alpha}}(k,\beta)^{m_\beta}(e^{(\alpha,x)}-e^{-(\alpha,x)})+ \\ +T_1(k)\Big(m_\alpha(\alpha,\alpha)(k,\alpha)^{m_\alpha-1}(e^{(\alpha,x)}+e^{-(\alpha,x)})\prod_{\substack{\beta\in A\\ \beta\neq\alpha}}(k,\beta)^{m_\beta}+(e^{(\alpha,x)}-e^{-(\alpha,x)})P_1\Big)+ \\ +O_1\big((k,\alpha)^{m_\alpha}\big)+R_1\Big\}e^{(k,x)}\equiv 0,$$

if $(k,\alpha)=0$. In the last formula

$$T_1(k)=\prod_{\substack{\beta\in A_+\\ \beta\neq\alpha}}\prod_{i=1}^{m_\beta}(k+i\beta,\beta),\qquad Q_1(k)=T_1(k+\alpha)\,T_1(k-\alpha),$$

and $O_1\big((k,\alpha)^{m_\alpha}\big)$ is a polynomial of degree $2\sum_{\beta\in A_+}m_\beta-1-m_\alpha$, which is divisible by $(k,\alpha)^{m_\alpha}$. And R_1 is some polynomial in k such that $\deg R_1<2\sum_{\beta\in A_+}m_\beta-1-m_\alpha$.

Going by induction we conclude that for an arbitrary s such that $m_\alpha\geqslant s>1$ one has

$$\frac{1}{Q_s(k)}\Bigg\{T_s(k)(k,\alpha)^{m_\alpha-s+1}\prod_{\substack{\beta\in A\\ \beta\neq\alpha}}(k,\beta)^{m_\beta}+ + T_s(k)\Bigg(c_s(m_\alpha)(\alpha,\alpha)(k,\alpha)^{m_\alpha-s}\prod_{\substack{\beta\in A\\ \beta\neq\alpha}}(k,\beta)^{m_\beta}\coth(\alpha,x)+\frac{P_1}{(k,\alpha)^{s-1}}\Bigg)+ + O_s\big((k,\alpha)^{m_\alpha-s+1}\big)+R_s\Bigg\}\equiv 0 \quad (2.3)$$

if $(k,\alpha)=0$. Here $T_s(k)=T_{s-1}(k)\,Q_{s-1}(k)$, $Q_s(k)=Q_{s-1}(k+\alpha)\,Q_{s-1}(k-\alpha)$, and the polynomial $O_s\big((k,\alpha)^{m_\alpha-s+1}\big)$ is divisible by $(k,\alpha)^{m_\alpha-s+1}$, $\deg O_s\leqslant\deg T_s+\sum_{\beta\in A_+}m_\beta-s$, $\deg R_s<\deg T_s+\sum_{\beta\in A}m_\beta-s$. It is important for us that $c_s(m_\alpha)=c_{s-1}(m_\alpha)+m_\alpha-s+1$. Consider now condition (2.3) with $s=m_\alpha$. As $T_s(k)\neq 0$ if $(k,\alpha)=0$ we conclude that

$$c_{m_\alpha}(m_\alpha)(\alpha,\alpha)\prod_{\substack{\beta\in A\\ \beta\neq\alpha}}(k,\beta)^{m_\beta}\coth(\alpha,x)+\frac{P_1}{(k,\alpha)^{m_\alpha-1}}=0 \qquad (2.4)$$

at $(\alpha,k)=0$, and

$$c_{m_\alpha}(m_\alpha)=m_\alpha+(m_\alpha-1)+\ldots+1=\frac{m_\alpha(m_\alpha+1)}{2}.$$

Now we remark that conditions (2.4) characterize the polynomial P_1 uniquely. Indeed the existence of a polynomial $\widetilde{P}_1$, $\deg\widetilde{P}_1=\deg P_1=\sum m_\beta-1$, satisfying (2.4) would mean that $\frac{P_1-\widetilde{P}_1}{(k,\alpha)^{m_\alpha-1}}=0$ at $(k,\alpha)=0$, thus $P_1-\widetilde{P}_1$ would be divisible by $(k,\alpha)^{m_\alpha}$. As any vector $\alpha\in A$ is an edge vector for a proper subsystem A_+, we get $P_1-\widetilde{P}_1\,\vdots\,\prod_{\alpha\in A}(k,\alpha)^{m_\alpha}$. But this is impossible as $\deg(P_1-\widetilde{P}_1)\leqslant\sum_{\alpha\in A}m_\alpha-1$. Further it is obvious that the polynomial

$$P_1=-\Bigg(\prod_{\alpha\in A}(k,\alpha)^{m_\alpha}\Bigg)\sum_{\alpha\in A}\frac{m_\alpha(m_\alpha+1)}{2}\,\frac{(\alpha,\alpha)}{(\alpha,k)}\coth(\alpha,x)$$

satisfies (2.4), therefore lemma 2 is proven. ∎

This completes the proof of the proposition. ∎

3 Bispectral duality and examples

By bispectral duality we mean the situation when a function $\psi(k,x)$ of two sets of variables k and x satisfies certain equations in each of the sets. In our case we will have the equations of the form

$$L(x,\partial_x)\psi(k,x) = k^2\psi(k,x), \qquad D\psi(k,x) = \lambda(x)\psi(k,x), \tag{3.1}$$

where D is some *difference* operator in k-variables, and ψ is the Baker-Akhiezer function. Originally the equations in the spectral parameter were considered by Duistermaat and Grunbaum [14] who analyzed in the one-dimensional situation the pair of equations (3.1) for a Sturm–Liouville operator L and a differential operator D.

One of the applications of the bispectrality is the following construction ([14], [2], [6]) allowing to obtain a commuting operator for D if a commuting operator for L is given. More exactly, assume we have some operator $M(x,\partial_x)$ satisfying

$$M(x,\partial_x)\psi(k,x) = q(k)\psi(k,x), \tag{3.2}$$

for some polynomial $q(k)$. Then from (3.1), (3.2) it follows

$$(\lambda M - M\lambda)\psi(k,x) = (qD - Dq)\psi(k,x).$$

Iterating this process we obtain

$$(ad^r_\lambda M)\psi(k,x) = (-1)^r(ad^r_D q)\psi(k,x),$$

where $ad_A B = A\circ B - B\circ A$ for any operators A,B. Now consider the difference operator $\widetilde{D}$ given by $\deg q$ iterations of the operation ad,

$$\widetilde{D} = ad_D^{\deg q} q(k).$$

As

$$a(x) = (-1)^{\deg q} ad_\lambda^{\deg q} M$$

becomes a polynomial in x, the function $\psi(k,x)$ is an eigenfunction for $\widetilde{D}$:

$$\widetilde{D}\psi(k,x) = a(x)\psi(k,x),$$

and therefore the commutativity relation holds:

$$[D,\widetilde{D}] = 0.$$

It happens that the difference operator D allows simple construction of the Baker–Akhiezer function itself. This method was introduced by Chalykh in [6] where such operators and the BA functions for the root systems and the $A_{n,1}(m)$ deformation were constructed. The formulas are as follows

$$\psi(k,x) = C(x)\,(D-\lambda(x))^M\left(Q(k)e^{(k,x)}\right) \tag{3.3}$$

where the number of iterations $M = \sum_{\alpha\in A} m_\alpha$, $Q(k)$ is the following polynomial in k

$$Q(k) = \prod_{\alpha\in A}\prod_{j=1}^{m_\alpha}(k+j\alpha,\alpha)(k-j\alpha,\alpha),$$

and $C(x)$ is a normalization function depending on x variables only. In the rational case such formulas for obtaining the Baker–Akhiezer functions through applying differential Calogero–Moser Hamiltonian were found earlier by Berest [2].

3.1 Root systems

Let $A = R = \{\alpha\}$ be a root system corresponding to a semisimple Lie algebra where we take exactly one of any pair of opposite roots. Let function $m(\alpha) = m_\alpha$ be invariant with respect to the action of the corresponding Weyl group.

Proposition 3. *For the system $\mathcal{R} = (R, m)$ there exists the Baker–Akhiezer function.*

Proof. Essentially this statement contains in [31]. More exactly, in [31] it was shown the existence of function $\psi(k, x)$ having the desired form (2.1) but satisfying conditions

$$\psi(k+s\alpha) = \psi(k-s\alpha), \tag{3.4}$$

at $(k,\alpha) = 0$ for all $\alpha \in R$, $s = 1,\ldots,m_\alpha$. It turns out that $\psi(k, x)$ also satisfies (2.2). Indeed, we have to check that for any $\alpha \in R$ and for any subsystem R_+ in $R \cup (-R)$ such that α is an edge vector one has

$$\prod_{\substack{\beta\in R_+\\ \beta\neq\alpha}}\prod_{i=1}^{m_\beta}(k+i\beta+s\alpha,\beta) = \prod_{\substack{\beta\in R_+\\ \beta\neq\alpha}}\prod_{i=1}^{m_\beta}(k+i\beta-s\alpha,\beta), \tag{3.5}$$

for $(k,\alpha) = 0$, $s = 1,\ldots,m_\alpha$. We remark that the condition that α is an edge vector for R_+ means that α is a simple root with respect to R_+. We show that the function $\prod_{\substack{\beta\in R_+\\ \beta\neq\alpha}}\prod_{i=1}^{m_\beta}(k+i\beta,\beta)$ is symmetric with respect to $(\alpha,k) = 0$, in particular that the identity (3.5) holds for arbitrary s. Indeed, if r_α is the reflection with respect to a root α then

$$r_\alpha \prod_{\substack{\beta\in R_+\\ \beta\neq\alpha}}\prod_{i=1}^{m_\beta}(k+i\beta,\beta) = \prod_{\substack{\beta\in R_+\\ \beta\neq\alpha}}\prod_{i=1}^{m_\beta}(r_\alpha k+i\beta,\beta) =$$

$$= \prod_{\substack{\beta\in R_+\\ \beta\neq\alpha}}\prod_{i=1}^{m_\beta}(k+ir_\alpha\beta, r_\alpha\beta) = \prod_{\substack{\gamma\in R_+\\ \gamma\neq\alpha}}\prod_{i=1}^{m_\gamma}(k+i\gamma,\gamma),$$

as for a simple root α the map $r_\alpha\colon R_+\setminus\alpha \to R_+\setminus\alpha$ is a one-to-one correspondence preserving the multiplicity function. ■

In order to construct the BA function let us first present the dual difference operator D. For the root system A_n this operator D was found by Ruijsenaars [24], and for an arbitrary root system the operators D were introduced by Macdonald [21]. For simplicity we will present here formulas for all reduced root systems except E_8, F_4, G_2. The last systems do not have the so called minuscule coweight but we need its existence for the formulas below. A minuscule coweight π is such a coweight that for any $\alpha \in R$ the scalar product (π, α) takes only three values 0,1, and -1 at most.

For example, the root system A_n consisting of the vectors $e_i - e_j$ in $\mathbb{R}^{n+1}$ has n non-trivial minuscule coweights given by the vectors $\pi_r = e_1 + ... + e_r$, where $1 \leqslant r \leqslant n$.

So we define following [21] the difference operator D_π by the formula

$$D_\pi = \sum_{\substack{\tau = w\pi \\ w \in W}} \left(\prod_{\substack{\alpha \in (R \cup (-R)) \\ (\alpha, \tau) = 1}} \left(1 - \frac{m_\alpha}{(\alpha, k)} \right) \right) T^\tau, \tag{3.6}$$

where W in the summation is the corresponding Weyl group, and the operator T^τ is the operator which shifts a function $f(k)$ to $f(k+\tau)$. In the following way the bispectral duality between the Calogero–Moser–Sutherland and Ruijsenaars–Macdonald systems was established by Chalykh.

Theorem 2. ([*6*]) *Let $\mathcal{A} = (A, m)$ be a positive part of any reduced root system of type A, B, C, D or E_6, E_7 with invariant multiplicity function. Let ψ be the corresponding Baker–Akhiezer function (2.1). Then the following two equations hold*

$$\left(\Delta - \sum_{\alpha \in A} \frac{m_\alpha (m_\alpha + 1)(\alpha, \alpha)}{\sinh^2 (\alpha, x)} \right) \psi = k^2 \psi,$$

$$D_\pi \psi = \sum_{w \in W} e^{(w\pi, x)} \psi,$$

where D_π is the difference operator (3.6) constructed for the root system $\frac{1}{2} A^\vee$ with a minuscule coweight π, and W is the corresponding Weyl group.

As it was shown in [6] the BA function can be expressed by formula (3.3) where D is the operator given by formula (3.6) constructed from the dual system $\frac{1}{2} A^\vee$ which means that we consider the set of vectors $\{\frac{\alpha}{(\alpha,\alpha)}\}$ instead of $\{\alpha\}$. And

$$C(x) = \left(\prod_{\alpha \in A} \left(\sum_{\substack{\tau = w\pi \\ w \in W}} (\alpha, \tau)(\alpha, \alpha) e^{(\tau, x)} \right)^{m_\alpha} \right)^{-1},$$

where π is a minuscule coweight for the root system $\{\frac{\alpha}{(\alpha,\alpha)}\}$. As to $\lambda(x)$ it is given by the formula

$$\lambda(x) = \sum_{\substack{\tau = w\pi \\ w \in W}} e^{(\tau, x)}.$$

3.2 Configuration $A_{n,1}(m)$

The system $A_{n,1}(m)$ consists of the vectors $e_p - e_q$, $p < q$, $p, q = 1, \dots, n$, $m_{e_p-e_q} = m$, and the vectors $e_p - \sqrt{m}\, e_{n+1}$, $p = 1, \dots, n$, $m_{e_p-\sqrt{m}\, e_{n+1}} = 1$. This configuration appeared in [30]. In [9] it was shown that the corresponding rational and trigonometric operators can be intertwined with the Laplacian thus they were algebraically integrable. In [10] it was shown that the rational version of the corresponding Schroedinger operator admits the corresponding (symmetric) Baker–Akhiezer function. The bispectral duality for the trigonometric version of this system as well as the existence of the BA function in the sense of [11] was obtained by Chalykh in [6].

Proposition 4. *There exists the Baker–Akhiezer function for the system $\mathcal{A} = A_{n,1}(m)$.*

Proof. In the paper [6] it was constructed a function $\psi(k, x)$ of the form (2.1), satisfying conditions (3.4) at $(k, \alpha) = 0$ for all $\alpha \in A$, $s = 1, \dots, m_\alpha$. It happens that as in the case of root systems (subsection 3.1), conditions (3.4) and (2.2) for the system $A_{n,1}(m)$ are equivalent. Indeed, if $\alpha = e_p - e_q$, then $\prod_{\beta\in A_+,\ \beta\neq\alpha}\prod_{i=1}^{m_\beta}(k + i\beta, \beta)$ is symmetric with respect to $(\alpha, k) = 0$. Consider now $\alpha = e_p - \sqrt{m}\, e_{n+1}$. In order to state (3.4) it is sufficient to check that in any two-dimensional plane π, $\pi \ni \alpha$ one has

$$\prod_{\substack{\beta\in A_+\cap\pi\\ \beta\neq\alpha}}\prod_{i=1}^{m_\beta}(k + i\beta + \alpha, \beta) = \prod_{\substack{\beta\in A_+\cap\pi\\ \beta\neq\alpha}}\prod_{i=1}^{m_\beta}(k + i\beta - \alpha, \beta) \tag{3.7}$$

at $k_p - \sqrt{m} k_{n+1} = 0$. There are two cases, either plane π contains only one vector $\beta \in A_+$, $\beta \neq \alpha$, or π contains two vectors β_1 and β_2. In the first case $(\alpha, \beta) = 0$ and relation (3.7) holds. In the second case the condition $\beta \in A_+$ allows to set $\beta_1 = e_q - e_p$, $\beta_2 = e_q - \sqrt{m}\, e_{n+1}$ or $\beta_1 = e_p - e_q$, $\beta_2 = \sqrt{m}\, e_{n+1} - e_q$ since $\alpha = e_p - \sqrt{m}\, e_{n+1}$ is an edge vector. For the first choice identity (3.7) takes the form

$$\begin{aligned}(k_q - k_p + 1)\dots(k_q - k_p + 2m - 1)(k_q - \sqrt{m}\, k_{n+1} + 2m + 1) &= \\ = (k_q - k_p + 3)\dots(k_q - k_p + 2m + 1)(k_q - \sqrt{m}\, k_{n+1} + 1),&\end{aligned}$$

which is valid at $k_p = \sqrt{m}\, k_{n+1}$. The second choice also gives a valid identity. ■

We present now the bispectral dual difference operator and the formula for the BA function both found by Chalykh in [6]. The operator is given by the following formulae

$$D = a_1 T_1 + \dots + a_n T_n + a_{n+1} T_{n+1}^{\sqrt{m}},$$

$$a_i = \left(1 - \frac{2}{k_i - \sqrt{m} k_{n+1} + 1 - m}\right)\prod_{j\neq i}^{n}\left(1 - \frac{2m}{k_i - k_j}\right), \quad i = 1, \dots, n, \tag{3.8}$$

$$a_{n+1} = \frac{1}{m}\prod_{i=1}^{n}\left(1 + \frac{2m}{k_i - \sqrt{m} k_{n+1} + 1 - m}\right),$$

where the operators T_i act on the functions $f(k)$ by shifting the ith argument k_i to $k_i + 2$, and $T_{n+1}^{\sqrt{m}} f(k_1, \dots, k_{n+1}) = f(k_1, \dots, k_{n+1} + 2\sqrt{m})$.

Theorem 3. ([6]) *Let $\psi(k,x)$ be the Baker–Akhiezer function for the system $A_{n,1}(m)$. Then $\psi(k,x)$ satisfies the following difference equation*

$$D\psi(k,x) = \lambda(x)\psi(k,x),$$

where the operator D is given by formulas (3.8), and

$$\lambda(x) = e^{2x_1} + \ldots + e^{2x_n} + \frac{1}{m} e^{2\sqrt{m}x_{n+1}}.$$

Also $\psi(k,x)$ itself can be expressed by the formula

$$\psi(k,x) = C(x)\,(D - \lambda(x))^M \left(Q(k)e^{(k,x)}\right)$$

with

$$C(x) = \left(2^M M! \prod_{i<j}^{n} (e^{2x_i} - e^{2x_j})^m \prod_{i=1}^{n} (e^{2x_i} - e^{2\sqrt{m}x_{n+1}})\right)^{-1}, \quad M = m\frac{n(n-1)}{2} + n,$$

$$Q(k) = \prod_{i<j}^{n} \prod_{s=1}^{m} \left((k_i - k_j)^2 - 4s^2\right) \prod_{i=1}^{n} \left((k_i - \sqrt{m}k_{n+1})^2 - (m+1)^2\right).$$

Remark 2. When $m = 1$ the system $A_{n,1}(m)$ coincides with the root system A_n with multiplicity $m = 1$, and the operator D degenerates to the corresponding Ruijsenaars–Macdonald operator (3.6) with coweight $\pi = e_1$.

4 Configuration $C_n(l,m)$

This system consists of the following vectors in $\mathbb{C}^n$ depending on two integer parameters l, m. The vectors $\sqrt{2m+1}\,e_i$ have multiplicities $m_i = l$, $i = 1,\ldots,n-1$, the vector $\sqrt{2l+1}\,e_n$ has multiplicity $m_n = m$, the vectors $\frac{\sqrt{2m+1}}{2}(e_i \pm e_j)$ have multiplicities $m_{ij} = \frac{2l+1}{2m+1}$, $1 \leqslant i < j \leqslant n-1$ (it is assumed that $\frac{2l+1}{2m+1} \in \mathbb{Z}$), and the vectors $\frac{\sqrt{2m+1}\,e_i \pm \sqrt{2l+1}\,e_n}{2}$ have multiplicities $m_{in} = 1$, $i = 1,\ldots,n-1$.

The configuration was introduced in [10] where the BA functions related to rational potentials corresponding to this system was under investigations. For the trigonometric version related to the $\mathcal{C}_2(m,l)$ system the intertwining operator to the pure Laplacian was constructed earlier in [9] (see also [30]).

We note at first that all the two-dimensional subsystems in $C_n(l,m)$ have the form either of the system $A_{2,1}(m)$ or the one of a root system or the form of the subsystem $C_2(l,m)$. We have noticed already that for a root system R and for the system $A_{n,1}(m)$ identity (3.5) holds. It also holds for the system $C_2(l,m)$ and therefore for the system $C_n(l,m)$. Thus for the system $C_n(l,m)$, as well as for the systems R, $A_{n,1}(m)$, conditions (2.2) for the Baker-Akhiezer function are equivalent to simpler conditions (3.4).

Now we start constructing the BA function for the system $C_n(l,m)$. The effective method we are going to use was found by Chalykh [6]. The method is based on finding a difference operator D with special properties. Then the BA function $\psi(k,x)$ is obtained by multiple application of such operator D to some initial function φ_0.

For the system $C_n(l,m)$ we define operator D by the following formulas

$$D = \sum_{i=1}^{n} a_i^+ T_i^+ + a_i^- T_i^-, \tag{4.1}$$

where $T_i^\pm$ are difference operators which act as follows

$$T_i^\pm f(k_1,\ldots,k_i,\ldots,k_n) = f(k_1,\ldots,k_i \pm \sqrt{2m+1},\ldots,k_n),$$
$$i = 1,\ldots,n-1,$$

$$T_n^\pm f(k_1,\ldots,k_n) = f(k_1,\ldots,k_n \pm \sqrt{2l+1}).$$

The coefficients $a_i^\pm$ are functions of k which are defined by the formulas

$$a_i^\pm = \prod_{j=1}^{n} a_{ij}^\pm, \qquad i = 1,\ldots,n,$$

where

$$a_{ij}^\pm = \left(1 - \frac{2l+1}{\pm\overline{k}_i + \overline{k}_j}\right)\left(1 - \frac{2l+1}{\pm\overline{k}_i - \overline{k}_j}\right), \qquad 1 \leqslant i,j \leqslant n-1,\ i \neq j,$$

$$a_{ii}^\pm = \frac{1}{2m+1}\left(1 - \frac{(2m+1)l}{\pm\overline{k}_i}\right), \qquad i = 1,\ldots,n-1,$$

$$a_{in}^\pm = \left(1 - \frac{2m+1}{\pm\overline{k}_i + \overline{k}_n - l + m}\right)\left(1 - \frac{2m+1}{\pm\overline{k}_i - \overline{k}_n - l + m}\right), \qquad i = 1,\ldots,n-1,$$

$$a_{nj}^\pm = \left(1 - \frac{2l+1}{\pm\overline{k}_n + \overline{k}_j + l - m}\right)\left(1 - \frac{2l+1}{\pm\overline{k}_n - \overline{k}_j + l - m}\right), \qquad j = 1,\ldots,n-1,$$

$$a_{nn}^\pm = \frac{1}{2l+1}\left(1 - \frac{(2l+1)m}{\pm\overline{k}_n}\right).$$

In the above formulas and throughout this section we use notation $\overline{k}_i = \sqrt{2m+1}\,k_i$ for $i = 1,\ldots,n-1$, and $\overline{k}_n = \sqrt{2l+1}\,k_n$.

Remark 3. When $m = l$ the system $\mathcal{C}_n(l,m)$ becomes the root system C_n consisting of the vectors $\sqrt{2m+1}e_i$ with multiplicities m and the vectors $\frac{\sqrt{2m+1}}{2}(e_i \pm e_j)$ with multiplicities 1. Then operator (4.1) is a $\frac{1}{2m+1}$ multiple of the corresponding Macdonald operator (3.6) written for the root system $B_n = \frac{1}{2}C_n^\vee$ consisting of the vectors $\frac{1}{\sqrt{2m+1}}e_i$ with multiplicity m, $\frac{1}{\sqrt{2m+1}}(e_i \pm e_j)$ with multiplicity 1, and the minuscule coweight $\pi = \sqrt{2m+1}e_1$.

The next step is to prove invariance of the space V of holomorphic functions $f(k)$ satisfying

$$f(k + s\alpha) = f(k - s\alpha) \quad \text{at } (k,\alpha) = 0 \tag{4.2}$$

for $s = 1, \dots, m_\alpha$, for all $\alpha \in \mathcal{C}_n(l,m)$, under the action of operator (4.1). Notice that for the system $C_n(l,m)$ conditions (4.2) can be rewritten in the following form. For $\alpha = \sqrt{2m+1}\, e_i, i \leqslant n-1$, and $\alpha = \sqrt{2l+1}\, e_n$

$$(T_i^+)^s f = (T_i^-)^s f \quad \text{at } \overline{k}_i = 0, \quad i = 1, \dots, n-1,\ s \leqslant l; \text{ and } \ i = n, s \leqslant m. \tag{4.3}$$

For $\alpha = \frac{\sqrt{2m+1}}{2}(e_i - e_j)$

$$\begin{gathered}(T_i^+)^s f = (T_j^+)^s f \\ \text{at } \overline{k}_i - \overline{k}_j = 0, \quad i,j = 1,\dots,n-1,\ s = 1,\dots,\frac{2l+1}{2m+1},\end{gathered} \tag{4.4}$$

or equivalently

$$\begin{gathered}(T_i^-)^s f = (T_j^-)^s f \\ \text{at } \overline{k}_i - \overline{k}_j = 0, \quad i,j = 1,\dots,n-1,\ s = 1,\dots,\frac{2l+1}{2m+1}.\end{gathered} \tag{4.4$'$}$$

For $\alpha = \frac{\sqrt{2m+1}}{2}(e_i + e_j)$

$$\begin{gathered}(T_i^+)^s f = (T_j^-)^s f \\ \text{at } \overline{k}_i + \overline{k}_j = 0, \quad i,j = 1,\dots,n-1,\ s = 1,\dots,\frac{2l+1}{2m+1},\end{gathered} \tag{4.5}$$

or equivalently

$$\begin{gathered}(T_i^-)^s f = (T_j^+)^s f \\ \text{at } \overline{k}_i + \overline{k}_j = 0, \quad i,j = 1,\dots,n-1,\ s = 1,\dots,\frac{2l+1}{2m+1}.\end{gathered} \tag{4.5$'$}$$

For $\alpha = \frac{\sqrt{2m+1}\, e_i - \sqrt{2l+1}\, e_n}{2}$

$$T_i^+ f = T_n^+ f \qquad \text{at } \overline{k}_i - \overline{k}_n - l + m = 0, \quad i = 1,\dots,n-1, \tag{4.6}$$

or equivalently

$$T_i^- f = T_n^- f \qquad \text{at } \overline{k}_i - \overline{k}_n + l - m = 0, \quad i = 1,\dots,n-1. \tag{4.6$'$}$$

Finally, for the case $\alpha = \frac{\sqrt{2m+1}\, e_i + \sqrt{2l+1}\, e_n}{2}$ conditions (4.2) may be represented as

$$T_i^+ f = T_n^- f \qquad \text{at } \overline{k}_i + \overline{k}_n - l + m = 0, \quad i = 1,\dots,n-1, \tag{4.7}$$

and also

$$T_i^- f = T_n^+ f \qquad \text{at } \overline{k}_i + \overline{k}_n + l - m = 0, \quad i = 1,\dots,n-1. \tag{4.7$'$}$$

The validity of the transformation from the form (4.2) to the form (4.3)–(4.7) can be simply established. For example, consider condition (4.2) for $\alpha = \frac{\sqrt{2m+1}\, e_i + \sqrt{2l+1}\, e_n}{2}$. Obviously it can be written as

$$(T_i^+ - T_n^-) f\left(k + \frac{-\sqrt{2m+1}\, e_i + \sqrt{2l+1}\, e_n}{2}\right) = 0, \qquad \text{at } \ \overline{k}_i + \overline{k}_n = 0.$$

We are left to point out that the set

$$\left\{k+\frac{-\sqrt{2m+1}\,e_i+\sqrt{2l+1}\,e_n}{2}\;\Big|\;\;\overline{k}_i+\overline{k}_n=0\right\}$$

is given by the equation $\overline{k}_i+\overline{k}_n+m-l=0$. Thus we arrive to record (4.7). Representing condition (4.2) in the form

$$(T_i^- - T_n^+)f\left(k+\frac{\sqrt{2m+1}\,e_i-\sqrt{2l+1}\,e_n}{2}\right)=0, \qquad \text{at}\;\; \overline{k}_i+\overline{k}_n=0.$$

we get record (4.7′). The form (4.6) is obtained analogously. The equivalence of conditions (4.3)–(4.5) to the corresponding conditions (4.2) is obvious.

Proposition 5. *Let D be operator* (4.1)*, let $f(k_1,\ldots,k_n)$ be any holomorphic function satisfying conditions* (4.3)–(4.7)*. Then the function $Df(k_1,\ldots,k_n)$ is also holomorphic.*

Proof. In principle the function $Df(k_1,\ldots,k_n)$ could have singularities at the hyperplanes where the operator D is singular. We will show that this doesn't happen by the subsequent consideration of the singularities of the operator D.

a) $k_i=0$, $i=1,\ldots,n$. We collect terms in $Df(k_1,\ldots,k_n)$ which are singular at $k_i=0$. We have

$$Df=\sum_{j=1}^{n}a_j^+T_j^+(f)+a_j^-T_j^-(f)=-\frac{\epsilon}{\overline{k}_i}\left(\prod_{j\neq i}a_{ij}^+T_i^+f-\prod_{j\neq i}a_{ij}^-T_i^-f\right)+f_i(k),$$

where $\epsilon=l$ for $i<n$ and $\epsilon=m$ for $i=n$; the functions $f_i(k)$ are holomorphic at $\overline{k}_i=0$. We note that $a_{ij}^+=a_{ij}^-$ at $\overline{k}_i=0$, therefore $a_{ij}^+=a_{ij}^-+\overline{k}_ih_{ij}(k)$ where $h_{ij}(k)$ are holomorphic at $\overline{k}_i=0$, and we obtain the relation

$$\sum_{j=1}^{n}a_j^+T_j^+f+a_j^-T_j^-f=-\Big(\epsilon\prod_{j\neq i}a_{ij}^+\Big)\frac{1}{\overline{k}_i}(T_i^+f-T_i^-f)+\widetilde{f}_i(k),$$

where $\widetilde{f}_i(k)$ is holomorphic at $\overline{k}_i=0$. Thus because of conditions (4.3) the function Df is non-singular at $\overline{k}_i=0$.

b) $\overline{k}_i-\overline{k}_j=0$, $i,j=1,\ldots,n-1$. For the appropriate functions f_{ij}, $\widetilde{f}_{ij}$, $\approx\!\!\!\!\!f_{ij}$ holomorphic at $\overline{k}_i = \overline{k}_j$ the following chain of equalities takes place

$$Df = a_i^+ T_i^+ f + a_j^+ T_j^+ f + a_i^- T_i^- f + a_j^- T_j^- f + f_{ij} =$$
$$= -\frac{2l+1}{\overline{k}_i - \overline{k}_j}\left(1 - \frac{2l+1}{\overline{k}_i + \overline{k}_j}\right)\frac{1}{2m+1}\left(1 - \frac{(2m+1)l}{\overline{k}_i}\right)\prod_{s\neq i,j} a_{is}^+ T_i^+ f -$$
$$-\frac{2l+1}{\overline{k}_j - \overline{k}_i}\left(1 - \frac{2l+1}{\overline{k}_i + \overline{k}_j}\right)\frac{1}{2m+1}\left(1 - \frac{(2m+1)l}{\overline{k}_j}\right)\prod_{s\neq i,j} a_{js}^+ T_j^+ f +$$
$$+\frac{2l+1}{\overline{k}_i - \overline{k}_j}\left(1 + \frac{2l+1}{\overline{k}_i + \overline{k}_j}\right)\frac{1}{2m+1}\left(1 + \frac{(2m+1)l}{\overline{k}_i}\right)\prod_{s\neq i,j} a_{is}^- T_i^- f +$$
$$+\frac{2l+1}{\overline{k}_j - \overline{k}_i}\left(1 + \frac{2l+1}{\overline{k}_i + \overline{k}_j}\right)\frac{1}{2m+1}\left(1 + \frac{(2m+1)l}{\overline{k}_j}\right)\prod_{s\neq i,j} a_{js}^- T_j^- f + \widetilde{f}_{ij} =$$
$$= -(2l+1)\left(1 - \frac{2l+1}{\overline{k}_i + \overline{k}_j}\right)\frac{1}{2m+1}\left(1 - \frac{(2m+1)l}{\overline{k}_i}\right)\prod_{s\neq i,j} a_{is}^+ \cdot \frac{1}{\overline{k}_i - \overline{k}_j}(T_i^+ f - T_j^+ f) +$$
$$+(2l+1)\left(1 + \frac{2l+1}{\overline{k}_i + \overline{k}_j}\right)\frac{1}{2m+1}\left(1 + \frac{(2m+1)l}{\overline{k}_i}\right)\prod_{s\neq i,j} a_{is}^+ \cdot \frac{1}{\overline{k}_i - \overline{k}_j}(T_i^- f - T_j^- f) + \approx{f}_{ij},$$

as one has $a_{is}^\pm = a_{js}^\pm$ at $k_i = k_j$ for $s \neq i,j$. Thus because of conditions (4.4), (4.4′) the function Df has no singularities at $\overline{k}_i - \overline{k}_j = 0$. Further it is easy to see the invariance of the operator D under reflections around $\overline{k}_j = 0$, $j = 1, \ldots, n$. But the hyperplane $\overline{k}_i - \overline{k}_j = 0$ is mapped to $\overline{k}_i + \overline{k}_j = 0$ under such a reflection. Therefore Df is non-singular also at the hyperplanes $\overline{k}_i + \overline{k}_j = 0$, $i, j = 1, \ldots, n-1$.

We are left to analyze the possible singularities of the function Df at the hyperplanes $\overline{k}_i \pm \overline{k}_n \pm (l - m) = 0$, $i = 1, \ldots, n-1$. Because of the mentioned symmetry of the operator D it is enough to restrict considerations to the hyperplanes $\overline{k}_i - \overline{k}_n + l - m = 0$.

c) $\overline{k}_i - \overline{k}_n + l - m = 0$, $i = 1, \ldots, n-1$. The coefficients of operator D which are singular at this hyperplane are a_i^-, a_n^-. We have

$$Df = a_i^- T_i^- f + a_n^- T_n^- f + f_{in} =$$
$$= \frac{1}{2m+1}\left(1 + \frac{(2m+1)l}{\overline{k}_i}\right)\left(1 + \frac{2m+1}{\overline{k}_i + \overline{k}_n + l - m}\right) \times$$
$$\times \prod_{j\neq i,n} a_{ij}^- \left(-\frac{2m+1}{-\overline{k}_i + \overline{k}_n - l + m}\right) T_i^- f +$$
$$+\frac{1}{2l+1}\left(1 + \frac{(2l+1)m}{\overline{k}_n}\right)\left(1 + \frac{2l+1}{\overline{k}_n + \overline{k}_i - l + m}\right) \times$$
$$\times \prod_{j\neq i,n} a_{nj}^- \left(-\frac{2l+1}{-\overline{k}_n + \overline{k}_i + l - m}\right) T_n^- f + \widetilde{f}_{in},$$

where f_{in}, $\widetilde{f}_{in}$ are some functions which are holomorphic at $\overline{k}_i - \overline{k}_n + l - m = 0$. Obviously one has $a^-_{ij} = a^-_{nj}$, $j \neq i, n$ at $\overline{k}_i - \overline{k}_n + l - m = 0$. Moreover, one has

$$\left(1 + \frac{(2m+1)l}{\overline{k}_i}\right)\left(1 + \frac{2m+1}{\overline{k}_i + \overline{k}_n + l - m}\right) =$$
$$= \frac{\overline{k}_i + (2m+1)l}{\overline{k}_i} \cdot \frac{\overline{k}_i + \overline{k}_n + l + m + 1}{\overline{k}_i + \overline{k}_n + l - m} =$$
$$= \frac{\overline{k}_n + (2l+1)m}{\frac{1}{2}(\overline{k}_i + \overline{k}_n - l + m)} \cdot \frac{\overline{k}_i + \overline{k}_n + l + m + 1}{2\overline{k}_n} =$$
$$= \left(1 + \frac{(2l+1)m}{\overline{k}_n}\right)\left(1 + \frac{2l+1}{\overline{k}_n + \overline{k}_i - l + m}\right)$$

at this hyperplane. Therefore we can extend the equality for Df as follows

$$Df = \left(1 + \frac{(2m+1)l}{\overline{k}_i}\right)\left(1 + \frac{2m+1}{\overline{k}_i + \overline{k}_n + l - m}\right) \times$$
$$\times \prod_{j \neq i,n} a^-_{ij} \cdot \frac{1}{\overline{k}_i - \overline{k}_n + l - m}(T^-_i f - T^-_n f) + \overset{\approx}{f}_{in}$$

for some function $\overset{\approx}{f}_{in}$ holomorphic at $\overline{k}_i - \overline{k}_n + l - m = 0$. Because of (4.6′) the function Df is non-singular at $\overline{k}_i - \overline{k}_n + l - m = 0$. Thus the proposition is fully proved. ■

Proposition 6. *For any holomorphic function $f(k_1, \ldots, k_n)$ satisfying conditions* (4.3)–(4.7) *the function $Df(k_1, \ldots, k_n)$ also satisfies conditions* (4.3)–(4.7) *if D is operator* (4.1).

Proof. We consider different hyperplanes and subsequently show that the operator D keeps axiomatics at any of the hyperplanes.

a) $\pi_i = \{k_i = 0\}$, $1 \leqslant i \leqslant n$.
We have

$$(T_i^{+s} - T_i^{-s})Df = (T_i^{+s} - T_i^{-s})\sum_{j=1}^{n}(a^+_j T^+_j + a^-_j T^-_j)f =$$
$$= \sum_{j \neq i}\left(T_i^{+s}(a^+_j)T^+_j T_i^{+s} f - T_i^{-s}(a^+_j)T^+_j T_i^{-s} f\right) +$$
$$+ \sum_{j \neq i}\left(T_i^{+s}(a^-_j)T^-_j T_i^{+s} f - T_i^{-s}(a^-_j)T^-_j T_i^{-s} f\right) +$$
$$+ T_i^{+s}(a^+_i)T_i^{+s+1} f - T_i^{-s}(a^-_i)T_i^{-s+1} f + T_i^{+s}(a^-_i)T_i^{+s-1} f - T_i^{-s}(a^+_i)T_i^{-s-1} f. \tag{4.8}$$

If $j \neq i$ then the functions $a^\pm_j$ are invariant with respect to reflection s_i around the hyperplane π_i. Therefore $T_i^{+s}(a^\pm_j)|_{\pi_i} = T_i^{-s}(a^\pm_j)|_{\pi_i}$. As $s_i(a^+_i) = a^-_i$ we get $s_i(T_i^{+s}(a^+_i)) =$

$T_i^{-s}(a_i^-)$ and in particular $T_i^{+s}(a_i^+)|_{\pi_i} = T_i^{-s}(a_i^-)|_{\pi_i}$. Thus the right-hand side of (4.8) can be rewritten in the form

$$\sum_{j\neq i} T_i^{+s}(a_j^+)T_j^+\left(T_i^{+s}-T_i^{-s}\right)f+\sum_{j\neq i} T_i^{+s}(a_j^-)T_j^-\left(T_i^{+s}-T_i^{-s}\right)f+$$
$$+T_i^{+s}(a_i^+)\big(T_i^{+s+1}-T_i^{-s+1}\big)f+T_i^{+s}(a_i^-)\big(T_i^{+s-1}-T_i^{-s-1}\big)f.$$

Because of conditions (4.3) at $s<m_i$ everything is proven. For $s=m_i$ we are left to notice that $T_i^{+s}(a_i^\pm)|_{\pi_i}=0$.

b) $\pi_{ij}=\{k_i=k_j\}$, $1\leqslant i<j<n$.
We have

$$(T_i^{+s}-T_j^{+s})Df=(T_i^{+s}-T_j^{+s})\sum_{q=1}^{n}(a_q^+T_q^+ + a_q^-T_q^-)f=$$
$$=\sum_{q\neq i,j}\big(T_i^{+s}(a_q^+)T_q^+T_i^{+s}f-T_j^{+s}(a_q^+)T_q^+T_j^{+s}f\big)+$$
$$+\sum_{q\neq i,j}\big(T_i^{+s}(a_q^-)T_q^-T_i^{+s}f-T_j^{+s}(a_q^-)T_q^-T_j^{+s}f\big)+$$
$$+\big(T_i^{+s}(a_i^+)T_i^{+s+1}f-T_j^{+s}(a_j^+)T_j^{+s+1}f\big)+$$
$$+\big(T_i^{+s}(a_i^-)T_i^{+s-1}f-T_j^{+s}(a_j^-)T_j^{+s-1}f\big)+$$
$$+\big(T_i^{+s}(a_j^+)T_i^{+s}T_j^+f-T_j^{+s}(a_i^+)T_j^{+s}T_i^+f\big)+$$
$$+\big(T_i^{+s}(a_j^-)T_i^{+s}T_j^-f-T_j^{+s}(a_i^-)T_j^{+s}T_i^-f\big). \quad (4.9)$$

We show that sum (4.9) vanishes at the hyperplane π_{ij}. For $q\neq i,j$ the functions $a_q^\pm$ are invariant with respect to reflection s_{ij} around the hyperplane $k_i=k_j$. Therefore $T_i^{+s}(a_q^\pm)|_{\pi_{ij}}=T_j^{+s}(a_q^\pm)|_{\pi_{ij}}$. As $s_{ij}(a_i^\pm)=a_j^\pm$ we get

$$s_{ij}(T_i^{+s}(a_i^\pm))=T_j^{+s}(a_j^\pm),\qquad s_{ij}(T_j^{+s}(a_i^\pm))=T_i^{+s}(a_j^\pm),$$

and in particular

$$T_i^{+s}(a_i^\pm)|_{\pi_{ij}}=T_j^{+s}(a_j^\pm)|_{\pi_{ij}},\qquad T_j^{+s}(a_i^\pm)|_{\pi_{ij}}=T_i^{+s}(a_j^\pm)|_{\pi_{ij}}.$$

Totally we conclude that the right-hand side of (4.9) can be rewritten as

$$\sum_{q\neq i,j}T_i^{+s}(a_q^+)T_q^+\left(T_i^{+s}-T_j^{+s}\right)f+\sum_{q\neq i,j}T_i^{+s}(a_q^-)T_q^-\left(T_i^{+s}-T_j^{+s}\right)f+$$
$$+T_i^{+s}(a_i^+)\left(T_i^{+s+1}-T_j^{+s+1}\right)f+T_i^{+s}(a_i^-)\left(T_i^{+s-1}-T_j^{+s-1}\right)f+$$
$$+T_i^{+s}(a_j^+)T_i^+T_j^+\left(T_i^{+s-1}-T_j^{+s-1}\right)f+T_i^{+s}(a_j^-)T_i^-T_j^-\left(T_i^{+s+1}-T_j^{+s+1}\right)f. \quad (4.10)$$

Because of conditions (4.4) the first two sums in (4.10) equal zero. As the shifts along the vectors $\overline{e}_i+\overline{e}_j$, $-\overline{e}_i-\overline{e}_j$ do not change $\overline{k}_i-\overline{k}_j$, the left four terms at $s<m_{ij}$ also vanish because of (4.4). If $s=m_{ij}$ then this is correct if we recall that $T_i^{+m_{ij}}(a_i^+)=T_i^{+m_{ij}}(a_j^-)=0$ at $k\in\pi_{ij}$.

c) $\pi_{in}=\{\overline{k}_n-\overline{k}_i+l-m=0\}$, $1\leqslant i<n$.
We have

$$
\begin{aligned}
&(T_n^+-T_i^+)Df=(T_n^+-T_i^+)\sum_{q=1}^{n}(a_q^+T_q^++a_q^-T_q^-)f=\\
&=\sum_{q\neq i,n}\left(T_n^+(a_q^+)T_q^+T_n^+f-T_i^+(a_q^+)T_q^+T_i^+f\right)+\\
&+\sum_{q\neq i,n}\left(T_n^+(a_q^-)T_q^-T_n^+f-T_i^+(a_q^-)T_q^-T_i^+f\right)+\\
&\qquad+\left(T_n^+-T_i^+\right)\left(a_n^+T_n^++a_n^-T_n^-+a_i^+T_i^++a_i^-T_i^-\right)f. \qquad (4.11)
\end{aligned}
$$

We notice that both sums in (4.11) vanish like in the case b) because $T_n^+(a_q^\pm)=T_i^+(a_q^\pm)$. Indeed,

$$
\begin{aligned}
&T_n^+(a_q^\pm)=\prod_{t\neq n}a_{qt}^\pm T_n^+(a_{qn}^\pm)=\\
&=\prod_{t\neq n,i}a_{qt}^\pm a_{qi}^\pm T_n^+\left(1-\frac{2m+1}{\pm\overline{k}_q+\overline{k}_n-l+m}\right)\left(1-\frac{2m+1}{\pm\overline{k}_q-\overline{k}_n-l+m}\right)=\\
&=\prod_{t\neq n,i}a_{qt}^\pm\left(1-\frac{2l+1}{\pm\overline{k}_q+\overline{k}_i}\right)\left(1-\frac{2l+1}{\pm\overline{k}_q-\overline{k}_i}\right)\times\\
&\qquad\times\left(1-\frac{2m+1}{\pm\overline{k}_q+\overline{k}_n+l+m+1}\right)\left(1-\frac{2m+1}{\pm\overline{k}_q-\overline{k}_n-3l+m-1}\right).
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
&T_i^+(a_q^\pm)=\prod_{t\neq n,i}a_{qt}^\pm\left(1-\frac{2m+1}{\pm\overline{k}_q+\overline{k}_n-l+m}\right)\left(1-\frac{2m+1}{\pm\overline{k}_q-\overline{k}_n-l+m}\right)\times\\
&\qquad\times\left(1-\frac{2l+1}{\pm\overline{k}_q+\overline{k}_i+2m+1}\right)\left(1-\frac{2l+1}{\pm\overline{k}_q-\overline{k}_i-2m-1}\right).
\end{aligned}
$$

It is easy to check that if $k \in \pi_{in}$ then one has

$$\left(1-\frac{2l+1}{\pm\overline{k}_q+\overline{k}_i}\right)\left(1-\frac{2l+1}{\pm\overline{k}_q-\overline{k}_i}\right)\left(1-\frac{2m+1}{\pm\overline{k}_q+\overline{k}_n+l+m+1}\right)\times$$
$$\times\left(1-\frac{2m+1}{\pm\overline{k}_q-\overline{k}_n-3l+m-1}\right)=$$
$$=\left(1-\frac{2m+1}{\pm\overline{k}_q+\overline{k}_n-l+m}\right)\left(1-\frac{2m+1}{\pm\overline{k}_q-\overline{k}_n-l+m}\right)\times$$
$$\times\left(1-\frac{2l+1}{\pm\overline{k}_q+\overline{k}_i+2m+1}\right)\left(1-\frac{2l+1}{\pm\overline{k}_q-\overline{k}_i-2m-1}\right).$$

Thus the right-hand side of (4.11) is simplified to the following expression

$$T_n^+(a_n^+){T_n^+}^2f-T_i^+(a_i^+){T_i^+}^2f+T_n^+(a_i^-)T_n^+T_i^-f-T_i^+(a_n^-)T_i^+T_n^-f+$$
$$+\big(T_n^+(a_i^+)-T_i^+(a_n^+)\big)T_i^+T_n^+f+\big(T_n^+(a_n^-)-T_i^+(a_i^-)\big)f. \quad (4.12)$$

We note that

$$T_n^+(a_{ni}^+)=T_i^+(a_{in}^+)=T_n^+(a_{in}^i)=T_i^+(a_{ni}^-)=0$$

at $k\in\pi_{in}$. We are left to check that

$$T_n^+(a_i^+)=T_i^+(a_n^+), \quad (4.13)$$
$$T_n^+(a_n^-)=T_i^+(a_i^-). \quad (4.14)$$

We note that if $t\neq i,n$ then

$$T_n^+(a_{it}^+)=a_{it}^+=a_{nt}^+=T_i^+(a_{nt}^+).$$

Therefore condition (4.13) is reduced to the condition $a_{ii}^+T_n^+(a_{in}^+)=a_{nn}^+T_i^+(a_{ni}^+)$, or

$$\frac{1}{2m+1}\left(1-\frac{(2m+1)l}{\overline{k}_i}\right)\left(1-\frac{2m+1}{\overline{k}_i+\overline{k}_n+l+m+1}\right)\times$$
$$\times\left(1-\frac{2m+1}{\overline{k}_i-\overline{k}_n-3l+m-1}\right)=$$
$$=\frac{1}{2l+1}\left(1-\frac{(2l+1)m}{\overline{k}_n}\right)\left(1-\frac{2l+1}{\overline{k}_n+\overline{k}_i+l+m+1}\right)\times$$
$$\times\left(1-\frac{2l+1}{\overline{k}_n-\overline{k}_i+l-3m-1}\right),$$

which is valid. We are left to check condition (4.14). We note that if $t\neq i,n$ then

$$T_n^+(a_{nt}^-)=T_n^+(1-\frac{2l+1}{-\overline{k}_n+\overline{k}_t+l-m})(1-\frac{2l+1}{-\overline{k}_n-\overline{k}_t+l-m})=$$
$$=(1-\frac{2l+1}{-\overline{k}_n+\overline{k}_t-l-m-1})(1-\frac{2l+1}{-\overline{k}_n-\overline{k}_t-l-m-1})=$$
$$=(1-\frac{2l+1}{-\overline{k}_i+\overline{k}_t-2m-1})(1-\frac{2l+1}{-\overline{k}_i-\overline{k}_t-2m-1})=$$
$$=T_i^+(1-\frac{2l+1}{-\overline{k}_i+\overline{k}_t})(1-\frac{2l+1}{-\overline{k}_i-\overline{k}_t})=T_i^+(a_{it}^-).$$

Therefore identity (4.14) is reduced to the following relation

$$T_n^+(a_{nn}^-a_{ni}^-) = T_i^+(a_{ii}^-a_{in}^-).$$

Substituting the corresponding expressions we get

$$\frac{1}{2l+1}\left(1+\frac{(2l+1)m}{\overline{k}_n+2l+1}\right)\left(1-\frac{2l+1}{-\overline{k}_n+\overline{k}_i-l-m-1}\right)\times$$
$$\times\left(1-\frac{2l+1}{-\overline{k}_n-\overline{k}_i-l-m-1}\right)=\frac{1}{2m+1}\left(1+\frac{(2m+1)l}{\overline{k}_i+2m+1}\right)\times$$
$$\times\left(1-\frac{2m+1}{-\overline{k}_i+\overline{k}_n-l-m-1}\right)\left(1-\frac{2m+1}{-\overline{k}_i-\overline{k}_n-l-m-1}\right),$$

equivalently,

$$\left(1+\frac{(2l+1)m}{\overline{k}_n+2l+1}\right)\left(1+\frac{2l+1}{\overline{k}_n+\overline{k}_i+l+m+1}\right)=$$
$$=\left(1+\frac{(2m+1)l}{\overline{k}_i+2m+1}\right)\left(1+\frac{2m+1}{\overline{k}_i+\overline{k}_n+l+m+1}\right),$$

which is valid for $k\in\pi_{in}$.

d) The fact that axiomatics at the hyperplanes $k_i+k_j=0$, $\overline{k}_n+\overline{k}_i+l-m=0$, $i,j=1,\dots,n-1$ is preserved can be checked analogously to the cases b) and c) correspondingly. Thus the proposition is proved. ■

Now we are ready to construct the Baker–Akhiezer function $\psi(k,x)$. We define the sequence of functions $\varphi_i(k,x)$ by the following formulas. Let

$$\varphi_0=\prod_{\alpha\in\mathcal{C}_n(l,m)}\prod_{s=1}^{m_\alpha}(k+s\alpha,\alpha)(k-s\alpha,\alpha)e^{(k,x)}. \tag{4.15}$$

More explicitly we have

$$\varphi_0=a\prod_{i=1}^{n-1}\prod_{s=1}^{l}(\overline{k}_i^2-s^2(2m+1)^2)\prod_{s=1}^{m}(\overline{k}_n^2-s^2(2l+1)^2)\times$$
$$\prod_{i=1}^{n-1}\left((\overline{k}_i+\overline{k}_n)^2-(m+l+1)^2\right)\left((\overline{k}_i-\overline{k}_n)^2-(m+l+1)^2\right)\times$$
$$\prod_{i<j}^{n-1}\prod_{s=1}^{\frac{2l+1}{2m+1}}\left((\overline{k}_i+\overline{k}_j)^2-s^2(2m+1)^2\right)\left((\overline{k}_i-\overline{k}_j)^2-s^2(2m+1)^2\right)e^{(k,x)},$$

where

$$a=2^{2(1-n)(2+(n-2)\frac{2l+1}{2m+1})}.$$

Then we define

$$\varphi_{i+1} = \left(D - \frac{2}{2m+1} \sum_{j=1}^{n-1} \cosh \sqrt{2m+1}\, x_j - \frac{2}{2l+1} \cosh \sqrt{2l+1}\, x_n \right) \varphi_i. \tag{4.16}$$

It turns out that at the step

$$M = \sum_{\alpha \in \mathcal{C}_n(l,m)} m_\alpha = (2+l)(n-1) + m + (n-1)(n-2)\frac{2l+1}{2m+1} \tag{4.17}$$

one gets the BA function. Before formulating the theorem let us introduce the abbreviations $\overline{x}_i = \sqrt{2m+1}x_i$ for $i = 1, \ldots, n-1$, and $\overline{x}_n = \sqrt{2l+1}x_n$.

Theorem 4. *The Baker–Akhiezer function is given by the formula*

$$\psi(k,x) = c^{-1}(x)\varphi_M,$$

where φ_M *is defined by formulas* (4.15), (4.16), (4.17), *and*

$$c(x) = M!\ (e^{\overline{x}_n} - e^{-\overline{x}_n})^m \times$$
$$\times \prod_{i=1}^{n-1} (e^{\overline{x}_i} - e^{-\overline{x}_i})^l \prod_{i<j}^{n-1} (e^{\overline{x}_i - \overline{x}_j} - e^{\overline{x}_j - \overline{x}_i})^{\frac{2l+1}{2m+1}} (e^{\overline{x}_i + \overline{x}_j} - e^{-\overline{x}_i - \overline{x}_j})^{\frac{2l+1}{2m+1}} \times$$
$$\times \prod_{i=1}^{n-1} (e^{\overline{x}_i - \overline{x}_n} - e^{\overline{x}_n - \overline{x}_i})(e^{\overline{x}_i + \overline{x}_n} - e^{-\overline{x}_i - \overline{x}_n}).$$

Proof. For the function φ_0 the axiomatic conditions (2.2′) are clearly satisfied. Therefore in view of propositions 5, 6 these conditions would also hold for all $\varphi_i(k,x)$, and $\varphi_i(k,x) = P_i(k,x)\, e^{(k,x)}$ where P_i is a polynomial in k. Further we use induction to find the highest term $P_i^0(k,x)$ of the polynomial P_i.

By definition for any $s \in \mathbb{N}$ we have

$$(P_{s+1}^0 + \text{ lower order terms in } P_{s+1})e^{(k,x)} =$$
$$= \left(D - \frac{1}{2m+1} \sum_{j=1}^{n-1} e^{\overline{x}_j} - \frac{1}{2m+1} \sum_{j=1}^{n-1} e^{-\overline{x}_j} - \frac{1}{2l+1} e^{\overline{x}_n} - \frac{1}{2l+1} e^{-\overline{x}_n} \right) \times$$
$$\times (P_s^0 + \text{lower order terms in } P_s)e^{(k,x)}. \tag{4.18}$$

In order to get the formulas for P_{s+1}^0 we represent the right-hand side of (4.18) as a fraction of two polynomials. In the denominator of (4.18) it will be the polynomial

$$N = \prod_{i<j}^{n-1} (\overline{k}_i + \overline{k}_j)(\overline{k}_i - \overline{k}_j) \times$$
$$\times \prod_{i=1}^{n-1} (\overline{k}_i + \overline{k}_n - l + m)(\overline{k}_i - \overline{k}_n - l + m)(-\overline{k}_i + \overline{k}_n - l + m)(-\overline{k}_i - \overline{k}_n - l + m) \prod_{i=1}^{n} \overline{k}_i.$$

We continue equality (4.18) using the formulas for the coefficients of the operator D given by (4.1). We introduce the notation $[Q(k)]^0$ for the highest homogeneous part of the polynomial $Q(k)$. Let N^1 denote the homogeneous component of the polynomial N of degree $\deg N-1$. Then up to the lower terms we have

$$\Big(D-\frac{1}{2m+1}\sum_{j=1}^{n-1}(e^{\overline{x}_j}+e^{-\overline{x}_j})-\frac{1}{2l+1}(e^{\overline{x}_n}+e^{-\overline{x}_n})\Big)(P_s^0+P_s^1+\ldots)e^{(k,x)}=$$

$$=\frac{1}{N}\Bigg\{\frac{1}{2m+1}\sum_{i=1}^{n-1}\Big(N^0+N^1-\Big[N\Big(\sum_{j\neq i}\Big(\frac{2l+1}{\overline{k}_i+\overline{k}_j}+\frac{2l+1}{\overline{k}_i-\overline{k}_j}\Big)+\frac{(2m+1)l}{\overline{k}_i}+$$
$$+\frac{2m+1}{\overline{k}_i+\overline{k}_n-l+m}+\frac{2m+1}{\overline{k}_i-\overline{k}_n-l+m}\Big)\Big]^0+\ldots\Big)T_i^+-$$
$$-(N^0+N^1+\ldots)e^{\overline{x}_i}+$$

$$+\frac{1}{2m+1}\sum_{i=1}^{n-1}\Big(N^0+N^1-\Big[N\Big(\sum_{j\neq i}\Big(\frac{2l+1}{-\overline{k}_i+\overline{k}_j}+\frac{2l+1}{-\overline{k}_i-\overline{k}_j}\Big)-\frac{(2m+1)l}{\overline{k}_i}+$$
$$+\frac{2m+1}{-\overline{k}_i+\overline{k}_n-l+m}+\frac{2m+1}{-\overline{k}_i-\overline{k}_n-l+m}\Big)\Big]^0+\ldots\Big)T_i^--$$
$$-(N^0+N^1+\ldots)e^{-\overline{x}_i}+$$

$$+\frac{1}{2l+1}\Big(N^0+N^1-\Big[\Big(\frac{(2l+1)m}{\overline{k}_n}+$$
$$+\sum_{j=1}^{n-1}\Big(\frac{2l+1}{\overline{k}_n+\overline{k}_j+l-m}+\frac{2l+1}{\overline{k}_n-\overline{k}_j+l-m}\Big)\Big)N\Big]^0+\ldots\Big)T_n^+-$$
$$-(N^0+N^1+\ldots)e^{\overline{x}_n}+$$

$$+\frac{1}{2l+1}\Big(N^0+N^1-\Big[\Big(\frac{(2l+1)m}{-\overline{k}_n}+$$
$$+\sum_{j=1}^{n-1}\Big(\frac{2l+1}{-\overline{k}_n+\overline{k}_j+l-m}+\frac{2l+1}{-\overline{k}_n-\overline{k}_j+l-m}\Big)\Big)N\Big]^0+\ldots\Big)T_n^--$$
$$-(N^0+N^1+\ldots)e^{-\overline{x}_n}\Bigg\}\times$$

$$\times\left(P_s^0+P_s^1+\ldots\right)e^{(k,x)}.$$

Applying operators $T_i^{\pm}$ we get the following expression

$$\frac{1}{2m+1}\sum_{i=1}^{n-1}\Big\{\Big[\Big(-\sum_{j\neq i}\Big(\frac{2l+1}{\overline{k}_i+\overline{k}_j}+\frac{2l+1}{\overline{k}_i-\overline{k}_j}\Big)- -\frac{2m+1}{\overline{k}_i+\overline{k}_n-l+m}-\frac{2m+1}{\overline{k}_i-\overline{k}_n-l+m}-\frac{(2m+1)l}{\overline{k}_i}\Big)N\Big]^0 e^{\overline{x}_i}P_s^0+ + e^{\overline{x}_i}N^0\frac{\partial P_s^0}{\partial \overline{k}_i}(2m+1)+\dots\Big\}\frac{e^{(k,x)}}{N}+$$

$$+\frac{1}{2m+1}\sum_{i=1}^{n-1}\Big\{\Big[\Big(-\sum_{j\neq i}\Big(\frac{2l+1}{-\overline{k}_i+\overline{k}_j}+\frac{2l+1}{-\overline{k}_i-\overline{k}_j}\Big)- -\frac{2m+1}{-\overline{k}_i+\overline{k}_n-l+m}-\frac{2m+1}{-\overline{k}_i-\overline{k}_n-l+m}+\frac{(2m+1)l}{\overline{k}_i}\Big)N\Big]^0 e^{-\overline{x}_i}P_s^0- - e^{-\overline{x}_i}N^0\frac{\partial P_s^0}{\partial \overline{k}_i}(2m+1)+\dots\Big\}\frac{e^{(k,x)}}{N}+$$

$$+\frac{1}{2l+1}\Big\{\Big[\Big(-\sum_{j=1}^{n-1}\Big(\frac{2l+1}{\overline{k}_n+\overline{k}_j+l-m}+\frac{2l+1}{\overline{k}_n-\overline{k}_j+l-m}\Big)- -\frac{(2l+1)m}{\overline{k}_n}\Big)N\Big]^0 e^{\overline{x}_n}P_s^0+e^{\overline{x}_n}N^0\frac{\partial P_s^0}{\partial \overline{k}_n}(2l+1)+\dots\Big\}\frac{e^{(k,x)}}{N}+$$

$$+\frac{1}{2l+1}\Big\{\Big[\Big(-\sum_{j=1}^{n-1}\Big(\frac{2l+1}{-\overline{k}_n+\overline{k}_j+l-m}+\frac{2l+1}{-\overline{k}_n-\overline{k}_j+l-m}\Big)+ +\frac{(2l+1)m}{\overline{k}_n}\Big)N\Big]^0 e^{-\overline{x}_n}P_s^0-e^{\overline{x}_n}N^0\frac{\partial P_s^0}{\partial \overline{k}_n}(2l+1)+\dots\Big\}\frac{e^{(k,x)}}{N}.$$

We assume now that P_s^0 has the following form

$$P_s^0=\sum_{\{\lambda\}}c_\lambda P^0_{s,\{\lambda\}},$$

where

$$P^0_{s,\{\lambda\}}=\prod_{i<j}^{n}\overline{k}_j^{\lambda_j}\,(\overline{k}_i+\overline{k}_j)^{\lambda^+_{ij}}\,(\overline{k}_i-\overline{k}_j)^{\lambda^-_{ij}}.$$

Then P^0_{s+1} being the ratio of the highest term in the numerator to the highest term in the denominator takes the following form

$$P^0_{s+1}=\sum_{\{\lambda\}}c_\lambda P^0_{s+1,\{\lambda\}},$$

where

$$P^0_{s+1,\{\lambda\}} = \sum_{i=1}^{n-1}(e^{\overline{x}_i} - e^{-\overline{x}_i})\Big(\frac{\lambda_i - l}{\overline{k}_i} + \sum_{j\neq i}^{n-1}\Big(\frac{\lambda^+_{ij} - \frac{2l+1}{2m+1}}{\overline{k}_i + \overline{k}_j} + \frac{\lambda^-_{ij} - \frac{2l+1}{2m+1}}{\overline{k}_i - \overline{k}_j}\Big) + \\ + \frac{\lambda^+_{in} - 1}{\overline{k}_i + \overline{k}_n} + \frac{\lambda^-_{in} - 1}{\overline{k}_i - \overline{k}_n}\Big)P^0_{s,\{\lambda\}} + \\ + (e^{\overline{x}_n} - e^{-\overline{x}_n})\Big(\frac{\lambda_n - m}{\overline{k}_n} + \sum_{j\neq i}\Big(\frac{\lambda^+_{jn} - 1}{\overline{k}_j + \overline{k}_n} + \frac{-\lambda^-_{jn} + 1}{\overline{k}_j - \overline{k}_n}\Big)\Big)P^0_{s,\{\lambda\}}$$

and we assume the notations $\lambda^{\pm}_{ij} = \lambda^{\pm}_{ji}$. Thus finally we have

$$P^0_{s+1,\{\lambda\}} = \sum_{i_0=1}^{n-1}(\lambda_{i_0} - l)(e^{\overline{x}_{i_0}} - e^{-\overline{x}_{i_0}})\overline{k}_{i_0}^{\lambda_{i_0}-1}\prod_{j\neq i_0}\overline{k}_j^{\lambda_j}\prod_{i<j}^{n}(\overline{k}_i + \overline{k}_j)^{\lambda^+_{ij}}(\overline{k}_i - \overline{k}_j)^{\lambda^-_{ij}} + \\ + (\lambda_n - m)(e^{\overline{x}_n} - e^{-\overline{x}_n})\overline{k}_n^{\lambda_n - 1}\prod_{j\neq n}\overline{k}_j^{\lambda_j}\prod_{i<j}^{n}(\overline{k}_i + \overline{k}_j)^{\lambda^+_{ij}}(\overline{k}_i - \overline{k}_j)^{\lambda^-_{ij}} + \\ + \sum_{i_0<j_0}^{n-1}\Big(\lambda^+_{i_0j_0} - \frac{2l+1}{2m+1}\Big)(e^{\overline{x}_{i_0}} - e^{-\overline{x}_{i_0}} + e^{\overline{x}_{j_0}} - e^{-\overline{x}_{j_0}})(\overline{k}_{i_0} + \overline{k}_{j_0})^{\lambda^+_{i_0j_0}-1}\times \\ \times\prod_{j=1}^{n}\overline{k}_j^{\lambda_j}\prod_{\substack{i<j\\(i,j)\neq(i_0,j_0)}}(\overline{k}_i + \overline{k}_j)^{\lambda^+_{ij}}\prod_{\substack{i<j\\(i,j)\neq(i_0,j_0)}}(\overline{k}_i - \overline{k}_j)^{\lambda^-_{ij}} + \\ + \sum_{i_0<j_0}^{n-1}\Big(\lambda^-_{i_0j_0} - \frac{2l+1}{2m+1}\Big)(e^{\overline{x}_{i_0}} - e^{-\overline{x}_{i_0}} - e^{\overline{x}_{j_0}} + e^{-\overline{x}_{j_0}})(\overline{k}_{i_0} - \overline{k}_{j_0})^{\lambda^-_{i_0j_0}-1}\times \\ \times\prod_{j=1}^{n}\overline{k}_j^{\lambda_j}\prod_{\substack{i<j\\(i,j)\neq(i_0,j_0)}}(\overline{k}_i + \overline{k}_j)^{\lambda^+_{ij}}\prod_{\substack{i<j\\(i,j)\neq(i_0,j_0)}}(\overline{k}_i - \overline{k}_j)^{\lambda^-_{ij}} + \\ + \sum_{i_0=1}^{n-1}(\lambda^+_{i_0n} - 1)(e^{\overline{x}_{i_0}} - e^{-\overline{x}_{i_0}} + e^{\overline{x}_n} - e^{-\overline{x}_n})(\overline{k}_{i_0} + \overline{k}_n)^{\lambda^+_{i_0n}-1}\times \\ \times\prod_{j=1}^{n}\overline{k}_j^{\lambda_j}\prod_{\substack{i<j\\(i,j)\neq(i_0,n)}}(\overline{k}_i + \overline{k}_j)^{\lambda^+_{ij}}\prod_{\substack{i<j\\(i,j)\neq(i_0,n)}}(\overline{k}_i - \overline{k}_j)^{\lambda^-_{ij}} + \\ + \sum_{i_0=1}^{n-1}(\lambda^-_{i_0n} - 1)(e^{\overline{x}_{i_0}} - e^{-\overline{x}_{i_0}} - e^{\overline{x}_n} + e^{-\overline{x}_n})(\overline{k}_{i_0} - \overline{k}_n)^{\lambda^-_{i_0n}-1}\times \\ \times\prod_{j=1}^{n}\overline{k}_j^{\lambda_j}\prod_{\substack{i<j\\(i,j)\neq(i_0,n)}}(\overline{k}_i + \overline{k}_j)^{\lambda^+_{ij}}\prod_{\substack{i<j\\(i,j)\neq(i_0,n)}}(\overline{k}_i - \overline{k}_j)^{\lambda^-_{ij}}. \quad (4.19)$$

We now follow the changes of P^0 starting from

$$\varphi_0 = \prod_{\alpha\in\mathcal{C}_n(l,m)} \prod_{j=1}^{m_\alpha} (k + j\alpha, \alpha)(k - j\alpha, \alpha) e^{(k,x)},$$

that is

$$P_0^0 = \prod_{i=1}^{n} \overline{k}_i^{2m_i} \prod_{i<j}^{n} (\overline{k}_i + \overline{k}_j)^{2m_{ij}} \prod_{i<j}^{n} (\overline{k}_i - \overline{k}_j)^{2m_{ij}}.$$

Formula (4.19) shows that for any s P_s^0 is a linear combination of monomials consisting of the products $\overline{k}_i^{\lambda_i}(\overline{k}_i + \overline{k}_j)^{\lambda_{ij}^+}(\overline{k}_i - \overline{k}_j)^{\lambda_{ij}^-}$, and the degree of monomials is decreasing by 1 at every application of the operator D. Besides this the coefficients in formula (4.19) show that the monomials with degrees $\lambda_i < m_i$ and $\lambda_{ij}^{\pm} < m_{ij}$ cannot appear. Thus we get

$$P^0_{\sum m_\alpha} = c(x) \prod_{i=1}^{n} \overline{k}_i^{m_i} \prod_{i<j} (\overline{k}_i + \overline{k}_j)^{m_{ij}} (\overline{k}_i - \overline{k}_n)^{m_{ij}}.$$

Therefore the function $c(x)^{-1}\varphi_{\sum m_\alpha}$ satisfies conditions (2.1), (2.2) of the BA function.

We are left to determine the coefficient $c(x)$. For this we analyze once again formula (4.19). At every step one of the terms $\overline{k}_i$, $\overline{k}_i \pm \overline{k}_j$ in the monomials is changed by the corresponding function of x with some coefficient. We begin with the monomial

$$\prod_{j=1}^{n} \overline{k}_j^{2m_j} \prod_{i<j} (\overline{k}_i + \overline{k}_j)^{2m_{ij}} (\overline{k}_i - \overline{k}_j)^{2m_{ij}}$$

and finish by the monomial

$$\prod_{j=1}^{n} \overline{k}_j^{m_j} \prod_{i<j} (\overline{k}_i + \overline{k}_j)^{m_{ij}} (\overline{k}_i - \overline{k}_j)^{m_{ij}}.$$

Therefore

$$\begin{aligned}
c(x) = c_0 \prod_{i=1}^{n-1} (e^{\overline{x}_i} - e^{-\overline{x}_i})^l (e^{\overline{x}_n} - e^{-\overline{x}_n})^m \times \\
\times \prod_{i<j}^{n-1} (e^{\overline{x}_i} - e^{-\overline{x}_i} + e^{\overline{x}_j} - e^{-\overline{x}_j})^{\frac{2l+1}{2m+1}} \prod_{i<j}^{n-1} (e^{\overline{x}_i} - e^{-\overline{x}_i} - e^{\overline{x}_j} + e^{-\overline{x}_j})^{\frac{2l+1}{2m+1}} \times \\
\times \prod_{i=1}^{n-1} (e^{\overline{x}_i} - e^{-\overline{x}_i} + e^{\overline{x}_n} - e^{-\overline{x}_n})(e^{\overline{x}_i} - e^{-\overline{x}_i} - e^{\overline{x}_n} + e^{-\overline{x}_n}) = \\
= c_0\, (e^{\overline{x}_n} - e^{-\overline{x}_n})^m \times \\
\times \prod_{i=1}^{n-1} (e^{\overline{x}_i} - e^{-\overline{x}_i})^l \prod_{i<j}^{n-1} (e^{\overline{x}_i - \overline{x}_j} - e^{\overline{x}_j - \overline{x}_i})^{\frac{2l+1}{2m+1}} (e^{\overline{x}_i + \overline{x}_j} - e^{-\overline{x}_i - \overline{x}_j})^{\frac{2l+1}{2m+1}} \times \\
\times \prod_{i=1}^{n-1} (e^{\overline{x}_i - \overline{x}_n} - e^{\overline{x}_n - \overline{x}_i})(e^{\overline{x}_i + \overline{x}_n} - e^{-\overline{x}_i - \overline{x}_n}).
\end{aligned}$$

It is left to determine the coefficient c_0. This is an integer equal to the total number of possible monomials. From (4.19) it easily follows that at the first step there appear $M = \sum m_i + 2\sum m_{ij}$ monomials, and after the second step there appear $M(M-1)$ monomials. In total we obtain $c_0 = M!$ and the theorem is proven. ■

In the end of this section we put the result on bispectrality.

Theorem 5. *The Baker–Akhiezer function $\psi(k,x)$ for the system $\mathcal{C}_n(l,m)$ satisfies the following equation in variables k:*

$$D\psi(k,x) = \left(\frac{2}{2m+1}\sum_{j=1}^{n-1}\cosh\sqrt{2m+1}\,x_j + \frac{2}{2l+1}\cosh\sqrt{2l+1}\,x_n\right)\psi(k,x),$$

where D is operator (4.1). For the polynomials $p(k) \in R_{\mathcal{C}_n(l,m)}$ the difference operators

$$D_p = ad_D^{\deg p}p(k)$$

commute with each other. These operators also commute with the operator D.

Proof. In the notations (4.15), (4.16) it follows from theorem 4 and propositions 5, 6 that $\varphi_{\sum m_\alpha+1}$ has the form $P(k,x)e^{(k,x)}$ where P is a polynomial in k of degree less than $\sum m_\alpha$, and it satisfies axiomatics (2.2). By lemma 1 it follows that $\varphi_{\sum m_\alpha+1} = 0$ which is equivalent to the first statement of the theorem.

Now, as it is explained in section 2 for any $p \in R_{\mathcal{C}_n(l,m)}$ there exists differential operator $L_p(x,\partial_x)$ such that

$$L_p(x,\partial_x)\psi(k,x) = p(k)\psi(k,x).$$

By the bispectrality arguments presented in section 3 we have

$$D_p\psi(k,x) = a_p(x)\psi(k,x)$$

for some function $a_p(x)$, therefore we have the relation

$$(D_{p_1}D_{p_2} - D_{p_2}D_{p_1})\psi(k,x) = (a_{p_1}a_{p_2} - a_{p_2}a_{p_1})\psi(k,x) = 0.$$

Because of the special form of ψ it follows that $D_{p_1}D_{p_2} - D_{p_2}D_{p_1} = 0$. ■

5 Configuration $A_{n,2}(m)$

The vectors and multiplicities forming this system in $\mathbb{C}^{n+1}$ are as follows. The vectors $\alpha_{0i} = \sqrt{-m-1}e_0 - e_i$, $e_i - \sqrt{m}e_n$ have multiplicities $m_{0i} = m_{in} = 1$, $i = 1,\ldots,n-1$. The vectors $\alpha_{ij} = e_i - e_j$ have multiplicities $m_{ij} = m$, $1 \leqslant i < j \leqslant n-1$, the vector $\alpha_{0n} = \sqrt{-m-1}\,e_0 - \sqrt{m}\,e_n$ has multiplicity $m_{0n} = 1$.

This configuration was introduced by Chalykh and Veselov in [12] as the one satisfying the rational locus conditions but not satisfying the $\vee$-conditions and thus not leading to a solution of the generalized WDVV equations (see [12]). In the case $n = 2$ the system contains three vectors all having multiplicity 1 thus the parameter m can be arbitrary complex rather than an integer. The corresponding elliptic operator was considered by

Hietarinta [16] (see also [8], [10]). The important for us feature of this configuration is the fact that the system does not admit the Baker–Akhiezer function in the sense of [31], that is satisfying the conditions

$$\psi(k+s\alpha,x)=\psi(k-s\alpha,x)$$

at $(\alpha,k)=0$, $s\leqslant m_\alpha$. But the system admits the BA function in the sense of our definition that is we impose conditions (2.2).

In order to construct the BA function we again follow the scheme of [6]. As a difference operator D we take

$$D=\sum_{i=0}^{n}\frac{1}{\overline{e}_i{}^2}\prod_{\substack{j=0\\ j\neq i}}^{n}(k-m_{ij}\alpha_{ij},\alpha_{ij})T_i\frac{1}{\prod_{\substack{j=0\\ j\neq i}}^{n}(k-\alpha_{ij},\alpha_{ij})},\tag{5.1}$$

where for this section we have introduced the following notations

$$\overline{e}_0=\sqrt{-m-1}\,e_0,\ \overline{e}_n=\sqrt{m}e_n,\qquad\text{and}\ \ \overline{e}_i=e_i\ \text{ for } 1\leqslant i\leqslant n-1.$$

Also for $0\leqslant i\leqslant n$ we denote

$$\overline{e}_i{}^2=(\overline{e}_i,\overline{e}_i),\ \overline{k}_i=(k,\overline{e}_i),\ \overline{x}_i=(x,\overline{e}_i)$$

for this section. Operators T_i act by the rule $T_i(f(k))=f(k+2\overline{e}_i)$, and we understand that $\alpha_{ij}=\overline{e}_i-\overline{e}_j$ also when $i>j$. Then operator D can be written as follows

$$D=-\frac{1}{m+1}\left(1+\frac{2(m+1)}{\overline{k}_0-\overline{k}_n-2m-1}\right)\prod_{j=1}^{n-1}\left(1+\frac{2(m+1)}{\overline{k}_0-k_j-m-2}\right)T_0+$$
$$\sum_{i=1}^{n-1}\left(1-\frac{2}{k_i-\overline{k}_0+m+2}\right)\left(1-\frac{2}{k_i-\overline{k}_n-m+1}\right)\prod_{j=1}^{n-1}\left(1-\frac{2m}{k_i-k_j}\right)T_i+$$
$$+\frac{1}{m}\left(1-\frac{2m}{\overline{k}_n-\overline{k}_0+2m+1}\right)\prod_{j=1}^{n-1}\left(1-\frac{2m}{\overline{k}_n-k_j+m-1}\right)T_n.$$

At first we rearrange conditions (2.2′) into more convenient for us form similarly to the case $C_n(l,m)$ system considered earlier. Namely, for $\alpha=e_i-e_j$, $1\leqslant i<j\leqslant n-1$ dropping A_+ in the notation $\psi_\alpha^{A_+}(k)$ for simplicity, the condition $\psi_\alpha(k+s\alpha)=\psi_\alpha(k-s\alpha)$ is equivalent to the condition

$$T_i^s\psi_\alpha=T_j^s\psi_\alpha\quad\text{at}\ \ k\in\pi_{ij}:k_i-k_j=0.\tag{5.2}$$

Now we move to the consideration of the condition for $\alpha=\overline{e}_i-\overline{e}_j$, where $i=0$ or $j=n$ or both. The identity $\psi_\alpha(k+\alpha)=\psi_\alpha(k-\alpha)$ is equivalent to the relation

$$T_i\psi_\alpha=T_j\psi_\alpha\quad\text{at}\ \ k\in\pi_{ij}:\overline{k}_i-\overline{k}_j+\overline{e}_i^2-\overline{e}_j^2=0,\tag{5.3}$$

where $\overline{k}_i=(k,\overline{e}_i)$. Indeed, let $k\in\pi_{ij}$, then

$$T_i\psi_\alpha(k)=\psi_\alpha(k+2\overline{e}_i)=\psi_\alpha(k+\alpha+\overline{e}_i+\overline{e}_j),$$
$$T_j\psi_\alpha(k)=\psi_\alpha(k+2\overline{e}_j)=\psi_\alpha(k-\alpha+\overline{e}_i+\overline{e}_j).$$

As $(\alpha,k+\overline{e}_i+\overline{e}_j)=0$ the conditions in the original form are equivalent to (5.3).

Proposition 7. *For any holomorphic function $f(k_0, k_1, \ldots, k_n)$ satisfying conditions* (5.2), (5.3) *the function $Df(k_0, \ldots, k_n)$ is also holomorphic if D is given by* (5.1).

Proof. In principle the function $Df(k_0, \ldots, k_n)$ could have singularities at the hyperplane π of the form

$$T_{i_0}(k - \alpha_{i_0 j_0}, \alpha_{i_0 j_0}) = (k + \overline{e}_{i_0} + \overline{e}_{j_0}, \overline{e}_{i_0} - \overline{e}_{j_0}) = 0, \tag{5.4}$$

$i_0 \neq j_0$. We will show that this does not happen. We collect terms in Df which possibly have singularities at $(k + \overline{e}_{i_0} + \overline{e}_{j_0}, \overline{e}_{i_0} - \overline{e}_{j_0}) = 0$. Since

$$T_{i_0}(k - \alpha_{i_0 j_0}, \alpha_{i_0 j_0}) = -T_{j_0}(k - \alpha_{j_0 i_0}, \alpha_{j_0 i_0}) = (k + \overline{e}_{i_0} + \overline{e}_{j_0}, \overline{e}_{i_0} - \overline{e}_{j_0}),$$

we get the sum of two terms

$$\frac{1}{(k + \overline{e}_{i_0} + \overline{e}_{j_0}, \overline{e}_{i_0} - \overline{e}_{j_0})} \left(\frac{1}{\overline{e}_{i_0}^2} \prod_{\substack{j=0 \\ j \neq i_0}}^{n} (k - m_{i_0 j}, \alpha_{i_0 j}) T_{i_0} \frac{f(k)}{\prod_{j \neq i_0, j_0}^{n} (k - \alpha_{i_0 j}, \alpha_{i_0 j})} - \right.$$
$$\left. - \frac{1}{\overline{e}_{j_0}^2} \prod_{\substack{i=0 \\ i \neq j_0}}^{n} (k - m_{j_0 i}, \alpha_{j_0 i}) T_{j_0} \frac{f(k)}{\prod_{i \neq i_0, j_0}^{n} (k - \alpha_{j_0 i}, \alpha_{j_0 i})} \right). \tag{5.5}$$

We have to show that the expression in brackets vanishes at $k \in \pi$ (5.4). We note that the vectors $A_{i_0 j_0} = \{\alpha_{i_0 j}, \alpha_{j_0 i} |\ j \neq i_0, i \neq j_0, i_0\}$ lie in some half-space in $\mathbb{C}^n \approx \mathbb{R}^{2n}$, and for any choice of the subsystem $B \subset A$ such that $B \cup A_{i_0 j_0}$ is a positive system A_+, the vector $\alpha_{i_0 j_0}$ is an edge vector in A_+. Therefore axiomatic conditions (5.2), (5.3) state, in particular, that

$$T_{i_0} \frac{f(k)}{\prod_{j \neq i_0, j_0} \prod_{s=1}^{m_{i_0 j}} (k - s\alpha_{i_0 j}, \alpha_{i_0 j}) \prod_{i \neq i_0, j_0} \prod_{s=1}^{m_{j_0 i}} (k - s\alpha_{j_0 i}, \alpha_{j_0 i}) \prod_{\beta \in B} \prod_{s=1}^{m_\beta} (k - s\beta, \beta)} =$$
$$= T_{j_0} \frac{f(k)}{\prod_{i \neq i_0, j_0}^{n} \prod_{s=1}^{m_{j_0 i}} (k - s\alpha_{j_0 i}, \alpha_{j_0 i}) \prod_{i \neq i_0, j_0} \prod_{s=1}^{m_{i_0 j}} (k - s\alpha_{i_0 j}, \alpha_{i_0 j}) \prod_{\beta \in B} \prod_{s} (k - s\beta, \beta)} \tag{5.6}$$

on the hyperplane π. Further we apply the shift operators to a part of the product in (5.6) and we use equality

$$T_{i_0} \prod_{s=2}^{m_{i_0 j}} (k - s\alpha_{i_0 j}, \alpha_{i_0 j}) = \prod_{s=1}^{m_{i_0 j} - 1} (k - s\alpha_{i_0 j}, \alpha_{i_0 j}),$$

which is non-trivial only if $m_{i_0 j} > 1$, that is for $1 \leqslant i_0, j \leqslant n - 1$. We get

$$\frac{T_{i_0} \frac{f(k)}{\prod_{j \neq i_0, j_0} (k - \alpha_{i_0 j}, \alpha_{i_0 j})}}{\prod_{j \neq i_0, j_0} \prod_{s=1}^{m_{i_0 j} - 1} (k - s\alpha_{i_0 j}, \alpha_{i_0 j}) \prod_{i \neq i_0, j_0} \prod_{s=1}^{m_{j_0 i}} (k - s\alpha_{j_0 i}, \alpha_{j_0 i})} =$$
$$= \frac{T_{j_0} \frac{f(k)}{\prod_{i \neq i_0, j_0} (k - \alpha_{j_0 i}, \alpha_{j_0 i})}}{\prod_{i \neq i_0, j_0}^{n} \prod_{s=1}^{m_{j_0 i} - 1} (k - s\alpha_{j_0 i}, \alpha_{j_0 i}) \prod_{j \neq i_0, j_0} \prod_{s=1}^{m_{i_0 j}} (k - s\alpha_{i_0 j}, \alpha_{i_0 j})}.$$

After necessary cancellations we obtain from above

$$\prod_{j\neq i_0,j_0}(k-m_{i_0j}\alpha_{i_0j},\alpha_{i_0j})T_{i_0}\frac{f(k)}{\prod_{j\neq i_0,j_0}(k-\alpha_{i_0j},\alpha_{i_0j})}=$$
$$=\prod_{i\neq i_0,j_0}(k-m_{j_0i}\alpha_{j_0i},\alpha_{j_0i})T_{j_0}\frac{f(k)}{\prod_{i\neq i_0,j_0}(k-\alpha_{j_0i},\alpha_{j_0i})}.$$

Simplifying (5.5) with the help of equality

$$\frac{(k-m_{i_0j_0}\alpha_{i_0j_0},\alpha_{i_0j_0})}{(\overline{e}_{i_0},\overline{e}_{i_0})}=\frac{(k-m_{j_0i_0}\alpha_{j_0i_0},\alpha_{j_0i_0})}{(\overline{e}_{j_0},\overline{e}_{j_0})}$$

which is valid for $k\in\pi$, we conclude that expression (5.5) has no singularities at the hyperplane π. ■

Proposition 8. *Let holomorphic function $f(k)$ satisfy conditions* (5.2), (5.3). *Then the function $Df(k)$ also satisfies* (5.2), (5.3) *if D is given by* (5.1).

Before we start proving the proposition we state a lemma which will be useful for us to work with axiomatic conditions (5.2), (5.3).

Lemma 3. *Let vector $\alpha_{ij}\in A_{n,2}(m)$ be an edge vector for two subsystems $A_+^{(1)}$ and $A_+^{(2)}$. Then the following condition for a holomorphic function $f(k)$*

$$(T_i^s-T_j^s)\frac{f(k)}{\prod_{\substack{\beta\in A_+^{(1)}\\ \beta\neq\alpha_{ij}}}\vec{\beta}}=0\qquad at\ \ k\in\pi_{ij},\ \ 1\leqslant s\leqslant m_{ij}$$

is equivalent to the condition

$$(T_i^s-T_j^s)\frac{f(k)}{\prod_{\substack{\beta\in A_+^{(2)}\\ \beta\neq\alpha_{ij}}}\vec{\beta}}=0\qquad at\ \ k\in\pi_{ij},\ \ 1\leqslant s\leqslant m_{ij}$$

where $\vec{\beta}=\prod_{l=1}^{m_\beta}(k+l\beta,\beta)$.

Proof. We denote for the brevity $\prod_t\vec{\beta}=\prod_{\substack{\beta\in A_+^{(t)}\\ \beta\neq\alpha_{ij}}}\vec{\beta}$, $t=1,2$. As

$$(T_i^s-T_j^s)\frac{f(k)}{\prod_2\vec{\beta}}=\left(\frac{T_i^sf(k)}{T_i^s\prod_1\vec{\beta}}\right)\left(\frac{T_i^s\prod_1\vec{\beta}}{T_i^s\prod_2\vec{\beta}}\right)-\left(\frac{T_j^sf(k)}{T_j^s\prod_1\vec{\beta}}\right)\left(\frac{T_j^s\prod_1\vec{\beta}}{T_j^s\prod_2\vec{\beta}}\right),$$

we have to show that

$$T_i^s\frac{\prod_1\vec{\beta}}{\prod_2\vec{\beta}}=T_j^s\frac{\prod_1\vec{\beta}}{\prod_2\vec{\beta}}\qquad \text{at}\ \ k\in\pi_{ij}.$$

This is equivalent to

$$(T_i^s-T_j^s)\frac{\prod_{\substack{\beta\in A_+^{(1)}\\ (\beta,\alpha_{ij})\neq 0}}\vec{\beta}}{\prod_{\substack{\beta\in A_+^{(2)}\\ (\beta,\alpha_{ij})\neq 0}}\vec{\beta}}=0\qquad \text{at}\ k\in\pi_{ij}. \tag{5.7}$$

Regrouping the product terms condition (5.7) takes the form

$$(T_i^s - T_j^s) \prod_{q\neq i,j} \prod_{t=1}^{m_{jq}} \prod_{s=1}^{m_{iq}} \frac{(k + s\varepsilon_{iq}\alpha_{iq}, \alpha_{iq})(k + t\varepsilon_{jq}\alpha_{jq}, \alpha_{jq})}{(k + s\delta_{iq}\alpha_{iq}, \alpha_{iq})(k + t\delta_{jq}\alpha_{jq}, \alpha_{jq})} = 0 \qquad \text{at } k \in \pi_{ij},$$

where $\varepsilon_{iq}, \delta_{iq} = \pm 1$. And it is sufficient to show that $\forall q \neq i, j$

$$(T_i^s - T_j^s) \prod_{t=1}^{m_{jq}} \prod_{s=1}^{m_{iq}} \frac{(k + s\varepsilon_{iq}\alpha_{iq}, \alpha_{iq})(k + t\varepsilon_{jq}\alpha_{jq}, \alpha_{jq})}{(k + s\delta_{iq}\alpha_{iq}, \alpha_{iq})(k + t\delta_{jq}\alpha_{jq}, \alpha_{jq})} = 0 \qquad \text{at } k \in \pi_{ij}. \tag{5.8}$$

This means that condition (5.7) is reduced to the two-dimensional identity (5.8) in the plane containing vectors $\alpha_{ij}, \alpha_{iq}, \alpha_{jq}$. And the condition that α_{ij} is an edge vector for $A_+^{(1)}$, $A_+^{(2)}$ means that $\alpha_{ij} = \pm(\varepsilon_{iq}\alpha_{iq} - \varepsilon_{jq}\alpha_{jq})$, that is $\varepsilon_{iq} = \varepsilon_{jq}$, analogously we have $\delta_{iq} = \delta_{jq}$. Therefore property (5.8) is reduced to the identity

$$(T_i^s - T_j^s) \prod_{t=1}^{m_{jq}} \prod_{s=1}^{m_{iq}} \frac{(k + s\alpha_{iq}, \alpha_{iq})(k + t\alpha_{jq}, \alpha_{jq})}{(k - s\alpha_{iq}, \alpha_{iq})(k - t\alpha_{jq}, \alpha_{jq})} = 0 \qquad \text{at } k \in \pi_{ij}. \tag{5.9}$$

Now we separately consider the arising cases

a) $1 \leqslant i, j \leqslant n-1$. At any q the product in (5.9) is invariant under the reflection $k_i \leftrightarrow k_j$. Therefore in particular property (5.9) holds.

Further we may assume that $s = 1$.

b) $i = 0$, $1 \leqslant j \leqslant n-1$. Consider firstly the case $q < n$. Identity (5.9) takes the form

$$(T_0 - T_j) \frac{(k + \alpha_{0q}, \alpha_{0q})}{(k - \alpha_{0q}, \alpha_{0q})} \frac{\prod_{t=1}^m (k + t\alpha_{jq}, \alpha_{jq})}{\prod_{t=1}^m (k - t\alpha_{jq}, \alpha_{jq})} = 0$$

at $\overline{k}_0 - \overline{k}_j + (\overline{e}_0, \overline{e}_0) - (\overline{e}_j, \overline{e}_j) = 0$. Or, more explicitly, we have

$$\frac{(k + 2e_0 + \alpha_{0q}, \alpha_{0q})}{(k + 2e_0 - \alpha_{0q}, \alpha_{0q})} \frac{\prod_{t=1}^m (k_j - k_q + 2t)}{\prod_{t=1}^m (k_j - k_q - 2t)} = \\ = \frac{(k + \alpha_{0q}, \alpha_{0q})}{(k - \alpha_{0q}, \alpha_{0q})} \frac{\prod_{t=1}^m (k_j - k_q + 2t + 2)}{\prod_{t=1}^m (k_j - k_q - 2t + 2)}.$$

Performing the cancellations and recalling that $\overline{k}_0 = \overline{k}_j - (\overline{e}_0, \overline{e}_0) + (\overline{e}_j, \overline{e}_j) = 0$ we get

$$\frac{(k_j - k_q - 2m)(k_j - k_q + 2)}{(k_j - k_q)(k_j - k_q - 2m)} = \frac{(k_j - k_q + 2)(k_j - k_q + 2m + 2)}{(k_j - k_q + 2(m+1))(k_j - k_q)},$$

which is obviously satisfied. Further we consider relation (5.9) at $q = n$. We have to check that

$$T_0 \frac{(k + \alpha_{0n}, \alpha_{0n})(k + \alpha_{jn}, \alpha_{jn})}{(k - \alpha_{0n}, \alpha_{0n})(k - \alpha_{jn}, \alpha_{jn})} = T_j \frac{(k + \alpha_{0n}, \alpha_{0n})(k + \alpha_{jn}, \alpha_{jn})}{(k - \alpha_{0n}, \alpha_{0n})(k - \alpha_{jn}, \alpha_{jn})}$$

at $\overline{k}_0 - \overline{k}_j + (\overline{e}_0, \overline{e}_0) - (\overline{e}_j, \overline{e}_j) = 0$. Applying the difference operators we have

$$\frac{(\overline{k}_0 - \overline{k}_n + 3\overline{e}_0^2 + \overline{e}_n^2)(\overline{k}_j - \overline{k}_n + \overline{e}_j^2 + \overline{e}_n^2)}{(\overline{k}_0 - \overline{k}_n + \overline{e}_0^2 - \overline{e}_n^2)(\overline{k}_j - \overline{k}_n - \overline{e}_j^2 - \overline{e}_n^2)} = \\ = \frac{(\overline{k}_0 - \overline{k}_n + \overline{e}_0^2 + \overline{e}_n^2)(\overline{k}_j - \overline{k}_n + 3\overline{e}_j^2 + \overline{e}_n^2)}{(\overline{k}_0 - \overline{k}_n - \overline{e}_0^2 - \overline{e}_n^2)(\overline{k}_j - \overline{k}_n + \overline{e}_j^2 - \overline{e}_n^2)}.$$

We substitute now $\overline{k}_0 = \overline{k}_j - \overline{e}_0^2 + \overline{e}_j^2$, $\overline{e}_0^2 = -m-1$, $\overline{e}_n^2 = m$, $\overline{e}_j^2 = 1$ and we get the obvious identity

$$\frac{(\overline{k}_j - \overline{k}_n - m - 1)(\overline{k}_j - \overline{k}_n + m + 1)}{(\overline{k}_j - \overline{k}_n - m + 1)(\overline{k}_j - \overline{k}_n - m - 1)} = \\ = \frac{(\overline{k}_j - \overline{k}_n + m + 1)(\overline{k}_j - \overline{k}_n + m + 3)}{(\overline{k}_j - \overline{k}_n + m + 3))(\overline{k}_j - \overline{k}_n - m + 1)}.$$

c) $i = 0$, $j = n$. We have to check that

$$T_0 \frac{(k + \alpha_{0q}, \alpha_{0q})(k + \alpha_{nq}, \alpha_{nq})}{(k - \alpha_{0q}, \alpha_{0q})(k - \alpha_{nq}, \alpha_{nq})} = T_n \frac{(k + \alpha_{0q}, \alpha_{0q})(k + \alpha_{nq}, \alpha_{nq})}{(k - \alpha_{0q}, \alpha_{0q})(k - \alpha_{nq}, \alpha_{nq})}$$

at $\overline{k}_0 - \overline{k}_n + \overline{e}_0^2 - \overline{e}_n^2 = 0$. Equivalently we have

$$\frac{(\overline{k}_0 - \overline{k}_q + 3\overline{e}_0^2 + \overline{e}_q^2)(\overline{k}_n - \overline{k}_q + \overline{e}_n^2 + \overline{e}_q^2)}{(\overline{k}_0 - \overline{k}_q + \overline{e}_0^2 - \overline{e}_q^2)(\overline{k}_n - \overline{k}_q - \overline{e}_n^2 - \overline{e}_q^2)} = \\ = \frac{(\overline{k}_0 - \overline{k}_q + \overline{e}_0^2 + \overline{e}_q^2)(\overline{k}_n - \overline{k}_q + 3\overline{e}_n^2 + \overline{e}_q^2)}{(\overline{k}_0 - \overline{k}_q - \overline{e}_0^2 - \overline{e}_q^2)(\overline{k}_n - \overline{k}_q + \overline{e}_n^2 - \overline{e}_q^2)}.$$

We express $\overline{k}_0$ through $\overline{k}_n$ and substitute the lengths of the vectors. We obtain the correct equality

$$\frac{(\overline{k}_n - \overline{k}_q - m - 1)(\overline{k}_n - \overline{k}_q + m + 1)}{(\overline{k}_n - \overline{k}_q + m - 1)(\overline{k}_n - \overline{k}_q - m - 1)} = \\ = \frac{(\overline{k}_n - \overline{k}_q + m + 1)(\overline{k}_n - \overline{k}_q + 3m + 1)}{(\overline{k}_n - \overline{k}_q + 3m + 1)(\overline{k}_n - \overline{k}_q + m - 1)}.$$

Finally, consider the last case

d) $1 \leqslant i \leqslant n-1$, $j = n$. Like in the case b) we have to consider the cases $q > 0$ and $q = 0$ separately. We assume at first that $q > 0$. We have to check that

$$(T_i - T_n) \frac{(k + \alpha_{nq}, \alpha_{nq})}{(k - \alpha_{nq}, \alpha_{nq})} \frac{\prod_{t=1}^m (k + t\alpha_{iq}, \alpha_{iq})}{\prod_{t=1}^m (k - t\alpha_{iq}, \alpha_{iq})} = 0 \tag{5.10}$$

at $\overline{k}_i - \overline{k}_n + \overline{e}_i^2 - \overline{e}_n^2 = 0$. We consider separately $(T_i - T_n)$ applied to the numerator of (5.10). We get

$$(\overline{k}_n - \overline{k}_q + m + 1) \prod_{t=1}^m (\overline{k}_i - \overline{k}_q + 2t + 2) - (\overline{k}_n - \overline{k}_q + 3m + 1) \prod_{t=1}^m (\overline{k}_i - \overline{k}_q + 2t) = \\ = \big((\overline{k}_i - \overline{k}_q + 2m + 2)(\overline{k}_i - \overline{k}_q + 2) - (\overline{k}_i - \overline{k}_q + 2)(\overline{k}_i - \overline{k}_q + 2m + 2)\big) \times \\ \times \prod_{t=2}^m (\overline{k}_i - \overline{k}_q + 2t) = 0.$$

Analogously we get

$$(T_i - T_n)(k - \alpha_{nq}, \alpha_{nq}) \prod_{t=1}^{m} (k - t\alpha_{iq}, \alpha_{iq}) = 0,$$

therefore condition (5.10) is satisfied. Finally let $q = 0$. We have to check that

$$(T_i - T_n)\frac{(k + \alpha_{i0}, \alpha_{i0})(k + \alpha_{n0}, \alpha_{n0})}{(k - \alpha_{i0}, \alpha_{i0})(k - \alpha_{n0}, \alpha_{n0})} = 0$$

if $\overline{k}_i - \overline{k}_n + 1 - m = 0$. We have

$$(T_i - T_n)\frac{(k + \alpha_{i0}, \alpha_{i0})(k + \alpha_{n0}, \alpha_{n0})}{(k - \alpha_{i0}, \alpha_{i0})(k - \alpha_{n0}, \alpha_{n0})} =$$
$$= \frac{(\overline{k}_i - \overline{k}_0 - m + 2)(\overline{k}_n - \overline{k}_0 - 1)}{(\overline{k}_i - \overline{k}_0 + m + 2)(\overline{k}_n - \overline{k}_0 + 1)} - \frac{(\overline{k}_i - \overline{k}_0 - m)(\overline{k}_n - \overline{k}_0 + 2m - 1)}{(\overline{k}_i - \overline{k}_0 + m)(\overline{k}_n - \overline{k}_0 + 2m + 1)} =$$
$$= \frac{\overline{k}_n - \overline{k}_0 - 1}{\overline{k}_i - \overline{k}_0 + m + 2} - \frac{\overline{k}_i - \overline{k}_0 - m}{\overline{k}_n - \overline{k}_0 + 2m + 1} = 0.$$

Lemma 3 is fully proven. ■

Now we are prepared for the proof of proposition 8.

Proof. At first we note that the operator D can be represented in the form

$$D = \sum_{p=0}^{n} \frac{1}{(\overline{e}_p, \overline{e}_p)} \prod_{\substack{q=0 \\ q \neq p}}^{n} \vec{\alpha}_{qp} T_p \frac{1}{\prod_{\substack{q=0 \\ q \neq p}}^{n} \vec{\alpha}_{qp}},$$

where

$$\vec{\alpha}_{qp} = \prod_{s=1}^{m_{qp}} (k + s\alpha_{qp}, \alpha_{qp}).$$

We have to prove that

$$(T_i^s - T_j^s)\frac{Df(k)}{\prod_{\substack{\beta \in A_+ \\ \beta \neq \alpha_{ij}}} \vec{\beta}} = 0 \qquad \text{at} \quad k \in \pi_{ij},$$

if

$$(T_i^s - T_j^s)\frac{f(k)}{\prod_{\substack{\beta \in A_+ \\ \beta \neq \alpha_{ij}}} \vec{\beta}} = 0 \qquad \text{at} \quad k \in \pi_{ij}, \tag{5.11}$$

$s = 1, \ldots, m_{ij}$.

We have

$$(T_i^s - T_j^s)\frac{Df(k)}{\prod \vec{\beta}} = (T_i^s - T_j^s)\frac{1}{\prod \vec{\beta}} \sum_{p=0}^{n} \frac{1}{(\overline{e}_p, \overline{e}_p)} \prod_{\substack{q=0 \\ q \neq p}}^{n} \vec{\alpha}_{qp} T_p \frac{1}{\prod_{\substack{q=0 \\ q \neq p}}^{n} \vec{\alpha}_{qp}} f(k).$$

We show that the terms in the last sum corresponding to $p\neq i,j$ vanish, that is we show that

$$(T_i^s-T_j^s)\frac{1}{\prod_{\substack{\beta\in A_+\\ \beta\neq\alpha_{ij}}}\vec\beta}\prod_{\substack{q=0\\ q\neq p}}^{n}\vec\alpha_{qp}T_p\frac{1}{\prod_{\substack{q=0\\ q\neq p}}^{n}\vec\alpha_{qp}}f(k)=0 \tag{5.12}$$

at $k\in\pi_{ij}$. According to lemma 3 conditions (5.12) are equivalent for different choices of A_+ such that α_{ij} is an edge vector. Therefore we can assume that A_+ contains the vectors $\alpha_{qp}, 0\leqslant q\leqslant n,\ q\neq p$. For such a choice of A_+ one can carry out the cancellations in (5.12) and continue the equality

$$(T_i^s-T_j^s)\frac{1}{\prod_{\substack{\beta\in A_+\\ \beta\neq\alpha_{ij},(\beta,e_p)=0}}\vec\beta}T_p\frac{f(k)}{\prod_{\substack{q=0\\ q\neq p}}^{n}\vec\alpha_{qp}}=T_p(T_i^s-T_j^s)\frac{f(k)}{\prod_{\substack{\beta\in A_+\\ \beta\neq\alpha_{ij}}}\vec\beta}=0$$

because of (5.11). Thus we get

$$\begin{aligned}(T_i^s-T_j^s)&\frac{1}{\prod_{\substack{\beta\in A_+\\ \beta\neq\alpha_{ij}}}}Df(k)=\\ &=(T_i^s-T_j^s)\frac{1}{\prod\vec\beta}\left(\frac{1}{(\overline{e}_i,\overline{e}_i)}\prod_q\vec\alpha_{qi}T_i\frac{1}{\prod_q\vec\alpha_{qi}}+\frac{1}{(\overline{e}_j,\overline{e}_j)}\prod_q\vec\alpha_{qj}T_j\frac{1}{\prod_q\vec\alpha_{qj}}\right)f(k).\end{aligned} \tag{5.13}$$

Because of lemma 3 it is again legal to prove the triviality of the last expression for a special choice of A_+ only. We choose A_+ containing the vectors $\alpha_{qi},\ \alpha_{qj},\ 0\leqslant q\leqslant n, q\neq i,j$. Then in equality (5.13) one can perform cancellations and commutation such that the equality continues as follows

$$\begin{aligned}(T_i^s-T_j^s)&\frac{1}{\prod_{\substack{\beta\in A_+\\ \beta\neq\alpha_{ij}}}}Df(k)=\\ &=(T_i^s-T_j^s)\left(\frac{\vec\alpha_{ji}}{(\overline{e}_i,\overline{e}_i)}T_i(\frac{1}{\vec\alpha_{ji}})T_i\frac{1}{\prod_{\substack{\beta\in A_+\\ \beta\neq\alpha_{ij}}}\vec\beta}+\frac{\vec\alpha_{ij}}{(\overline{e}_j,\overline{e}_j)}T_j(\frac{1}{\vec\alpha_{ij}})T_j\frac{1}{\prod_{\substack{\beta\in A_+\\ \beta\neq\alpha_{ij}}}\vec\beta}\right)f(k)=\\ &=\frac{1}{(\overline{e}_i,\overline{e}_i)}\frac{T_i^s\vec\alpha_{ji}}{T_i^{s+1}\vec\alpha_{ji}}T_i^{s+1}\left(\frac{f(k)}{\prod\vec\beta}\right)-\frac{1}{(\overline{e}_j,\overline{e}_j)}\frac{T_j^s\vec\alpha_{ij}}{T_j^{s+1}\vec\alpha_{ij}}T_j^{s+1}\left(\frac{f(k)}{\prod\vec\beta}\right)-\\ &\qquad-\frac{1}{(\overline{e}_i,\overline{e}_i)}\frac{T_j^s\vec\alpha_{ji}}{T_j^sT_i\vec\alpha_{ji}}T_iT_j^s\left(\frac{f(k)}{\prod\vec\beta}\right)+\frac{1}{(\overline{e}_j,\overline{e}_j)}\frac{T_i^s\vec\alpha_{ij}}{T_i^sT_j\vec\alpha_{ij}}T_jT_i^s\left(\frac{f(k)}{\prod\vec\beta}\right).\end{aligned} \tag{5.14}$$

In order to check that the obtained expression is equal to zero we analyze the possible cases. If $m_{ij}=1$ then $s=1$ and $T_i\vec\alpha_{ji}=T_i(k+\alpha_{ji},\alpha_{ji})=\overline{k}_j-\overline{k}_i+(\overline{e}_j,\overline{e}_j)-(\overline{e}_i,\overline{e}_i)=0$ if $k\in\pi_{ij}$. Analogously $T_i\vec\alpha_{ij}=0$, thus the first two terms in (5.14) vanish. Also the last two terms in (5.14) cancel pairwise as

$$\begin{aligned}\frac{1}{(\overline{e}_i,\overline{e}_i)}\frac{T_j\vec\alpha_{ji}}{T_jT_i\vec\alpha_{ji}}=\frac{T_j(k+\alpha_{ji},\alpha_{ji})}{(\overline{e}_i,\overline{e}_i)T_jT_i(k+\alpha_{ji},\alpha_{ji})}&=\frac{\overline{k}_j-\overline{k}_i+3e_j^2+e_i^2}{(\overline{e}_i,\overline{e}_i)(\overline{k}_j-\overline{k}_i+3e_j^2-e_i^2)}=\\ &=\frac{2(e_i^2+e_j^2)}{2e_i^2e_j^2}=\frac{1}{(\overline{e}_j,\overline{e}_j)}\frac{T_i\vec\alpha_{ij}}{T_iT_j\vec\alpha_{ij}}\end{aligned}$$

at $k \in \pi_{ij}$. Now, if $m_{ij} = m$, that is $1 \leqslant i, j \leqslant n-1$, then at $k_i = k_j$ because of the symmetry we obviously have

$$\frac{1}{(\overline{e}_i, \overline{e}_i)} \frac{T_i^s \vec{\alpha}_{ji}}{T_i^{s+1} \vec{\alpha}_{ji}} = \frac{1}{(\overline{e}_j, \overline{e}_j)} \frac{T_j^s \vec{\alpha}_{ij}}{T_j^{s+1} \vec{\alpha}_{ij}} = g(k),$$

and also

$$\frac{1}{(\overline{e}_i, \overline{e}_i)} \frac{T_j^s \vec{\alpha}_{ji}}{T_j^s T_i \vec{\alpha}_{ji}} = \frac{1}{(\overline{e}_j, \overline{e}_j)} \frac{T_i^s \vec{\alpha}_{ij}}{T_i^s T_j \vec{\alpha}_{ij}} = h(k).$$

Thus relation (5.14) can be rewritten as

$$\left(T_i^s - T_j^s\right) \frac{1}{\prod \vec{\beta}} D f(k) = g(k) \left(T_i^{s+1} \frac{f(k)}{\prod \vec{\beta}} - T_j^{s+1} \frac{f(k)}{\prod \vec{\beta}} \right) + \\ + h(k) T_i T_j \left(T_i^{s-1} \frac{f(k)}{\prod \vec{\beta}} - T_j^{s-1} \frac{f(k)}{\prod \vec{\beta}} \right) + O(k_i - k_j) = 0,$$

since conditions (5.11) hold, here we have $1 \leqslant s < m_\alpha$. In the case $s = m_\alpha$ the previous equality also takes place as in this case $g(k) = 0$. The proposition is proven. ■

Theorem 6. *Let*

$$\varphi_0 = \left((\overline{k}_0 - \overline{k}_n)^2 - 1\right) \prod_{\substack{i,j=1 \\ i<j}}^{n-1} \prod_{s=1}^{m} \left((k_i - k_j)^2 - 4s^2\right) \times \\ \prod_{i=1}^{n-1} \left((\overline{k}_0 - k_i)^2 - m^2\right)\left((k_i - \overline{k}_n)^2 - (m+1)^2\right), \quad (5.15)$$

and let

$$\varphi_{t+1} = (D - \lambda(x))\varphi_t,$$

where D is operator (5.1), $\lambda(x) = \sum_{i=0}^{n} \frac{1}{\overline{e}_i^2} e^{2\overline{x}_i}$, *and $t = 0, 1, 2, \ldots$ Then*

$$\psi(k, x) = \left[2^M M! \prod_{i<j} (e^{2\overline{x}_i} - e^{2\overline{x}_j})^{m_{ij}}\right]^{-1} \varphi_M(k, x)$$

is the Baker–Akhiezer function for the configuration $A_{n,2}(m)$ if $M = \frac{m(n-1)(n-2)}{2} + 2n - 1$.

Proof. We note at first that the function φ_0 has in fact the following form

$$\varphi_0 = \prod_{\alpha \in A_{n,2}(m)} \prod_{i=1}^{m_\alpha} (k + i\alpha, \alpha)(k - i\alpha, \alpha) e^{(k,x)}.$$

Therefore propositions 7, 8 guarantee that for any s the function $\varphi_s(k, x)$ has the form $\varphi_s = P_s(k, x) e^{(k,x)}$ where P_s is a polynomial in k with the highest term P_s^0, and also φ_s satisfies axiomatics (5.2), (5.3). Thus we have to show that if $s = M$ then the first condition of the BA function definition (2.1) holds, that is $P_M^0 = \prod_{i<j} (\overline{k}_i - \overline{k}_j)^{m_{ij}}$. For that we analyze how P_s^0 changes while one applies operator $D - \lambda(x)$.

Lemma 4. *Let* $(D-\lambda(x))(Q_1(k,x)e^{(k,x)}) = Q_2(k,x)e^{(k,x)}$, *where* Q_1, Q_2 *are polynomials in* k *with the highest terms* Q_1^0, Q_2^0. *Then*

$$Q_2^0 = 2\sum_{i=0}^{n} e^{2\overline{x}_i}\frac{\partial Q_1^0}{\partial \overline{k}_i} + \left(\sum_{i=0}^{n} e^{2\overline{x}_i}\sum_{\substack{j=0\\ j\neq i}}^{n}\frac{-2m_{ij}}{\overline{k}_i-\overline{k}_j}\right)Q_1^0. \tag{5.16}$$

To prove the lemma we rewrite the operator D in the form

$$D = \sum_{i=0}^{n}\frac{1}{\overline{e}_i^2}\prod_{\substack{j=0\\ j\neq i}}^{n}\frac{\overline{k}_i-\overline{k}_j-m_{ij}(\overline{e}_i^2+\overline{e}_j^2)}{\overline{k}_i-\overline{k}_j+\overline{e}_i^2-\overline{e}_j^2}T_i.$$

Now the arguments analogous to the ones given in the proof of theorem 4, show that

$$Q_2^0 = \sum_{i=0}^{n}\frac{e^{2\overline{x}_i}}{\overline{e}_i^2}\frac{\partial Q_1^0}{\partial k_i}2\sqrt{(\overline{e}_i,\overline{e}_i)}+ \\ +\sum_{i=0}^{n}\frac{e^{2\overline{x}_i}}{\overline{e}_i^2}\sum_{j\neq i}\frac{1}{\overline{k}_i-\overline{k}_j}\left(-(m_{ij}+1)\overline{e}_i^2-(m_{ij}-1)\overline{e}_j^2\right)Q_1^0.$$

And it is easy to notice that the obtained expression coincides with the one in formula (5.16). In particular, if Q_1^0 is a linear combination of monomials, $Q_1^0 = \sum_{\{\lambda\}}\prod_{i<j}(\overline{k}_i-\overline{k}_j)^{\lambda_{ij}}$, then

$$Q_2^0 = \sum_{\{\lambda\}}\sum_{i_0<j_0}2(\lambda_{i_0j_0}-m_{i_0j_0})(e^{2\overline{x}_{i_0}}-e^{2\overline{x}_{j_0}})(\overline{k}_{i_0}-\overline{k}_{j_0})^{\lambda_{i_0j_0}-1}\prod_{(i,j)\neq(i_0,j_0)}(\overline{k}_i-\overline{k}_j)^{\lambda_{ij}}. \tag{5.17}$$

Thus in order to construct φ_i we start with the monomial $P_0^0 = \prod_{i<j}(k_i-k_j)^{2m_{ij}}$, and at every step i the highest term P_i^0 is a linear combination of monomials of the form $\prod(\overline{k}_i-\overline{k}_j)^{\lambda_{ij}}$. From formula (5.17) it can be seen that $\lambda_{ij}\geqslant m_{ij}$, therefore at the step with the number $M=\sum m_\alpha$ it is necessarily that $P_M^0 = C(x)\prod_{i<j}(\overline{k}_i-\overline{k}_j)^{m_{ij}}$. The combinatorial arguments similar to the ones given in the proof of theorem 4 for the system $\mathcal{C}_n(l,m)$ show that $C(x) = 2^M M!\prod_{i<j}(e^{2\overline{x}_i}-e^{2\overline{x}_j})^{m_{ij}}$, thus the theorem is proven. ∎

In the end of this section we put the result on bispectrality.

Theorem 7. *The Baker–Akhiezer function* $\psi(k,x)$ *for the system* $A_{n,2}(m)$ *satisfies the following equation in variables* k:

$$D\psi(k,x) = \sum_{i=0}^{n}\frac{1}{\overline{e}_i^2}e^{2\overline{x}_i}\psi(k,x),$$

where D *is operator* (5.1). *For the polynomials* $p(k)\in R_{A_{n,2}(m)}$ *the difference operators*

$$D_p = ad_D^{\deg p}p(k)$$

commute. These operators also commute with operator D.

Proof. By propositions 7, 8 the function

$$(D - \sum_{i=0}^{n} \frac{1}{\overline{e}_i^2} e^{2\overline{x}_i})\psi(k,x)$$

has the form $P(k,x)e^{(k,x)}$ where P is a polynomial in k of degree less than $\sum m_\alpha$, and satisfies axioms (5.2), (5.3). By lemma 1 we have $P = 0$ which is the required equation. The proof of the second part of the theorem is also identical to the proof of the corresponding theorem 5 about the configuration $\mathcal{C}_n(l,m)$.

■

6 Trigonometric locus conditions

In this section we obtain the restrictions for a configuration $\mathcal{A} = (A, m)$ to admit the Baker–Akhiezer function. We obtain them from the Scroedinger equation which holds for the BA function, the restrictions turn out to be quite strong, they also have clear geometrical sense.

By Proposition 2 we have the following equation for the Baker–Akhiezer function $\psi(k,x)$:

$$\left(\Delta - \sum_{\alpha\in A} \frac{m_\alpha(m_\alpha+1)(\alpha,\alpha)}{\sinh^2(\alpha,x)}\right)\psi(k,x) = k^2\psi(k,x). \tag{6.1}$$

In paper [10] such an equation was considered for an arbitrary meromorphic potential and a function ψ of the form $\psi = P(k,x)e^{(k,x)}$ where P is a polynomial in k. As it was shown in [10] (see also [15]) the potential should satisfy the so called locus conditions. Regarding the form (6.1) these conditions have the form

$$\partial_\alpha^{2s-1} \sum_{\substack{\beta\in A\\ \beta\neq\alpha}} \frac{m_\beta(m_\beta+1)(\beta,\beta)}{\sinh^2(\beta,x)} = 0 \qquad \text{at} \quad \sinh(\alpha,x) = 0, \tag{6.2}$$

$s = 1, \ldots, m_\alpha$.

We take vectors β forming a positive subsystem A_+, with α one of the edge vectors. Then projections

$$\widehat{\beta} = \beta - a_\beta\alpha, \quad a_\beta = \frac{(\alpha,\beta)}{(\alpha,\alpha)} \tag{6.3}$$

of the vectors β to the hyperplane $\Pi : (\alpha,x) = 0$ belong to a half-space in this hyperplane. Indeed, otherwise we have a non-trivial dependence $\sum_{\beta\in A_+\backslash\alpha} r_\beta\widehat{\beta} = 0$ with some non-negative real coefficients r_β. Then $\sum_{\beta\in A_+\backslash\alpha} r_\beta\beta = \lambda\alpha$ for some $\lambda \in \mathbb{C}$. Since all the vectors from A_+ belong to some lattice it follows that the coefficient λ must be real. In order for α to belong to the same half space as all β it must be $\lambda > 0$ which contradicts the condition that α is an edge vector. So the projections $\widehat{\beta}$ must belong to a half-space. We denote by σ the border of this half-space.

The cone $K = \{\mathrm{Re}(\widehat{\beta},x) < 0 |\ \beta \in A_+\backslash\alpha\}$ has a non-empty intersection with Π. Indeed, we consider a generic extension to $\mathbb{C}^n$ of the $(2n-3)-$plane σ to form a $(2n-1)-$

hyperplane. Let it have the equation $Re(u,x)=0$ for some $u\in\mathbb{C}^n$ so that $Re(u,\widehat{\beta})<0$ for all projections $\widehat{\beta}$. Now consider $\widehat{u}=u-\frac{(u,\alpha)}{(\alpha,\alpha)}\alpha$. One has $\widehat{u}\in\Pi$, and also $(\widehat{u},\widehat{\beta})=(u,\widehat{\beta})$ thus $\widehat{u}\in K$ so the intersection $K\cap\Pi$ is non-empty.

In the cone K we can expand $\sinh(\beta,x)$ into the corresponding series so that conditions (6.2) take the form

$$\partial_\alpha^{2s-1}\sum_{\substack{\beta\in A_+\\ \beta\neq\alpha}}4m_\beta(m_\beta+1)(\beta,\beta)\sum_{j=1}^{\infty}je^{2j(\beta,x)}=0\qquad\text{at }\sinh(\alpha,x)=0.$$

More explicitly we obtain

$$\sum_{\substack{\beta\in A_+\\ \beta\neq\alpha}}\sum_{j=1}^{\infty}m_\beta(m_\beta+1)(\beta,\beta)(j(\alpha,\beta))^{2s-1}je^{2j(\beta,x)}=0\tag{6.4}$$

at $\Pi_n:\{(\alpha,x)=\pi in\}$; $n\in\mathbb{Z}$. We note that the intersection $K\cap\Pi_n$ is non-empty for any n. Indeed, we represent x in the form $x=\frac{\pi in}{(\alpha,\alpha)}\alpha+y$ for some vector y. The condition $x\in\Pi_n\cap K$ takes the form $(\alpha,y)=0$ and $\mathrm{Re}(\beta,x)<0$, $\beta\in A_+\backslash\alpha$. In terms of y we get

$$\mathrm{Re}(\beta,x)=\mathrm{Re}\Big(\widehat{\beta}+a_\beta\alpha,\ \frac{\pi in}{(\alpha,\alpha)}\alpha+y\Big)=\mathrm{Re}(\widehat{\beta},y)+\mathrm{Re}(a_\beta\pi in)<0.$$

Thus x belongs to $\Pi_n\cap K$ if and only if the corresponding $y=x-\frac{\pi in}{(\alpha,\alpha)}\alpha$ satisfies $(\alpha,y)=0$ and also $\mathrm{Re}(\widehat{\beta},y)<-\mathrm{Re}(a_\beta\pi in)$. The last intersection is non-empty as it contains a real multiple of any vector from the cone $\Pi\cap K$.

Now in the cone $\widehat{K}_n=\big\{y|\ (\alpha,y)=0,\ \mathrm{Re}(\widehat{\beta},y)<-\mathrm{Re}(a_\beta\pi in)\big\}$ the following form of conditions (6.4) takes place:

$$\sum_{\substack{\beta\in A_+\\ \beta\neq\alpha}}\sum_{j=1}^{\infty}m_\beta(m_\beta+1)(\beta,\beta)\,(j(\alpha,\beta))^{2s-1}\,j\,e^{2ja_\beta\pi in}\,e^{2j(\hat{\beta},y)}=0.\tag{6.5}$$

We notice that the vectors $\widehat{\beta}$ belong to a lattice of rank $n-1$ in the hyperplane Π. Indeed, considering if necessary a sub-lattice of the original lattice in $\mathbb{C}^n$ we may assume that the vector α is an integer multiple of a basis vector of this lattice. The projections of other $n-1$ basis vectors to Π will generate a lattice in Π containing the vectors $\widehat{\beta}$.

Let $e_1^*,\dots,e_{n-1}^*$ be a basis of this lattice. We claim that after collecting the terms the coefficients at each particular exponent $(p_1e_1^*+\ldots+p_{n-1}e_{n-1}^*,x)$ in (6.5) equal zero. Indeed, the cone $\widehat{K}_n$ contains a parallelepiped of the form

$$B_\lambda=\big\{x|\ (x,e_j^*)\in[\lambda_j,\lambda_j+2\pi it_j],\ 0\leqslant t_j\leqslant 1,\ j=1,\dots,n-1\big\}\tag{6.6}$$

for some $\lambda_j\in\mathbb{C}$. Multiplying the series (6.5) by $exp(-p_1e_1^*-\ldots-p_{n-1}e_{n-1}^*,x)$ and integrating it over B_λ (which can be done term by term as (6.5) is uniformly convergent on B_λ) we conclude that all the terms are zero except the coefficient at the chosen exponent, thus it should vanish as well.

Consider now the set of vectors $B^1 = \{\beta_1, \dots, \beta_p\} \subset A_+$ such that $\widehat{\beta}_1 = \dots = \widehat{\beta}_p$, and $j\widehat{\beta} \neq \widehat{\beta}_1$ for any $\beta \in A_+$, $j \in \mathbb{N}$, $j > 1$. By the previous argument we conclude

$$\sum_{\beta\in B^1} m_\beta(m_\beta+1)(\beta,\beta)(\alpha,\beta)^{2s-1}e^{2a_\beta\pi in} = 0.$$

Since $n \in \mathbb{Z}$ is arbitrary the set B^1 is decomposed into the subsets $B^1 = B^1_1 \cup \dots \cup B^1_t$ such that $\forall\beta,\gamma \in B^1_l$ one has $e^{2a_\beta\pi i} = e^{2a_\gamma\pi i}$ and

$$\sum_{\beta\in B^1_l} m_\beta(m_\beta+1)(\beta,\beta)(\alpha,\beta)^{2s-1} = 0. \tag{6.7}$$

We note that the condition $e^{2a_\beta\pi i} = e^{2a_\gamma\pi i}$ is equivalent to $a_\beta - a_\gamma = n_{\beta\gamma} \in \mathbb{Z}$. Using $\widehat{\beta} = \widehat{\gamma}$ and recalling (6.3) we get

$$\beta - \gamma = n_{\beta\gamma}\alpha. \tag{6.8}$$

Further, for any $j \geqslant 1$ we clearly have

$$\sum_{\beta\in B^1} m_\beta(m_\beta+1)(\beta,\beta)\,(j(\alpha,\beta))^{2s-1}\,j\,e^{2ja_\beta\pi in} = 0,$$

and therefore identity (6.5) is valid with the summation over $\beta \in A_+\backslash\alpha\backslash B^1$. Thus the system $A_+\backslash\alpha$ can be presented as a union of subsystems $B^1 \sqcup B^2 \sqcup \dots$ for each of which it is valid (6.7), (6.8). We have proven the following

Theorem 8. *Let configuration $\mathcal{A} = (A, m)$ admit the Baker–Akhiezer function. Let $A_+ \subset (A \cup (-A))$ be a positive subsystem and let vector $\alpha \in A_+$ be an edge vector. Then the system of vectors $A_+\backslash\alpha$ can be represented as a disjoint union of "series" $A_+\backslash\alpha = B_1 \sqcup \dots \sqcup B_N$ such that for all l, $1 \leqslant l \leqslant N$ one has*

1) *for any $\beta,\gamma \in B_l$ the difference $\beta - \gamma = n_{\beta\gamma}\alpha$, with $n_{\beta\gamma} \in \mathbb{Z}$;*

2) $\sum_{\beta\in B_l} m_\beta(m_\beta+1)(\beta,\beta)(\alpha,\beta)^{2s-1} = 0$, *where $1 \leqslant s \leqslant m_\alpha$.*

These properties are equivalent to locus conditions (6.2).

Acknowledgments. *I am very grateful to O A Chalykh for numerous stimulating and helpful discussions, and to A P Veselov for useful discussions and interest to the work. On the final stage the work was supported by the Chapman Fellowship (Department of Mathematics, Imperial College) and by the European grant "ENIGMA" (MRTN-CT-2004-5652).*

References

[1] Airault H, McKean H P, Moser J, Rational and elliptic solutions of the Korteweg-de Vries equation and a related many-body problem, *Comm. Pure Appl. Math.* **30** (1977), 95–178.

[2] Berest Yu, Huygens' principle and the bispectral problem, *CRM Proceedings and Lecture Notes* **14** (1998), 11–30.

[3] Burchnall J L and Chaundy T W, Commutative ordinary differential operators, *Proc. London Math. Soc.* **21** (1922), 420–440; *Proc. Royal Soc. London.* **118** (1928), 557–583.

[4] Calogero F, Solution of the one-dimensional n-body problem with quadratic and/or inversely quadratic pair potential, *J. Math. Phys.* **12** (1971), 419–436.

[5] Chalykh O A, Darboux transformations for multidimensional Schrödinger operators, *Russian Math. Surveys.* **53** (1998), 167–168.

[6] Chalykh O, Bispectrality for quantum Ruijsenaars model and its integrable deformations, *J. Math. Phys.* **41** (2000), 5139–5167.

[7] Chalykh O A, Macdonald polynomials and algebraic integrability, *Adv. Math.* **166** (2002), 193–259.

[8] Chalykh O, Etingof P and Oblomkov A, Generalized Lame operators, *Commun. Math. Phys.* **239** (2003), 115–153.

[9] Chalykh O A, Feigin M V and Veselov A P, New integrable generalisations of Calogero–Moser quantum problem, *J. Math. Phys.* **39** (1998), 695–703.

[10] Chalykh O A, Feigin M V, Veselov A P, Multidimensional Baker–Akhiezer Functions and Huygens' Principle, *Commun. Math. Phys.* **206** (1999), 533–566.

[11] Chalykh O A, Veselov A P, Commutative rings of partial differential operators and Lie algebras, *Commun. Math. Phys.* **126** (1990), 597–611.

[12] Chalykh O A and Veselov A P, Locus configurations and $\vee$-systems, *Phys. Lett. A* **285** (2001), 339–349.

[13] Cherednik I, Macdonald's evaluation conjectures and difference Fourier transform, *Invent. Math.* **122** (1995), 119–145.

[14] Duistermaat J J and Grünbaum F A, Differential equations in the spectral parameter, *Comm. Math. Phys.* **103** (1986), 177–240.

[15] Feigin M, Multidimensional integrable Schroedinger operators, Candidate of Science (PhD) thesis, Moscow State University, 2001.

[16] Hietarinta J, Pure quantum integrability, *Phys. Lett. A* **246** (1998), 97–104.

[17] Koornwinder T H, unpublished manuscript, 1988.

[18] Koornwinder T H, Askey–Wilson polynomials for root systems of type BC, *Contemp. Math.* **138** (1992), 189–204.

[19] Krichever I M, Methods of algebraic geometry in the theory of non-linear equations, *Russian Math. Surveys* **32** (1977), 183–208.

[20] Macdonald I G, Symmetric Functions and Hall Polynomials, Clarendon, Oxford, 1995.

[21] Macdonald I G, Orthogonal polynomials associated with root systems, 1988, Preprint.

[22] Moser J, Three integrable hamiltonian systems connected with isospectral deformations, *Adv. Math.* **16** (1975), 197–220.

[23] Olshanetsky M A and Perelomov A M, Quantum integrable systems related to Lie algebras, *Phys. Rep.* **94** (1983), 313–404.

[24] Ruijsenaars S N M, Complete integrability of relativistic Calogero–Moser systems and elliptic functions identities, *Commun. Math. Phys.* **110** (1987), 191–213.

[25] Ruijsenaars S N M, Finite-dimensional soliton systems, in Integrable and Superintegrable Systems, Editor: B. Kupershmidt, World Scientific, Singapore, 1990, 165–206.

[26] Ruijsenaars S N M and Sneider H, A new class of integrable systems and its relation to solitons, *Ann. Phys.* **170** (1986), 370–405.

[27] Sahi S, Nonsymmetric Koornwinder polynomials and duality, *Ann. Math.* **150** (1999), 267–282.

[28] Sutherland B, Exact results for a quantum many-body problem in one dimension, *Phys. Rev. A* **4** (1971), 2019–2021.

[29] van Diejen J F, Self-dual Koornwinder-Macdonald polynomials, *Invent. Math.* **126** (1996), 319–339.

[30] Veselov A P, Feigin M V and Chalykh O A, New integrable deformations of quantum Calogero-Moser problem, *Russian Math Surveys* **51** (1996), 185–186.

[31] Veselov A P, Styrkas K L and Chalykh O A, Algebraic integrability for Schrodinger equation and finite reflection k groups, *Theor. Math. Phys.* **94** (1993), 253–275.