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Non-isospectral lattice hierarchies in $2+1$ dimensions and generalized discrete Painlevé hierarchies

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Abstract

In a recent paper we introduced a new $2+1$-dimensional non-isospectral extension of the Volterra lattice hierarchy, along with its corresponding hierarchy of underlying linear problems. Here we consider reductions of this lattice hierarchy to hierarchies of discrete equations, which we obtain once again along with their hierarchy of underlying linear problems. We obtain a generalized discrete first Painlevé hierarchy which includes as special cases, after further summation, both the standard discrete first Painlevé hierarchy and a new extended version of the discrete thirty-fourth Painlevé hierarchy.

1 Introduction

Non-isospectral scattering problems were introduced very soon after isospectral scattering problems were used to solve the Korteweg-de Vries (KdV) [5] and nonlinear Schrödinger [28] equations. The first example of a nonlinear equation having such a scattering problem is due to Calogero [2]; since then a great many papers have been published on such systems, both continuous, e.g. [17]—[22], and discrete, e.g. [19, 20]. Of particular relevance here is the link, observed in [21], between non-isospectral scattering problems for PDEs or lattice equations, and linear problems for ODEs or discrete equations, respectively. It is this link that we have exploited (and generalized) in our work on continuous and discrete Painlevé hierarchies.

In a series of recent papers [6]—[8] (see also [9]) a method has been developed which allows the construction, starting from a partial differential equation (PDE) having a non-isospectral scattering problem of a certain kind in $2+1$ dimensions, of a whole hierarchy of PDEs having non-isospectral scattering problems in $2+1$ dimensions. This involves interpreting the PDE and its non-isospectral scattering problem as defining a recursion relation between successive members of the hierarchy and their corresponding scattering problems. Reductions of the resulting hierarchies then yield hierarchies of PDEs in fewer dimensions (isospectral or non-isospectral), and also hierarchies of ordinary differential equations (ODEs), all along with corresponding underlying linear problems. In this way a variety of hierarchies of ODEs having as first member a Painlevé equation, or a higher order analogue thereof, were successfully obtained. Such Painlevé hierarchies are a topic of much intensive current research.

In a still more recent paper [10] the current authors have extended the above-mentioned approach to the discrete case. In particular we obtained a new non-isospectral extension of the Volterra hierarchy to $2+1$ dimensions — with two continuous independent variables, t and y, and one discrete independent variable, n — along with its underlying linear problem. A variety of reductions of this hierarchy were also considered, to lattice hierarchies, differential-delay hierarchies, and also to a discrete hierarchy. It was noted in particular that the first non-trivial member of this discrete hierarchy contained as special cases, after further summation, the discrete first Painlevé (dP_I) equation and an extended version of the known discrete thirty-fourth Painlevé equation: we thus referred to our hierarchy as a generalized dP_I hierarchy, and argued that it is our extended discrete thirty-fourth Painlevé equation — which we called the dP_{34} equation — which should properly be understood as the analogue of the well-known continuous P_{34} equation. We also observed that the second non-trivial member of our hierarchy contained as a special case the known fourth-order dP_I equation, as obtained in [4]. We remark that, in fact, in [4] a dP_I hierarchy and a discrete second Painlevé (dP_{II}) hierarchy were found, and it was also shown how, from a special case of this last, a discrete thirty-fourth Painlevé hierarchy can be constructed.

The aim of the present paper is to explore further the generalized dP_I hierarchy found in [10]. We claim that this hierarchy corresponds to a discrete version of the ODE hierarchy (3.29) of [7], but with an additional parity-dependent term, and that it contains as special cases, after further summation, the dP_I hierarchy of [4] and an extended version of the discrete thirty-fourth Painlevé hierarchy of [4]. In order to support this claim, we consider in detail the first two non-trivial members of our hierarchy. We also give an alternative approach to the construction of Bäcklund transformations, which allows us to relate our dP_{34} hierarchy to the general case of the dP_{II} hierarchy. In addition, we illustrate how for a special case of our hierarchy, the linear problem can be used to obtain constants of summation.

2 A $2+1$ non-isospectral Volterra lattice hierarchy

Our $2+1$-dimensional non-isospectral Volterra lattice hierarchy in $u^{(n)} = u(n,t,y)$, as obtained in [10], has the form

$$u^{(n)}_{t_m} = Q^{(n)}_m = \left(\mathcal{R}^{(\mathrm{n})}\right)^m u^{(n)}_y + \sum_{j=0}^{m-1} \alpha_{m-j} \left(\mathcal{R}^{(\mathrm{n})}\right)^j K^{(n)}_1 + \sum_{j=0}^{m} \beta_{m-j} \left(\mathcal{R}^{(\mathrm{n})}\right)^j u^{(n)}, \quad (2.1)$$

where α_k and β_k are functions of (t_m, y), any m, $\mathcal{R}^{(\mathrm{n})}$ is the recursion operator of the Volterra lattice,

$$\mathcal{R}^{(\mathrm{n})} = u^{(n)}\left(1+E^{-1}\right)\left(u^{(n)} - u^{(n+1)}E^2\right)(E-1)^{-1}\left(u^{(n)}\right)^{-1}, \tag{2.2}$$

where E is the shift operator ($Ez^{(n)} = z^{(n+1)}$), and

$$K_1^{(n)} = u^{(n)}\left(u^{(n-1)} - u^{(n+1)}\right). \tag{2.3}$$

The corresponding hierarchy of underlying linear problems, with associated non-isospectral condition

$$\lambda_{t_m} = \lambda^m \lambda_y + \sum_{j=0}^{m} \lambda^{m+1-j}\beta_j, \tag{2.4}$$

is

$$E\phi^{(n)} = \begin{pmatrix} 1 & u^{(n)} \\ 1/\lambda & 0 \end{pmatrix}\phi^{(n)}, \tag{2.5}$$

$$\begin{aligned} \phi^{(n)}_{t_m} &= \Gamma_m \phi^{(n)}_y + H^{(n)}_m \phi^{(n)} \\ &= \lambda^m \phi^{(n)}_y + \left(\sum_{j=1}^{m} \lambda^{m-j} G^{(n)}_j\right)\phi^{(n)}, \end{aligned} \tag{2.6}$$

with each $G^{(n)}_j$ given by

$$G^{(n)}_j = \begin{pmatrix} A^{(n)}_j & B^{(n)}_j \\ C^{(n)}_j & D^{(n)}_j \end{pmatrix}, \tag{2.7}$$

where for $j > 1$,

$$A^{(n)}_j = (E-1)^{-1}\left(u^{(n)} - u^{(n+1)}E^2\right)\left[\alpha_j + (E-1)^{-1}\left(\frac{Q^{(n)}_{j-1}}{u^{(n)}}\right)\right], \tag{2.8}$$

$$B^{(n)}_j = \lambda u^{(n)}\left[\alpha_j + (E-1)^{-1}\left(\frac{Q^{(n+1)}_{j-1}}{u^{(n+1)}}\right)\right], \tag{2.9}$$

$$C^{(n)}_j = \left[\alpha_j + (E-1)^{-1}\left(\frac{Q^{(n)}_{j-1}}{u^{(n)}}\right)\right], \tag{2.10}$$

$$\begin{aligned} D^{(n)}_j &= (E-1)^{-1}\left(u^{(n-1)} - u^{(n)}E^2\right)\left[\alpha_j + (E-1)^{-1}\left(\frac{Q^{(n-1)}_{j-1}}{u^{(n-1)}}\right)\right] \\ &\quad -\lambda\left[\alpha_j + (E-1)^{-1}\left(\frac{Q^{(n)}_{j-1}}{u^{(n)}}\right)\right] - \beta_j, \end{aligned} \tag{2.11}$$

and

$$A_1^{(n)} = (E-1)^{-1}\left(u^{(n)} - u^{(n+1)}E^2\right)\left[\alpha_1 + (n-1)\beta_0 + (E-1)^{-1}\left(\frac{u_y^{(n)}}{u^{(n)}}\right)\right], \tag{2.12}$$

$$B_1^{(n)} = \lambda u^{(n)}\left[\alpha_1 + n\beta_0 + (E-1)^{-1}\left(\frac{u_y^{(n+1)}}{u^{(n+1)}}\right)\right], \tag{2.13}$$

$$C_1^{(n)} = \left[\alpha_1 + (n-1)\beta_0 + (E-1)^{-1}\left(\frac{u_y^{(n)}}{u^{(n)}}\right)\right], \tag{2.14}$$

$$\begin{aligned} D_1^{(n)} &= (E-1)^{-1}\left(u^{(n-1)} - u^{(n)}E^2\right)\left[\alpha_1 + (n-2)\beta_0 + (E-1)^{-1}\left(\frac{u_y^{(n-1)}}{u^{(n-1)}}\right)\right] \\ &\quad -\lambda\left[\alpha_1 + (n-1)\beta_0 + (E-1)^{-1}\left(\frac{u_y^{(n)}}{u^{(n)}}\right)\right] - \beta_1 - \lambda\beta_0. \end{aligned} \tag{2.15}$$

The first term of the right-hand-side of equation (2.1) corresponds to our non-isospectral extension of the Volterra lattice hierarchy to $2+1$ dimensions; the second term consists of a sum of standard (isospectral) Volterra lattice flows [25, 14, 24]. The third term consists of additional $1+1$-dimensional non-isospectral terms which in the general case are both non-autonomous (depend explicitly on n) and non-local. To the best of our knowledge the $2+1$-dimensional hierarchy (2.1) is new, although $1+1$-dimensional non-isospectral modifications of Volterra lattice flows, or indeed such terms alone, have been considered before, e.g. in [21, 23, 29].

For $m = 1$ we obtain the equation

$$u_{t_1}^{(n)} = Q_1^{(n)} = \mathcal{R}^{(\mathrm{n})} u_y^{(n)} + \alpha_1 K_1^{(n)} + \beta_0 \mathcal{R}^{(\mathrm{n})} u^{(n)} + \beta_1 u^{(n)}, \tag{2.16}$$

which, noting that

$$\mathcal{R}^{(\mathrm{n})} u^{(n)} = u^{(n)}\left((n-2)u^{(n-1)} - u^{(n)} - (n+1)u^{(n+1)}\right), \tag{2.17}$$

we can rewrite as

$$\begin{aligned} u_{t_1}^{(n)} &= u^{(n)}\left(u^{(n)}w^{(n)} + u^{(n-1)}w^{(n-1)} - u^{(n)}w^{(n+1)} - u^{(n+1)}w^{(n+2)}\right) \\ &\quad +\alpha_1 u^{(n)}\left(u^{(n-1)} - u^{(n+1)}\right) + \beta_0 u^{(n)}\left((n-2)u^{(n-1)} - u^{(n)}\right. \\ &\quad \left. -(n+1)u^{(n+1)}\right) + \beta_1 u^{(n)}, \end{aligned} \tag{2.18}$$

where we have also used a potential:

$$w^{(n+1)} - w^{(n)} = \frac{u_y^{(n)}}{u^{(n)}}. \tag{2.19}$$

This equation has the linear problem

$$E\phi^{(n)} = \begin{pmatrix} 1 & u^{(n)} \\ 1/\lambda & 0 \end{pmatrix} \phi^{(n)}, \tag{2.20}$$

$$\phi^{(n)}_{t_1} = \lambda\phi^{(n)}_y + \begin{pmatrix} v^{(n)} & \lambda u^{(n)}\left(\alpha_1 + n\beta_0 + w^{(n+1)}\right) \\ \alpha_1 + (n-1)\beta_0 + w^{(n)} & \begin{array}{c} v^{(n-1)} - \beta_1 - \lambda\beta_0 \\ -\lambda\left(\alpha_1 + (n-1)\beta_0 + w^{(n)}\right) \end{array} \end{pmatrix} \phi^{(n)} \tag{2.21}$$

where

$$v^{(n+1)} - v^{(n)} = \left(u^{(n)} - u^{(n+1)}E^2\right)\left(\alpha_1 + (n-1)\beta_0 + w^{(n)}\right) \tag{2.22}$$

and $\lambda = \lambda(t_1, y)$ satisfies the non-isospectral condition

$$\lambda_{t_1} = \lambda\lambda_y + \beta_1\lambda + \beta_0\lambda^2. \tag{2.23}$$

In [10] we considered reductions of our hierarchy (2.1) to lattice and differential-delay hierarchies, and to a hierarchy of discrete equations. Here we concentrate on this last case, and the resulting generalized discrete Painlevé hierarchies.

3 Generalized discrete Painlevé hierarchies

If in the $2+1$ lattice hierarchy (2.1) we now take the reduction $\partial_{t_m} = \partial_y = 0$, we obtain a hierarchy of discrete equations:

$$\sum_{j=0}^{m-1} \alpha_{m-j}\left(\mathcal{R}^{(\mathrm{n})}\right)^j K_1^{(n)} + \sum_{j=0}^{m} \beta_{m-j}\left(\mathcal{R}^{(\mathrm{n})}\right)^j u^{(n)} = 0. \tag{3.1}$$

This hierarchy arises as the compatibility condition

$$\left(\sum_{j=0}^{m} \lambda^{m+1-j}\beta_j\right) F_\lambda^{(n)} + F^{(n)}H_m^{(n)} - H_m^{(n+1)}F^{(n)} = 0 \tag{3.2}$$

of the associated hierarchy of linear problems

$$E\phi^{(n)} = F^{(n)}\phi^{(n)}, \tag{3.3}$$

$$\left(\sum_{j=0}^{m} \lambda^{m+1-j}\beta_j\right)\phi_\lambda^{(n)} = H_m^{(n)}\phi^{(n)}, \tag{3.4}$$

where

$$F^{(n)} = \begin{pmatrix} 1 & u^{(n)} \\ 1/\lambda & 0 \end{pmatrix}, \qquad H_m^{(n)} = \sum_{j=1}^{m} \lambda^{m-j} G_j^{(n)} \tag{3.5}$$

and the matrices $G_j^{(n)}$ are obtained from those of Section 2 in the appropriate way. Here all α_k and β_k are now constants.

This hierarchy contains, in the general case, non-local terms: although we could consider this hierarchy in its full generality, by introducing auxiliary potential functions, we prefer here to consider the case where $\beta_k = 0$, $k = 0, 1, \ldots, m-2$, i.e.

$$P_m^{(\mathrm{n})} \equiv \frac{1}{u^{(n)}} \left(\sum_{j=0}^{m-1} \alpha_{m-j} \left(\mathcal{R}^{(\mathrm{n})}\right)^j K_1^{(n)} + \beta_{m-1} \mathcal{R}^{(\mathrm{n})} u^{(n)} + \beta_m u^{(n)} \right) = 0. \tag{3.6}$$

We claim that (3.6), for $\beta_{m-1} = 0$, sums to the dP_I hierarchy given in [4], and for $\beta_{m-1} \neq 0$, sums to give an extended version of the discrete thirty-fourth Painlevé hierarchy of [4]. We now consider the cases $m = 1$, $m = 2$ and $m = 3$.

3.1 The case $\mathbf{m} = 1$

For $m = 1$, (3.6) gives the linear equation

$$P_1^{(\mathrm{n})} \equiv \alpha_1 \left(u^{(n-1)} - u^{(n+1)}\right) + \beta_0 \left((n-2)u^{(n-1)} - u^{(n)} - (n+1)u^{(n+1)}\right) + \beta_1 = 0, \tag{3.7}$$

which is readily solved. The solution of this equation is in fact useful when we come to consider cases having $m > 1$. First we observe that this equation can be written in the form

$$\begin{aligned} &(E^2 - 1)\left[-\alpha_1 u^{(n-1)} + \left(\frac{1}{2}\beta_1\right) n\right] \\ &-\beta_0 (E+1)\left[u^{(n-1)} + (E-1)\left((n-1)u^{(n-1)}\right)\right] = 0, \end{aligned} \tag{3.8}$$

and so it is straightforward to pass to the first order equation

$$\begin{aligned} \tilde{P}_1^{(n)} \equiv\ & (E-1)\left[-\alpha_1 u^{(n-1)} + \frac{1}{2}\beta_1 n\right] - \beta_0 \left[u^{(n-1)} + (E-1)\left((n-1)u^{(n-1)}\right)\right] \\ & -\omega_1 (-1)^n = 0, \end{aligned} \tag{3.9}$$

where ω_1 is an arbitrary constant. The case $\beta_0 = 0$, which was dealt with in [4], gives (after a shift on n) the solution defined by the equation

$$-\alpha_1 u^{(n)} + \frac{1}{2}\beta_1 n - \nu_1 - \mu_1 (-1)^n = 0, \tag{3.10}$$

where $\mu_1 = \frac{1}{2}\omega_1$ and ν_1 is a second arbitrary constant. The case $\beta_0 \neq 0$ has the solution defined by the equation

$$\begin{aligned} D_1^{(n)} \equiv\ & u^{(n-1)} (-\beta_0 n - \alpha_1 + \beta_0)(-\beta_0(n-1) - \alpha_1 + \beta_0) \\ & -\frac{\beta_1}{\beta_0}\left(-C_1 - \frac{1}{2}\beta_0(n-1) + \frac{1}{2}(-\alpha_1 + \beta_0)\right)\left(C_1 - \frac{1}{2}\beta_0 n \right. \\ & \left. + \frac{1}{2}(-\alpha_1 + \beta_0)\right) - \omega_1 \left(\frac{1}{2}(-\alpha_1 + \beta_0)(-1)^{n-1} + B_1^{(n)}\right) = 0, \end{aligned} \tag{3.11}$$

where C_1 is an arbitrary constant of summation (included as above in order to make clear the relationship with our later results for $m > 1$), and

$$B_1^{(n)} = \begin{cases} \frac{1}{2}\beta_0 n, & n \text{ even}, \\ \\ -\frac{1}{2}\beta_0(n-1), & n \text{ odd}. \end{cases} \tag{3.12}$$

Here we have made use of the summing factor $(-\beta_0 n - \alpha_1 + \beta_0)$:

$$\big(-\beta_0 n - \alpha_1 + \beta_0\big)\tilde{P}_1^{(n)} = (E-1)D_1^{(n)}. \tag{3.13}$$

3.2 The case $\mathbf{m} = 2$

For $m = 2$, (3.6) gives the first of our nonlinear equations,

$$\begin{aligned} P_2^{(\mathrm{n})} \equiv{}& \alpha_1 \left[u^{(n+1)}\left(u^{(n)} + u^{(n+1)} + u^{(n+2)}\right) - u^{(n-1)}\left(u^{(n)} + u^{(n-1)} + u^{(n-2)}\right)\right] \\ &+\alpha_2\left(u^{(n-1)} - u^{(n+1)}\right) + \beta_1\left((n-2)u^{(n-1)} - u^{(n)} - (n+1)u^{(n+1)}\right) + \beta_2 = 0. \end{aligned} \tag{3.14}$$

This last equation has the linear problem formed by (3.3) together with

$$(\beta_1\lambda^2 + \beta_2\lambda)\phi_\lambda^{(n)} = \begin{pmatrix} -\alpha_1\lambda u^{(n)} + v^{(n)} & \begin{array}{c}\lambda^2\alpha_1 u^{(n)} + \lambda u^{(n)}\big(\alpha_2 + n\beta_1 \\ -\alpha_1(u^{(n+1)} + u^{(n)})\big)\end{array} \\ \\ \begin{array}{c}\lambda\alpha_1 + \alpha_2 + (n-1)\beta_1 \\ -\alpha_1(u^{(n)} + u^{(n-1)})\end{array} & \begin{array}{c} v^{(n-1)} - \beta_2 - \lambda\beta_1 \\ -\lambda\big(\alpha_2 + (n-1)\beta_1 \\ -\alpha_1(u^{(n)} + u^{(n-1)})\big) \\ -\alpha_1\lambda u^{(n-1)} - \alpha_1\lambda^2 \end{array} \end{pmatrix} \phi^{(n)}, \tag{3.15}$$

where $v^{(n)}$ satisfies the equation

$$v^{(n+1)} - v^{(n)} = \left(u^{(n)} - u^{(n+1)}E^2\right)\left(\alpha_2 + (n-1)\beta_1 - \alpha_1\left(u^{(n)} + u^{(n-1)}\right)\right). \tag{3.16}$$

As we see shortly, equation (3.14) is a generalization of a well-known discrete first Painlevé (dP_I) equation. Thus our hierarchy of discrete equations (3.6) corresponds to a new generalized dP_I hierarchy. First we note that (3.14) can be written in the form

$$\begin{aligned} (E^2-1)&\left[\alpha_1 u^{(n-1)}\left(u^{(n)} + u^{(n-1)} + u^{(n-2)}\right) - \alpha_2 u^{(n-1)} + \left(\frac{1}{2}\beta_2\right)n\right] \\ &-\beta_1(E+1)\left[u^{(n-1)} + (E-1)\left((n-1)u^{(n-1)}\right)\right] = 0, \end{aligned} \tag{3.17}$$

and so we can sum to obtain

$$\begin{aligned} \tilde{P}_2^{(\mathrm{n})} \equiv (E-1)&\left[\alpha_1 u^{(n-1)}\left(u^{(n)} + u^{(n-1)} + u^{(n-2)}\right) - \alpha_2 u^{(n-1)} + \left(\frac{1}{2}\beta_2\right)n\right] \\ &-\beta_1\left[u^{(n-1)} + (E-1)\left((n-1)u^{(n-1)}\right)\right] - \omega_2(-1)^n = 0, \end{aligned} \tag{3.18}$$

where ω_2 (labelled using m) is an arbitrary constant. We now see that this third order discrete equation corresponds to the case $n = 1$ of equation (3.29) in [7], but with an additional parity-dependent term. It should therefore be expected that, in two different cases, it can be summed to give an analogue of P_I or of P_{34}. This is precisely what happens: for each of the cases $\beta_1 = 0$ and $\beta_1 \neq 0$, we obtain a second order equation, maintaining moreover the parity-dependence of (3.18).

3.2.1 Continuum limit for $\omega_2 = 0$

We now consider the continuum limit of equation (3.18) in the special case $\omega_2 = 0$,

$$\tilde{P}_2^{(\mathrm{n})} \equiv (E-1)\left[\alpha_1 u^{(n-1)}\left(u^{(n)} + u^{(n-1)} + u^{(n-2)}\right) - \alpha_2 u^{(n-1)} + \left(\frac{1}{2}\beta_2\right) n\right]$$
$$-\beta_1\left[u^{(n-1)} + (E-1)\left((n-1)u^{(n-1)}\right)\right] = 0. \qquad (3.19)$$

Setting

$$u^{(n)} = 1 + h^2 y(x), \qquad x = nh, \qquad (3.20)$$

we obtain, with the identifications

$$\alpha_2 = \alpha_1(6 - a_1 h^2), \qquad \beta_1 = -2\alpha_1 g_0 h^3, \qquad \beta_2 = 2\alpha_1(g_1 h^5 - 4g_0 h^3), \qquad (3.21)$$

the continuum limit

$$y''' + 6yy' + a_1 y' + 2g_0(xy' + 2y) + g_1 = 0. \qquad (3.22)$$

This last corresponds to (3.29) in [7] in the case $n = 1$ (although in (3.29) of [7] we have assumed that $a_1 = 0$, since this can always be done using a shift on y).

This continuum limit is of interest since we expect that our hierarchy corresponds to a discrete version of (3.29) in [7], but with an additional parity-dependent term.

3.2.2 Summation to dP_I

In the case $\beta_1 = 0$, we obtain from (3.18) (after a shift on n) the second order discrete equation

$$\alpha_1 u^{(n)}\left(u^{(n+1)} + u^{(n)} + u^{(n-1)}\right) - \alpha_2 u^{(n)} + \left(\frac{1}{2}\beta_2\right) n - \nu_2 - \mu_2(-1)^n = 0, \qquad (3.23)$$

where $\mu_2 = \frac{1}{2}\omega_2$ and ν_2 is a second arbitrary constant (again labelled using m). Equation (3.23) is the version of dP_I, containing a parity-dependent term, which includes both the first and second Painlevé equations amongst its continuum limits; see [11], as well as [21] and [4]. In our framework this equation corresponds to the special case $\beta_1 = 0$ of our more general integrable discrete equation (3.14) [or (3.18)].

We note that it is straightforward, using our above results, to give a linear problem for (3.23) not involving the potential $v^{(n)}$, whose compatibility condition is precisely equation (3.23) rather than the higher order equivalents (3.14) or (3.18) with $\beta_1 = 0$. This linear problem consists of (3.3) together with

$$\beta_2\lambda\phi_\lambda^{(n)} = \begin{pmatrix} \begin{array}{c} -\alpha_1\lambda u^{(n)} - \frac{1}{2}\beta_2 n \\ +\nu_2 + \mu_2(-1)^n \end{array} & \begin{array}{c} \lambda^2\alpha_1 u^{(n)} + \lambda\big(\alpha_1 u^{(n)} u^{(n-1)} \\ +\frac{1}{2}\beta_2 n - \nu_2 - \mu_2(-1)^n\big) \end{array} \\ & \\ \begin{array}{c} \lambda\alpha_1 + \alpha_2 \\ -\alpha_1(u^{(n)} + u^{(n-1)}) \end{array} & \begin{array}{c} -\frac{1}{2}\beta_2(n+1) + \nu_2 + \mu_2(-1)^{n-1} \\ -\lambda\big(\alpha_2 - \alpha_1(u^{(n)} + u^{(n-1)})\big) \\ -\alpha_1\lambda u^{(n-1)} - \alpha_1\lambda^2 \end{array} \end{pmatrix} \phi^{(n)}. \qquad (3.24)$$

Of course, linear problems for equation (3.23) are well known.

Here we have made use of the fact that, when $\beta_1 = 0$, we can solve (3.16) to obtain

$$v^{(n)} = \alpha_1 u^{(n)} \left(u^{(n+1)} + u^{(n)} + u^{(n-1)} \right) - \alpha_2 u^{(n)}, \tag{3.25}$$

which by virtue of our equation (3.23) then gives

$$v^{(n)} = -\frac{1}{2}\beta_2 n + \nu_2 + \mu_2 (-1)^n. \tag{3.26}$$

We do not here consider the continuum limit of (3.23), since this has been considered in [4]. In the case $\omega_2 = 0$, one limit is P_I, whose derivative corresponds to a special case of (3.22).

3.2.3 Summation to dP_{34}

In the case $\beta_1 \neq 0$, we obtain the second order discrete equation

$$\begin{aligned} D_2^{(n)} \equiv \; & u^{(n-1)}\big(\alpha_1 u^{(n)} + \alpha_1 u^{(n-1)} - \beta_1 n + 2\gamma\big)\big(\alpha_1 u^{(n-1)} + \alpha_1 u^{(n-2)} - \beta_1 (n-1) \\ & +2\gamma\big) - \tilde{\beta}_2\big(\alpha_1 u^{(n-1)} - C_2 - \frac{1}{2}\beta_1 (n-1) + \gamma\big)\big(\alpha_1 u^{(n-1)} + C_2 - \frac{1}{2}\beta_1 n \\ & +\gamma\big) - \omega_2\big(\alpha_1 u^{(n-1)}(-1)^{n-1} + \gamma(-1)^{n-1} + B_2^{(n)}\big) = 0, \end{aligned} \tag{3.27}$$

where C_2 is an arbitrary constant of summation,

$$\gamma = \frac{1}{2}\left(-\alpha_2 + \beta_1 + \alpha_1 \frac{\beta_2}{\beta_1} \right), \qquad \tilde{\beta}_2 = \frac{\beta_2}{\beta_1}, \tag{3.28}$$

and

$$B_2^{(n)} = \begin{cases} \frac{1}{2}\beta_1 n, & n \text{ even}, \\ -\frac{1}{2}\beta_1 (n-1), & n \text{ odd}. \end{cases} \tag{3.29}$$

Here we have made use of the summing factor $(\alpha_1 u^{(n)} + \alpha_1 u^{(n-1)} - \beta_1 n + 2\gamma)$:

$$\big(\alpha_1 u^{(n)} + \alpha_1 u^{(n-1)} - \beta_1 n + 2\gamma\big)\tilde{P}_2^{(\mathrm{n})} = (E-1)\big(D_2^{(n)}\big). \tag{3.30}$$

Equation (3.27), which we will call the dP_{34} equation, is a generalized version of the known discrete thirty-fourth Painlevé equation. We note that when summing we have retained the parity-dependent terms which appear in (3.18). In the special case $\omega_2 = 0$ this equation yields the discrete thirty-fourth Painlevé equation of [4].

We see from the above discussion that we have obtained a new generalized dP_I hierarchy (3.6): the first member of this hierarchy is a generalized dP_I equation since, for the choice $\beta_1 = 0$, it sums to a known dP_I equation. In addition, we have seen that this first member, for the choice $\beta_1 \neq 0$, sums to a new dP_{34} equation.

3.2.4 Obtaining the Bäcklund transformation to dP_{II}

Our dP_{34} equation is related to the discrete second Painlevé equation (dP_{II}) in its more general form [21, 4, 12], i.e.

$$\big(1-(q^{(n)})^2\big)\big(q^{(n+1)}+q^{(n-1)}\big)-(A_1+A_2n)q^{(n)}-A_3-A_4(-1)^n=0, \tag{3.31}$$

via the Bäcklund transformation (BT)

$$u^{(n)} = \frac{1}{4}\tilde{\beta}_2\big(1-q^{(n)}\big)\big(1+q^{(n+1)}\big), \tag{3.32}$$

$$q^{(n)} = \frac{4(u^{(n-1)}-u^{(n)})+\tilde{\beta}_2(A_3+A_4(-1)^n)}{4(u^{(n-1)}+u^{(n)})-\tilde{\beta}_2(A_1+A_2n)}, \tag{3.33}$$

where

$$\beta_1 = \frac{1}{4}\alpha_1\tilde{\beta}_2A_2, \tag{3.34}$$

$$\omega_2 = -\frac{1}{4}\alpha_1\tilde{\beta}_2^2A_4, \tag{3.35}$$

$$\gamma = -\frac{1}{8}\alpha_1\tilde{\beta}_2A_1, \tag{3.36}$$

and where $C_2=(\alpha_1\tilde{\beta}_2/8)K_2$ and A_3 are related by the equation

$$K_2^2-A_2K_2+A_4^2+A_2A_4-A_3^2+A_2A_3=0. \tag{3.37}$$

This BT between our dP_{34} equation (3.27) and the dP_{II} equation (3.31) is of course new. Given that (3.18) is the discrete analogue, but now including parity-dependent terms, of equation (3.29) of [7], the existence of this BT should be expected. We note that in the special case $\omega_2=A_4=0$, this BT is as given in [4].

We do not here consider the continuum limit of (3.27), since this equation is related by the above BT to (3.31), and the continuum limit of this last is discussed in [4].

We now comment briefly on how this BT may be found. We seek a relation between $u^{(n)}$ and $q^{(n)}$ of the form (3.32),

$$u^{(n)}=\theta\big(1-q^{(n)}\big)\big(1+q^{(n+1)}\big), \tag{3.38}$$

where θ is a constant to be determined. (Thus of course we choose not to ignore what we already know about our dP_{34} equation in the special case $\omega_2=0$. In fact the approach which we outline here is the analogue of that which may be used to obtain the BT between the continuous P_{II} and P_{34} hierarchies: shortly we will see the result of its application to a new fourth order dP_{34} equation.)

We solve (3.38) for $q^{(n+1)}$,

$$q^{(n+1)}=-1+\frac{u^{(n)}}{\theta\big(1-q^{(n)}\big)}, \tag{3.39}$$

and a shifted version ($n\to n-1$) for $q^{(n-1)}$,

$$q^{(n-1)}=1-\frac{u^{(n-1)}}{\theta\big(1+q^{(n)}\big)}. \tag{3.40}$$

Substitution into (3.31) then leads to an algebraic equation in $q^{(n)}$ which turns out to be *linear*, and so allows us to easily obtain the inverse BT,

$$q^{(n)} = \frac{(u^{(n-1)} - u^{(n)}) + \theta(A_3 + A_4(-1)^n)}{(u^{(n-1)} + u^{(n)}) - \theta(A_1 + A_2 n)}. \tag{3.41}$$

Elimination between (3.38) and (3.41) then yields equation (3.31) (by construction) and, with $\theta = \tilde{\beta}_2/4$ and the relations (3.34)—(3.37) satisfied, equation (3.27). We note that obtaining a linear equation for $q^{(n)}$ after substitution of $q^{(n+1)}$ and $q^{(n-1)}$ above into (3.31) is analogous to obtaining a linear algebraic equation for the P_{II} hierarchy dependent variable V after setting $V' - V^2 = Y$ in that hierarchy (see [3]).

It is worth making here some comments in order to clarify terminology. The transformation (3.32) is of course the Miura transformation which relates the Volterra lattice to the modified Volterra lattice, and has long been referred to as such within the realm of lattice equations; here we refer to [26]—[16]. It is thus distinguished from the notion of auto-BT for lattice equations. We, however, have followed standard practice within the context of Painlevé equations, where, under reduction from higher dimensional equations, Miura maps often become invertible and the resulting pair of equations is referred to as a BT: it is thus that we call (3.32), (3.33) a BT.

3.2.5 Constants of summation from the linear problem

Here we consider the linear problem in the case $\beta_1 = 0$ and $\beta_2 = 0$. Substituting (3.25) into (3.15), and setting also $\beta_2 = 0$, we obtain the matrix

$$\hat{H}_2^{(n)} = \begin{pmatrix} \begin{matrix} -\alpha_1\lambda u^{(n)} - \alpha_2 u^{(n)} \\ +\alpha_1 u^{(n)}\big(u^{(n+1)} + u^{(n)} + u^{(n-1)}\big) \end{matrix} & \begin{matrix} \lambda^2\alpha_1 u^{(n)} + \lambda u^{(n)}\big(\alpha_2 \\ -\alpha_1(u^{(n+1)} + u^{(n)})\big) \end{matrix} \\ \begin{matrix} \lambda\alpha_1 + \alpha_2 \\ -\alpha_1(u^{(n)} + u^{(n-1)}) \end{matrix} & \begin{matrix} \alpha_1 u^{(n-1)}\big(u^{(n)} + u^{(n-1)} \\ +u^{(n-2)}\big) - \alpha_2 u^{(n-1)} - \lambda\big(\alpha_2 \\ -\alpha_1(u^{(n)} + u^{(n-1)})\big) \\ -\alpha_1\lambda u^{(n-1)} - \alpha_1\lambda^2 \end{matrix} \end{pmatrix}.$$

As in the continous case, the coefficients of λ in the determinant of this matrix then give constants of integration (now summation) for the corresponding equation,

$$\begin{aligned} \hat{P}_2^{(\mathrm{n})} \equiv\ & \alpha_1 \left[u^{(n+1)}\left(u^{(n)} + u^{(n+1)} + u^{(n+2)}\right) - u^{(n-1)}\left(u^{(n)} + u^{(n-1)} + u^{(n-2)}\right)\right] \\ & +\alpha_2\left(u^{(n-1)} - u^{(n+1)}\right) = 0. \end{aligned} \tag{3.42}$$

For this particular example, we find

$$\det\left[\hat{H}_2^{(n)}\right] = \Omega_0 + \Omega_1\lambda \tag{3.43}$$

where

$$
\begin{aligned}
\Omega_0 &= u^{(n)}u^{(n-1)}\Big(\alpha_1\big(u^{(n+1)}+u^{(n)}+u^{(n-1)}\big)-\alpha_2\Big)\Big(\alpha_1\big(u^{(n)}+u^{(n-1)} \\
&\quad +u^{(n-2)}\big)-\alpha_2\Big), \qquad (3.44)\\
\Omega_1 &= -\alpha_1 u^{(n)}u^{(n-1)}\Big(\alpha_1\big(u^{(n+1)}+u^{(n)}+u^{(n-1)}+u^{(n-2)}\big)-\alpha_2\Big). \qquad (3.45)
\end{aligned}
$$

Thus here we have that $(E-1)\Omega_0 = 0$ and $(E-1)\Omega_1 = 0$ when (3.42) holds.

The above illustrates a general method, analogous to the continuous case, of obtaining constants of summation when $\beta_m = \beta_{m-1} = 0$ in the hierarchy (3.6).

3.3 The case $\mathbf{m} = 3$

We now consider the case $m = 3$ of (3.6), which may be written

$$
\begin{aligned}
(E^2-1)\Big\{ &-\alpha_1 u^{(n-1)}\Big(u^{(n)}u^{(n+1)}+{u^{(n)}}^2+2u^{(n-1)}u^{(n)}+u^{(n-2)}u^{(n-3)} \\
&+{u^{(n-2)}}^2+2u^{(n-1)}u^{(n-2)}+{u^{(n-1)}}^2+u^{(n-2)}u^{(n)}\Big) \\
&+\alpha_2 u^{(n-1)}\big(u^{(n)}+u^{(n-1)}+u^{(n-2)}\big)-\alpha_3 u^{(n-1)}+\left(\frac{1}{2}\beta_3\right)n\Big\} \\
&-\beta_2(E+1)\Big[u^{(n-1)}+(E-1)\Big((n-1)u^{(n-1)}\Big)\Big]=0. \qquad (3.46)
\end{aligned}
$$

The linear problem for this equation may also be given explicitly, using the results given earlier, but we choose not to do so here. We can sum this last to obtain

$$
\begin{aligned}
\tilde{P}_3^{(n)} \equiv (E-1)\Big\{ &-\alpha_1 u^{(n-1)}\Big(u^{(n)}u^{(n+1)}+{u^{(n)}}^2+2u^{(n-1)}u^{(n)}+u^{(n-2)}u^{(n-3)} \\
&+{u^{(n-2)}}^2+2u^{(n-1)}u^{(n-2)}+{u^{(n-1)}}^2+u^{(n-2)}u^{(n)}\Big) \\
&+\alpha_2 u^{(n-1)}\big(u^{(n)}+u^{(n-1)}+u^{(n-2)}\big)-\alpha_3 u^{(n-1)}+\left(\frac{1}{2}\beta_3\right)n\Big\} \\
&-\beta_2\Big[u^{(n-1)}+(E-1)\Big((n-1)u^{(n-1)}\Big)\Big]-\omega_3(-1)^n=0, \qquad (3.47)
\end{aligned}
$$

where ω_3 is an arbitrary constant. We will see that this fifth order discrete equation can be summed once again in order to give, for each of the cases $\beta_2 = 0$ and $\beta_2 \neq 0$, a fourth order equation. These are, of course, respectively the fourth-order dP_I of [4] and — analogously to the case $m = 2$ — an extended version of the fourth-order discrete thirty-fourth Painlevé equation of [4].

3.3.1 Summation to fourth-order dP_I

In the case $\beta_2 = 0$, we obtain (after a shift on n) the fourth order discrete equation

$$
\begin{aligned}
&-\alpha_1 u^{(n)}\left(u^{(n+1)}u^{(n+2)} + {u^{(n+1)}}^2 + 2u^{(n)}u^{(n+1)} + u^{(n-1)}u^{(n-2)} + {u^{(n-1)}}^2\right.\\
&\left.+2u^{(n)}u^{(n-1)} + {u^{(n)}}^2 + u^{(n-1)}u^{(n+1)}\right) + \alpha_2 u^{(n)}\left(u^{(n+1)} + u^{(n)} + u^{(n-1)}\right)\\
&\qquad\qquad -\alpha_3 u^{(n)} + \left(\frac{1}{2}\beta_3\right) n - \nu_3 - \mu_3(-1)^n = 0, \qquad (3.48)
\end{aligned}
$$

where $\mu_3 = \frac{1}{2}\omega_3$ and ν_3 is a second arbitrary constant. This last is, as expected, the fourth-order dP_I equation, containing a parity-dependent term, given in [4].

We note that the continuum limit of (3.48) has been considered in [4].

3.3.2 Summation to fourth-order dP_{34}

In the case $\beta_2 \neq 0$, we obtain the fourth order discrete equation

$$
\begin{aligned}
D_3^{(n)} \equiv\ & u^{(n-1)}\Big(\alpha_1\big[u^{(n+1)}u^{(n)} + (u^{(n)})^2 + 2u^{(n)}u^{(n-1)} + (u^{(n-1)})^2\\
&+u^{(n-1)}u^{(n-2)}\big] - \epsilon\big[u^{(n)} + u^{(n-1)}\big] + \beta_2 n - 2\delta\Big)\Big(\alpha_1\big[u^{(n)}u^{(n-1)}\\
&+(u^{(n-1)})^2 + 2u^{(n-1)}u^{(n-2)} + (u^{(n-2)})^2 + u^{(n-2)}u^{(n-3)}\big] - \epsilon\big[u^{(n-1)}\\
&+u^{(n-2)}\big] + \beta_2(n-1) - 2\delta\Big) - \tilde{\beta}_3\Big[\alpha_1\big(u^{(n)}u^{(n-1)} + (u^{(n-1)})^2\\
&+u^{(n-1)}u^{(n-2)}\big) - \epsilon u^{(n-1)} + \frac{1}{2}\beta_2 n - \delta\Big]\Big[\alpha_1\big(u^{(n)}u^{(n-1)} + (u^{(n-1)})^2\\
&+u^{(n-1)}u^{(n-2)}\big) - \epsilon u^{(n-1)} + \frac{1}{2}\beta_2(n-1) - \delta\Big] + \frac{1}{64}\tilde{\beta}_3^3\beta_2\alpha_1 C_3\\
&+\frac{1}{1024}\tilde{\beta}_3^5\alpha_1^2 C_3^2 - \omega_3\Big(-\alpha_1\big[(u^{(n-1)})^2 + u^{(n)}u^{(n-1)}\\
&+u^{(n-1)}u^{(n-2)}\big](-1)^{n-1} + \epsilon u^{(n-1)}(-1)^{n-1} + \delta(-1)^{n-1} + B_3^{(n)}\Big) = 0
\end{aligned}
\qquad (3.49)
$$

where C_3 is an arbitrary constant of summation (included as above in order to ease comparison with the results of [4]),

$$
\delta = \frac{1}{2}\left(-\alpha_3 + \beta_2 + \alpha_2\left(\frac{\beta_3}{\beta_2}\right) - \alpha_1\left(\frac{\beta_3}{\beta_2}\right)^2\right), \quad \epsilon = \alpha_2 - \alpha_1\frac{\beta_3}{\beta_2}, \quad \tilde{\beta}_3 = \frac{\beta_3}{\beta_2}, \qquad (3.50)
$$

and

$$
B_3^{(n)} = \begin{cases} \frac{1}{2}\beta_2 n, & n \text{ even},\\[2ex] -\frac{1}{2}\beta_2(n-1), & n \text{ odd}. \end{cases} \qquad (3.51)
$$

Once again we have made use of a summing factor S:

$$S\tilde{P}_3^{(n)} = (E-1)D_3^{(n)}, \tag{3.52}$$

where

$$\begin{aligned} S &= -\alpha_1\big[u^{(n+1)}u^{(n)} + (u^{(n)})^2 + 2u^{(n)}u^{(n-1)} + (u^{(n-1)})^2 + u^{(n-1)}u^{(n-2)}\big] \\ &\quad +\epsilon\big(u^{(n)} + u^{(n-1)}\big) - \beta_2 n + 2\delta. \end{aligned} \tag{3.53}$$

Equation (3.49) is new. It is this equation that we present as a fourth order dP_{34} equation. In the case $\omega_3 = 0$ this equation reduces to that given in [4].

3.3.3 Obtaining the Bäcklund transformation to fourth-order dP_{II}

We are also able to give a BT which relates our new fourth order dP_{34} equation to the fourth order dP_{II} equation given in [4], i.e.

$$\begin{aligned} &\big(1-(q^{(n)})^2\big)\Bigg(\big[1-(q^{(n+1)})^2\big]q^{(n+2)} + \big[1-(q^{(n-1)})^2\big]q^{(n-2)} \\ &-\big[q^{(n+1)} + q^{(n-1)}\big]^2 q^{(n)}\Bigg) - A_5\big(1-(q^{(n)})^2\big)\big(q^{(n+1)} + q^{(n-1)}\big) \\ &-(A_1 + A_2 n)q^{(n)} - A_3 - A_4(-1)^n = 0. \end{aligned} \tag{3.54}$$

The BT which relates (3.49) and (3.54) is

$$u^{(n)} = \frac{1}{4}\tilde{\beta}_3\big(1-q^{(n)}\big)\big(1+q^{(n+1)}\big), \tag{3.55}$$

$$q^{(n)} = \frac{\pi}{\Pi}, \tag{3.56}$$

where

$$\begin{aligned} \pi &= 16\big[u^{(n)}u^{(n+1)} + (u^{(n)})^2 - (u^{(n-1)})^2 - u^{(n-1)}u^{(n-2)}\big] \\ &\quad +4\tilde{\beta}_3(A_5+2)\big[u^{(n-1)} - u^{(n)}\big] - \tilde{\beta}_3^2(A_3 + A_4(-1)^n), \end{aligned} \tag{3.57}$$

$$\begin{aligned} \Pi &= -16\big[u^{(n)}u^{(n+1)} + (u^{(n)})^2 + 2u^{(n)}u^{(n-1)} + (u^{(n-1)})^2 + u^{(n-1)}u^{(n-2)}\big] \\ &\quad +4\tilde{\beta}_3(A_5+2)\big[u^{(n)} + u^{(n-1)}\big] + \tilde{\beta}_3^2(A_1 + A_2 n), \end{aligned} \tag{3.58}$$

and

$$\beta_2 = -\frac{1}{16}\alpha_1\tilde{\beta}_3^2 A_2, \tag{3.59}$$

$$\omega_3 = \frac{1}{16}\alpha_1\tilde{\beta}_3^3 A_4, \tag{3.60}$$

$$\delta = \frac{1}{32}\alpha_1\tilde{\beta}_3^2 A_1, \tag{3.61}$$

$$\epsilon = \frac{1}{4}\alpha_1\tilde{\beta}_3(A_5+2), \tag{3.62}$$

and where C_3 and A_3 are related by the equation

$$C_3^2 - A_2C_3 + A_4^2 + A_2A_4 - A_3^2 + A_2A_3 = 0. \tag{3.63}$$

In the case $\omega_3 = A_4 = 0$, this BT, which is new, reduces to that given in [4].

We do not here consider the continuum limit of (3.49), since this equation is related via the above BT to (3.54), and the continuum limit of this last has been considered in [4].

This BT can be obtained as indicated earlier in the case of our second order dP_{34} equation: starting once again with (3.38), we use shifted versions of (3.39) and (3.40),

$$q^{(n+2)} = -1 + \frac{u^{(n+1)}}{\theta\big(1 - q^{(n+1)}\big)}, \tag{3.64}$$

$$q^{(n-2)} = 1 - \frac{u^{(n-2)}}{\theta\big(1 + q^{(n-1)}\big)}, \tag{3.65}$$

to replace $q^{(n+2)}$ and $q^{(n-2)}$ in (3.54), and then (3.39) and (3.40) themselves to replace $q^{(n+1)}$ and $q^{(n-1)}$. The result is an algebraic equation in $q^{(n)}$ which, as in our previous case, turns out to be linear, and so easily solvable in order to obtain the inverse BT. Elimination between (3.38) and this inverse BT then yields (3.54), which follows by construction, and, with $\theta = \tilde{\beta}_3/4$ and the relations (3.59)—(3.63) satisfied, our fourth order dP_{34} equation (3.49). Thus we see that our new fourth order dP_{34} equation is related to the fourth order dP_{II} equation (3.54) in the same way as obtained earlier for the corresponding second order equations (3.27) and (3.31).

This approach can also be used for higher members of the dP_{II} hierarchy, in order to obtain a BT to (and find) the corresponding member of the dP_{34} hierarchy. We expect the expression for $u^{(n)}$ in terms of $q^{(n)}$ in this BT always to be of the form (3.38).

4 Conclusions

We have given a new non-isospectral generalization of the Volterra hierarchy in $2+1$ dimensions, and have considered a reduction to a generalized dP_I hierarchy. This last, whose corresponding linear problem follows easily from that given for our $2+1$-dimensional hierarchy, corresponds to a discrete version of the ODE hierarchy (3.29) in [7], but with an additional parity-dependent term. It contains as special cases both the known dP_I hierarchy and an extended version of the known discrete thirty-fourth Painlevé hierarchy. We have also seen how BTs can be constructed between members of our dP_{34} hierarchy and those of the known dP_{II} hierarchy.

There is of course still much scope for future work on our Volterra hierarchy in $2+1$ dimensions and its reductions to lattice and differential-delay hierarchies [10]. One particularly interesting question is the form of the multi-soliton solutions of the members of our $2+1$-dimensional hierarchy, and in particular for equation (2.16). The soliton solutions of this $2+1$-dimensional non-local lattice equation are presumably discrete analogues of the "breaking soliton" solutions of Calogero's equation [2], as given in [1]. A complete analysis of such solutions could be undertaken, for example, by deriving an auto-BT for equation (2.16), either related to the underlying linear problem of (2.16), or for its bilinear form. In the case of the Volterra equation itself, these approaches can be found in [15] and [13], respectively. The extension of such results to our $2+1$-dimensional case will be the subject of future papers.

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