An Efficient Difference Algorithm for Black-Scholes Equation with Payment of Dividend

Wu Lifei
Dept. of Mathematics and Physics
North China Electric Power University
Beijing, China
e-mail: wulf@ncepu.edu.cn

Yang Xiaozhong
Dept. of Mathematics and Physics
North China Electric Power University
Beijing, China
e-mail: yxiaozh@ncepu.edu.cn

***Abstract*—Black-Scholes equation is the basic equation of option pricing in financial mathematics, it is important to study its numerical solution in financial market. This paper constructs a new kind of high order accuracy numerical algorithm (Three-layer difference scheme) for Black-Scholes equation with payment of dividend. Secondly, it gives the convergence of scheme. Thirdly, the stability and error estimates are analyzed. Finally, the numerical examples show the feasibility and effectiveness of the scheme. The truncation error of Three-layer scheme is little worse than Crank-Nicolson scheme and computational cost is little better than Crank-Nicoslon scheme. Therefore, the scheme is better suitable for applying to calculate the option pricing in the demanding high level of instantaneity.**

Keywords-component; Black-Scholes equation; Three-layer difference scheme; calculation stability; error estimate; numerical example

I. INTRODUCTION

In the financial market, the option is a kind of important financial derivatives and the core of risk management tools for reducing investment risk. Along with the development of the financial market, the investors put forward new requirements for the option pricing modeling. For example, they begin to consider paying a dividend transaction costs and other practical factors. Therefore, the numerical method for Black-Scholes equation with payment of dividend is researched in this paper.

Currently based on a series of extended assumptions, another differential equation is well-know about the pricing of stock options with paying dividends[1,2]:

$$\frac{\partial V}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}+(r-q)S\frac{\partial V}{\partial S}-rV=0 \qquad (1)$$

Here, V is option price; S is native asset price; r is risk free rate; q is dividend yield; σ and t represent volatility and time. The explicit expression of Black-Scholes equation with paying dividend:

$$V(S,t)=Ke^{-r(T-t)}N(-d_2)-Se^{-q(T-t)}N(-d_1) \qquad (2)$$

$$d_1=\frac{\ln\frac{S}{K}+(r+\frac{\sigma^2}{2}-q)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2=d_1-\sigma\sqrt{T-t}.$$

Where, $N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{\frac{-\omega^2}{2}}d\omega$ is called the standard normal cumulative probability distribution function in probability.

Although analytical solution of expression the payment of dividends under the Black-Scholes equation, it cannot meet the effective requirement in option pricing.

In practice, the numerical method has been widely used, such as the Monte Carlo method and the Binary Tree method [4], but the two methods have less accuracy than the finite difference method. Therefore, we usually adopt finite difference method. Yang Xiaozhong, Liu Yangguo(2007) put forward a new kind of universal difference schemes for solving Black-Scholes equation[5], but not involved in the case of payment dividends. Wu Lifei (2011) proposed semi-implicit difference scheme (asymmetric difference scheme) for solving Black-Scholes equation with paying dividends[6]. The computational cost of asymmetric scheme is approximately 95% less than Crank-Nicolson scheme, but it has less accuracy.

For these reason, in this paper, taking European put option pricing as an example, we put forward a kind of high order accuracy difference schemes (Three-layer difference shcheme) for solving the payment of dividends under the Black-Scholes equation. And the analyses of stability and convergence have been given; finally, numerical examples demonstrate the effectiveness of the schemes.

II. CONSTRUCTION OF DIFFERENCE SCHEME

A. The Definite of Solution Problem

In order to get the value of a European put option, equation (1) must be integrated with the boundary conditions for numerical solution. There are three boundary conditions:

*1)*The profit and loss status: $V(S,t)=(K-S)^+$. This condition is quite clear, the profit and loss when the option expires is its price.

*2)*when S (native asset prices) is zero, the option price is close to Ke^{-rt}.

*3)*when S is sufficiently great, the option is approximate to zero.

So on the European put option pricing is to solve the following equation:

$$\begin{cases} \frac{\partial V}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}+(r-q)S\frac{\partial V}{\partial S}-rV=0 \\ V(S,T)=\max\{K-S,0\} \end{cases} \quad (3)$$

Boundary conditions:

$$V(0,t)=Ke^{-r(T-t)},$$

$$\lim_{S\to+\infty}V(S,t)=0.$$

Solution region:

$$\Sigma=\{0\le S<\infty, 0\le t\le T\}.$$

Eq. (3) is anti-variable coefficients parabolic equation. So make the following coordinate transform:

$$x=InS, \tau=T-t.$$

Eq.(3) converse into constant coefficients parabolic equation Cauchy problem, as follows:

$$\begin{cases} \frac{\partial V}{\partial \tau}-\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2}-(r-q-\frac{\sigma^2}{2})\frac{\partial V}{\partial x}+rV=0 \\ V(x,0)=\max\{K-e^x,0\} \end{cases} \quad (4)$$

Boundary conditions converse into:

$$\lim_{x\to-\infty}V(x,\tau)=Ke^{-r\tau},$$

$$\lim_{x\to+\infty}V(x,\tau)=0.$$

Solution region converse into:

$$\Sigma_0=\{-\infty\le x<+\infty, 0\le\tau\le T\}.$$

B. Construction of Three-layer Difference Scheme

Let us first discretize the region Σ_0 as a uniform grid with space step h and time step k,

$$\begin{cases} x_j=jh, j=0,\pm1,\pm2,\cdots,M \\ \tau_n=nk, n=0,1,2,\cdots,N(N=[T/k]) \end{cases}$$

Let us denote the numerical approximation of the solution by $V_j^n=V(x_j,\tau_n)$. At the point (x_j,τ_n), we adopt 1-order and 2-order central difference scheme for space derivatives Eq.(4) in the spatial direction.

$$\left[\frac{\partial V}{\partial x}\right]_j^n=\frac{V_{j+1}^{n+1}-V_{j-1}^{n+1}}{2h},$$

$$\left[\frac{\partial^2 V}{\partial x^2}\right]_j^n=\frac{V_{j+1}^{n+1}-2V_j^{n+1}+V_{j-1}^{n+1}}{h^2}.$$

For space derivatives of Eq.(4), we adopt its approximation value

$$\frac{\partial V}{\partial \tau}=\frac{3}{2}(\frac{V_j^{k+1}-V_j^k}{k})-\frac{1}{2}(\frac{V_j^k-V_j^{k-1}}{k}) \quad (5)$$

Then the equation (4) will be discredited into

$$\frac{3}{2}(\frac{V_j^{k+1}-V_j^k}{k})-\frac{1}{2}(\frac{V_j^k-V_j^{k-1}}{k}) = [\frac{\sigma^2}{2h^2}\delta_x^2+(r-q-\frac{\sigma^2}{2})\frac{1}{2h}\delta_x-r]V_j^{k+1} \quad (6)$$

III. Truncation Error Analysis

In order to analysis the accuracy of scheme (6), the solution converse into:

$$\Sigma_1=\{-M\le x\le+M, 0\le\tau\le T\}.$$

Boundary conditions converse into:

$$V_j^0=\max\{K-e^{jh},0\},(j=0,\pm1,\pm2,\cdots,\pm J)$$

$$V_J^n=0$$

$$V_{-J}^n=\mathrm{Ke}^{-nkr},(n=0,1,2,\cdots,N)$$

The truncation error of a difference equation is the difference between its left side and right side after the approximate value V_j^n is replaced by the analytic solution $V(x_j,t_n)$ of the original differential equation.

The truncation error of the scheme (6)

$$T(k,h)=\frac{3}{2}(\frac{V_j^{k+1}-V_j^k}{k})-\frac{1}{2}(\frac{V_j^k-V_j^{k-1}}{k})-[\frac{\sigma^2}{2h^2}\delta_x^2+(r-q-\frac{\sigma^2}{2})\frac{1}{2h}\delta_x-r]V_j^{k+1}$$

Expand the terms of $T(k,h)$ as Taylor expansion at the point (x_j,τ_n). Because $V(x,t)$ is the analytic solution of the Eq. (4), we have the preceding four term of $T(k,h)$

$$\begin{aligned} T(k,h)=&\frac{\partial V}{\partial t}+k\frac{\partial^2 V}{\partial t^2}-\frac{\sigma^2}{2}(\frac{\partial^2 V}{\partial x^2}+k\frac{\partial^3 V}{\partial x^2\partial t}) \\ &-(r-q-\frac{\sigma^2}{2})(\frac{\partial V}{\partial x}+k\frac{\partial^2 V}{\partial x\partial t})+r(V+t\frac{\partial V}{\partial t}) \\ &+o(k^2+x^2) \end{aligned} \quad (7)$$

Besides, based on the functional difference theory and the equation (4),

$$\frac{\partial^2 V}{\partial \tau^2} = \frac{\sigma^2}{2}\frac{\partial^3 V}{\partial x^2 \partial \tau} - (r - q - \frac{\sigma^2}{2})\frac{\partial^2 V}{\partial x \partial \tau} + r\frac{\partial V}{\partial \tau} \quad (8)$$

Substituting equation (8) into (7),

$$T(k,h) = O(h^2 + k^2)$$

Therefore, we have

Theorem 1. The truncation error of the Three-layer difference scheme (6) for solving the payment of dividend Black-Scholes equation is $O(h^2 + k^2)$, the truncation error of the scheme (6) has 2-order accuracy.

IV. Analysis of the Stability and Convergence

Now, we'll consider the stability and convergence of the schemes (6). We firstly make Three-layer difference scheme format into an equivalent two levels difference scheme.

$$\begin{cases} (r_2 - r_1)V_{j-1}^k + (\frac{3}{2} + 2r_1 + rk)V_j^k - (r_1 + r_2)V_{j-1}^k = 2V_j^k - \frac{1}{2}U_j^k \\ U_j^{k+1} = V_j^k \end{cases}$$

Here $r_1 = \frac{\sigma^2 k}{2h^2}$, $r_2 = (r - q - \frac{\sigma^2}{2})\frac{k}{2h}$.

Let $W = [V,U]^T$, the above equation can be written as

$$\begin{pmatrix} 2 & \frac{1}{2} \\ 1 & 0 \end{pmatrix} W_j^k = \begin{pmatrix} -r_1 + r_2 & 0 \\ 0 & 0 \end{pmatrix} W_{j-1}^{k+1} + \begin{pmatrix} \frac{3}{2} + 2r_1 + rk & 0 \\ 0 & 1 \end{pmatrix} W_j^{k+1} + \begin{pmatrix} -r_1 - r_2 & 0 \\ 0 & 0 \end{pmatrix} W_{j+1}^{k+1}$$

Denote $V_j^n = v^n e^{ijQh}$, where $i = \sqrt{-1}$ is the imaginary unit, Q is the wave number. Hence, we get the growth matrix of the Three-layer difference scheme (6),

$$G(k,Q) = \begin{bmatrix} \frac{2}{\alpha} & \frac{1}{2\alpha} \\ 1 & 0 \end{bmatrix},$$

Here

$$\alpha = \frac{5}{2} + 2r_1(1 - \cos Qh) + rk - i2r_2 \sin Qh.$$

The infinite norm of growth matrix is

$$\|G\|_\infty = \max\{1, \left|\frac{5}{2\alpha}\right|\}.$$

If we want $\|G\|_\infty \le 1$, we should meet the condition $5 \le |2\alpha| = |5 + 4r_1(1 - \cos Qh) + 2rk - i4r_2 \sin Qh|$.

When $\min\{r_1, r_2\} \ge 1$, we have $\left|\frac{5}{2\alpha}\right| \le 1 \Leftrightarrow \|G\|_\infty \le 1$ and $\rho(G) \le \|G\|_\infty \le 1$. In other words, the two characteristic root of the growth matrix are less than or equal 1. Therefore, von Neumann condition is satisfied，the Three-layer difference scheme conditional stability.

Through comprehensive analysis, we have

Theorem 2. When $\min\{r_1, r_2\} \ge 1$, the Three-layer difference scheme (6) of the payment of dividend Black-Scholes equation is stable.

Hereby, based on the Lax theorem [10], we obtain

Corollary. When $\min\{r_1, r_2\} \ge 1$, the Three-layer difference scheme (6) of the payment of dividend Black-Scholes equation is convergent.

V. Numerical Example

In this section, we present and analyses some numerical results for pricing option, which was calculated by the Three-layer difference scheme and other difference schemes in MATLAB 7.6.

Example 1[8]: Suppose Europe call option which is stock, the strike price is 50$, the risk-free nominal interest rate is 6%, the dividend yield is 1% (with the unit of time being one year), and the stock's volatility is 0.4. Considering paying-dividend stock European call options those in 6 months.

Solution: $K = 50,\ T = 0.5,\ \sigma = 0.40,$
$r = 0.06,\ q = 0.01$

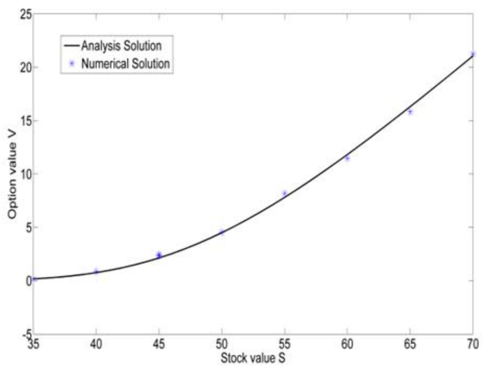

Figure 1. Comparison between numerical solution and analysis solution

Example 2: Suppose Europe call option which is stock, the strike price is 50$, the risk-free nominal interest rate is

6%, the dividend yield is 4% (with the unit of time being one year), and the stock's volatility is 0.2. Considering paying-dividend stock European call options those in 6 months, the stock current price is 50$.

Solution: $S = 50,\ K = 50,\ r = 0.06,$
$\sigma = 0.2,\ q = 0.04,\ T = 0.5.$

Figure 1 demonstrates that the Three-layer difference scheme's solution can approximate to analytic solution.

Form the Table 1, we can obtain that the five difference schemes all can better approximate to analysis solution. The truncation error of Asymmetric scheme is the worst in the five difference schemes. Due to the semi-implicit character of Asymmetric scheme (implicit scheme, explicit calculate), the computational cost is the least in the five schemes. From Table 1, we can see that the computational cost (CPU time) of Asymmetric scheme can save approximately 95% for Crank-Nicolson scheme. We can also see that, the calculation accuracy of the Explicit-Implicit difference scheme is close to the famous Crank-Nicolson scheme, but the computation (Running Time) of those can save about 50% of Crank-Nicolson scheme's.

Three-layer scheme is closed to Crank-Nicolson scheme from truncation error and computational cost. The truncation error of Three-layer scheme is little worse than Crank-Nicolson scheme and computational cost is little better than Crank-Nicoslon scheme.

VI. CONCLUSION

In this paper, we construct a kind of high order accuracy difference scheme (Three-layer difference scheme) for solving the Black-Schloes equation with payment of dividends, make the analysis of truncation error, stability and convergent for Three-layer scheme; at final the numerical examples show the well stability and calculation accuracy for the Three-layer scheme. Three-layer scheme is closed to Crank-Nicolson scheme from truncation error and computational cost. Our future work is to develop higher order computational precision and faster computer algorithm for option pricing.

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Table 1. COMPARISON BETWEEN ASYMMETRIC SCHEME AND OTHER SCHEMES

Numerical example	0.5	Relative error	CPU
Analysis solution[1]	**6.344806**		
Crank-Nicolson scheme[5]	6.344796	0.002%	6.32
Asymmetric scheme[6]	6.306952	0.59%	0.21
Explicit-Implicit scheme[7]	6.344786	0.003%	3.98
Three-layer scheme	6.346942	0.034%	4.53

Note: X=6, M=500, N=3000.