Improved Additive Operator Splitting Algorithms for Basket Option Pricing Model

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***Abstract*-In order to solve Black-Scholes equation of basket option pricing model by numerical method. This paper used Additive Operator Splitting (AOS) algorithm to split the multi-dimensional Black-Scholes equation into equivalent one-dimensional equation set, and constructed 'Explicit-Implicit' and 'Implicit-Explicit' schemes to solve it. Then compatibility, stability and convergence of those schemes were analyzed. Finally, this paper compared computation time and precision of the schemes through numerical experiments. 'Explicit-Implicit' and 'Implicit-Explicit' schemes of AOS algorithms have both higher accuracy and faster computing speed and them have practical significance in solving basket option pricing model.**

Keywords-basket option pricing model; Additive Operator Splitting (AOS); difference schem; stability; numerical experiment.

I. INTRODUCTION

Option is significant financial derivatives. Multi-assets option is the most widely used option in the real financial market. Basket option is a kind of multi-asset option, and its price depends on the average price of two or more assets. Basket options are widely used in hedging. According to the portfolio theory, volatility of basket risk asset is relatively small. Option premium of basket option is less than the sum of every option. Therefore, it has practical significance to research the pricing of basket option [1].

Assume that $S_i(i=1,2,\cdots,n)$ is the exchange rate of n currency, and they obey geometry Brown motion. Through δ Hedge principle, the Black-Scholes equation of basket option price $f(S_1, S_2,\cdots S_n,t)$ is as follows [2]:

$$\frac{\partial f}{\partial t}+\frac{1}{2}\sum_{i,j=1}^{n}a_{ij}S_iS_j\frac{\partial^2 f}{\partial S_i\partial S_j}+\sum_{i=1}^{n}(r-q_i)S_i\frac{\partial f}{\partial S_i}-rf=0. \quad (1)$$

Here, r is risk-free interest rate of our own country, and $q_i(i=1,2,\cdots,n)$ is risk-free interest rate of the country using currency i. If the profit of basket option on the expiry date is geometrical average of n kind of underlying assets, namely:

$$f=\left[\prod_{i=1}^{n}S_i^{\alpha_i}-X\right]^+.$$

Where,

$$\alpha_i\geq 0, \sum_{i=1}^{n}\alpha_i=1.$$

And analytical solution of (1) is that:

$$f(S_1,S_2,\cdots,S_n,t)=e^{-\hat{q}(T-t)}S_1^{\alpha_1}\cdots S_n^{\alpha_n}N(\hat{d}_1)-e^{-r(T-t)}XN(\hat{d}_2),$$

$$\hat{d}_1=\frac{\ln\dfrac{S_1^{\alpha_1},S_2^{\alpha_2},\cdots,S_n^{\alpha_n}}{X}+\left[r+\hat{q}+\dfrac{\hat{\sigma}^2}{2}\right](T-t)}{\hat{\sigma}\sqrt{T-t}},$$

$$\hat{d}_2=\hat{d}_1-\hat{\sigma}\sqrt{T-t}.$$

This paper discussed basket option pricing model, and committed to find new difference algorithm for multi-asset option pricing problem. Weickert Joachim (1998) firstly used the Additive Operator Splitting (AOS) method to solve the multi-dimensional partial differential equations in image processing [5]. Yi Zhang and Xiaozhong Yang (2010) proposed the accelerated AOS schemes for non-linear diffusion filtering [6], this method reduce the computation time and storage space. Xiaozhong Yang and Gaoxin Zhou (2011) have applied AOS algorithm on solving dual currency option pricing model [8], which has achieved good effect in the practical application. In this paper, AOS method is applied into basket option pricing model, and it will get the same good result as before.

II. DETERMINE INITIAL-BOUNDARY VALUE FOR BLACK-SCHOLES EQUATION OF BASKET OPTION PRICING MODEL

In theory, the solving area of this equation is:

$$\{(S_i,t)|0\leq S_i\leq+\infty, i=1,2,\cdots,n, t=[0,T]\}.$$

But in actual transaction, the price of the underlying asset will not always appear to be zero or infinity. Therefore, the financial institution provides a small enough value $S_{i\min}(S_{i\min}>0)$ as the lower bound and a large enough value $S_{i\max}(S_{i\max}<+\infty)$ as the upper bound. Then the pricing problem can be solved in a bounded area:

$$\Omega=\{(S_i,t)|S_{i\min}\leq S_i\leq S_{i\max}, i=1,2,\cdots,n, t=[0,\mathrm{T}]\}.$$

To construct the difference scheme for Black-Scholes equation of basket option pricing, this paper must gain the boundary condition of (1). Take foreign call option for example. For the reason that option pricing is a backward problem, the initial condition is the value at the time: $t=T$. Suppose that the profit of basket option on the expiry date is geometrical average of n kind of the underlying assets. Then the boundary condition is that:

$$f(S_1,S_2,\cdots,S_{i\min},\cdots S_n,t)=0,$$

$$f(S_1,S_2,\cdots,S_{i\max},\cdots,S_n,t)=0, i=1,2,\cdots,n.$$

To solve (1), we must replace its variables:

$$x_i=\ln S_i, i=1,2,\cdots,n,$$

$$\tau = T - t.$$

Then (1) is transformed into initial boundary value problem of constant coefficient parabolic equation:

$$\frac{\partial f}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial \eta^2} - \left[r - \hat{q} - \frac{1}{2}\sigma^2 \right]\frac{\partial f}{\partial \eta} + rf = 0. \qquad (2)$$

$$\eta = \sum_{i=1}^{n} \alpha_i \beta_i .$$

III. CONSTRUCT AOS ALGORITHM FOR BASKET OPTION PRICING MODEL

Additive Operator Splitting (AOS) algorithm is an effective method to solve multi-dimensional partial differential equations. This method firstly split the multi-dimensional Black-Scholes equation of basket option pricing model into equivalent one dimensional equation set. Then compute value of one dimensional equation set by 'Explicit-Implicit' and 'Implicit-Explicit' scheme. Finally, take the arithmetic mean value of one dimensional equation set as the final value.

Make use of AOS algorithm to split (2) into equivalent equation set on the direction of $x_1, x_2, \cdots, x_n$.

$$\frac{\partial f}{\partial \tau} - n\sigma^2 \frac{\partial^2 f}{\partial(\alpha_i^2 x_i^2)} - n\sigma^2 \frac{\partial^2 f}{\partial \sum\limits_{i,j=1, i\neq j}^{n} \alpha_i \alpha_j x_i x_j} - n\left[r - \hat{q} - \frac{1}{2}\sigma^2 \right]\frac{\partial f}{\partial(\alpha_i x_i)} + rf = 0 \qquad (3)$$

$i = 1, 2, \cdots, n.$

Then, 'Explicit-Implicit' and 'Implicit-Explicit' scheme is constructed to solve the one dimensional equation set.

Firstly, we construct 'Explicit-Implicit' scheme. The method is to adopt the explicit scheme at the odd number floor, and implicit scheme at the even number floor. Then, the i equation of (3) became into:

$$\begin{cases} \dfrac{f_i^{2n+1} - f_i^{2n}}{k} = n\sigma^2 \dfrac{f_{i+1}^{2n} - 2f_i^{2n} + f_{i-1}^{2n}}{2\alpha_i^2 h_i^2} + n\sigma^2 F_{ij}^{2n} + na\dfrac{f_{i+1}^{2n} - f_{i-1}^{2n}}{2\alpha_i h_i} - rf_i^{2n} \\ \dfrac{f_i^{2n+2} - f_i^{2n+1}}{k} = n\sigma^2 \dfrac{f_{i+1}^{2n+2} - 2f_i^{2n+2} + f_{i-1}^{2n+2}}{2\alpha_i^2 h_i^2} + n\sigma^2 F_{ij}^{2n+2} + na\dfrac{f_{i+1}^{2n+2} - f_{i-1}^{2n+2}}{2\alpha_i h_i} - rf_i^{2n+2} \end{cases}$$

$$a = r - \hat{q} - \frac{1}{2}\hat{\sigma}^2 \qquad (4)$$

Then, solve the above equation and denote the result on the direction x_i as f_{x_i}.

Finally, compute arithmetic mean of n time layers, and take it as the new time layer result.

$$f_i = \frac{f_{x_1} + f_{x_2} + \cdots f_{x_{n-1}} + f_{x_n}}{n}$$

Similarly, if we adopt the implicit scheme at the odd number floor, and explicit scheme at the even number floor, we can construct the 'Implicit-Explicit' scheme:

$$\begin{cases} \dfrac{f_i^{2n+1} - f_i^{2n}}{k} = n\sigma^2 \dfrac{f_{i+1}^{2n+1} - 2f_i^{2n+1} + f_{i-1}^{2n+1}}{2\alpha_i^2 h_i^2} + n\sigma^2 F_{ij}^{2n+1} + na\dfrac{f_{i+1}^{2n+1} - f_{i-1}^{2n+1}}{2\alpha_i h_i} - rf_i^{2n+1} \\ \dfrac{f_i^{2n+2} - f_i^{2n+1}}{k} = n\sigma^2 \dfrac{f_{i+1}^{2n+1} - 2f_i^{2n+1} + f_{i-1}^{2n+1}}{2\alpha_i^2 h_i^2} + n\sigma^2 F_{ij}^{2n+1} + na\dfrac{f_{i+1}^{2n+1} - f_{i-1}^{2n+1}}{2\alpha_i h_i} - rf_i^{2n+1} \end{cases}$$

$$a = r - \hat{q} - \frac{1}{2}\hat{\sigma}^2 \qquad (5)$$

Then, take the same method to compute arithmetic mean of n time layers as the new time layer result.

When the traditional AOS scheme is adopted to calculate, it needs to solve an equation set that contains a triple diagonal matrix every step. Generally, we use the thomas method to solve it, and the computation is $O(M_{x_1} \times M_{x_2} \times \cdots M_{x_n} \times N)$. However, if we use AOS algorithm to construct the 'Explicit-Implicit' and 'Implicit-Explicit' scheme, it only needs to solve the triple diagonal matrix every two step in the x_i axis direction. Therefore, the total computation of the accelerated AOS scheme can be reduced greatly.

IV. ANALYSIS OF COMPATIBILITY AND ACCURACY OF AOS ALGORITHM FOR BASKET OPTION PRICING MODEL

Firstly, 'Explicit-Implicit' and 'Implicit-Explicit' scheme of AOS algorithm is to be considered. Take 'Explicit-Implicit' scheme for example. Add up the two equations of equation set (4) to eliminate f_i^{2n+1} on the direction of x_i.

$$\frac{f_i^{2n+2} - f_i^{2n}}{k} = n\sigma^2\left(G^{2n} + G^{2n+2}\right) + n\sigma^2\left(F_{ij}^{2n} + F_{ij}^{2n+2}\right) + na\left(H^{2n} + H^{2n+2}\right) - r\left(f^{2n} + f^{2n+2}\right)$$

$$G^n = \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{2\alpha_i^2 h_i^2},$$

$$H^n = \frac{f_{i+1}^n - f_{i-1}^n}{2\alpha_i h_i}. \qquad (6)$$

Substitute $f(x_1, x_2, \cdots, x_n, t)$ by f_i^n in (6), then make difference between the two side of the equation, and we will get the truncation error $T(x_1, x_2, \cdots, x_n, t)$. And expand $T(x_1, x_2, \cdots, x_n, t)$ as the Taylor series at the point $(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n, t)$, and take the arithmetic mean of $x_i (i = 1, 2, \cdots, n)$ direction:

$$\mathrm{T} = \frac{1}{n}\left(T_{x_1} + T_{x_2} + \cdots + T_{x_n}\right) = O\left(h^2 + k^2\right)$$

Similarly, 'Implicit-Explicit' scheme will get the same result. Then we can get that:

Theorem 1: 'Explicit-Implicit' and 'Implicit-Explicit' scheme of AOS algorithm of basket option pricing model (4) and (5) has two-order space and time accuracy. And they are compatible with Black-Scholes equation (3) unconditionally.

V. ANALYSIS OF STABILITY AND CONVERGENCE OF AOS ALGORITHM FOR BASKET OPTION PRICING MODEL

Take the Fourier transformation on the two sides of the equation (4), and simplify it to get:

$$\left[1 - \mathrm{rk} - \frac{\mathrm{nk}\hat{\sigma}^2}{2\alpha_i^2 h_i^2}\left(e^{i\xi} - 2 + e^{-i\xi}\right) + \frac{nk\left(r - \hat{q} - \frac{1}{2}\hat{\sigma}^2\right)}{2\alpha_i h_i}\left(e^{i\xi} - e^{-i\xi}\right) \right]\tilde{f}^{2n+2}(\xi) =$$

$$\left[1+\mathrm{rk}+\frac{\mathrm{nk}\,\hat{\sigma}^2}{2\alpha_i^2 h_i^2}\left(e^{i\xi}-2+e^{-i\xi}\right)-\frac{nk\left(r-\hat{q}-\frac{1}{2}\hat{\sigma}^2\right)}{2\alpha_i h_i}\left(e^{i\xi}-e^{-i\xi}\right)\right]\tilde{f}^{\,2n}(\xi)$$

Therefore, the growth factor is:

$$G(\xi)=\frac{1-P+Qi}{1+P-Qi},$$

$$P=rk+\frac{nk\hat{\sigma}^2}{2\alpha_i^2 h_i^2}\sin^2\left(\frac{\xi}{2}\right),$$

$$Q=\frac{nk\left(r-\hat{q}-\frac{1}{2}\hat{\sigma}^2\right)}{2\alpha_i h_i}\sin(\xi).$$

Because that:

$$\left|G(\xi)\right|^2=\frac{(1-P)^2+Q^2}{(1+P)^2+Q^2}\le 1.$$

By the Von Neumann Theorem, we can get that the ’Explicit-Implicit’ scheme of AOS algorithm of basket option pricing model is stable unconditionally. In addition, due to the Lax Theorem, we can get that the scheme is convergent. Therefore we can get the following theorems.

Theorem 2: ’Explicit-Implicit’ scheme of AOS algorithm of basket option pricing model (4) is stable and convergent unconditionally.

Similarly, ’Implicit-Explicit’ scheme of AOS algorithm will get the same result. Then we can get that:

Theorem 3: ’Implicit-Explicit’ scheme of AOS algorithm of basket option pricing model (5) is stable and convergent unconditionally.

VI. NUMERICAL EXAMPLE

Here, we consider one American basket Option, take the option is call option for example. The dividend rate is 0.03, the volatility is 0.2, the risk-free interest rate of American is 0.08, the strike price of option is 30000$. Consider the deadline of the option is 3, 6, 9 and 12 months, and the final exchange rate is the spot exchange rate.

The numerical experiment is done in MATLAB 2008 environment. The comparison among analytical solution and numerical solution is as follows:

TABLE I. THE COMPASION OF ANALYTICAL AND NUMERICAL SOLUTION TABLE

Time (month)	**3**	**6**	**9**	**12**	**relative error**
Analytical solution	42.7913	45.9223	48.7165	49.5046	0
’Explicit-Implicit’ scheme	42.5428	45.6596	48.4428	49.5929	0.00288
’Implicit-Explicit’ scheme	42.5428	45.6596	48.4428	49.5929	0.00288

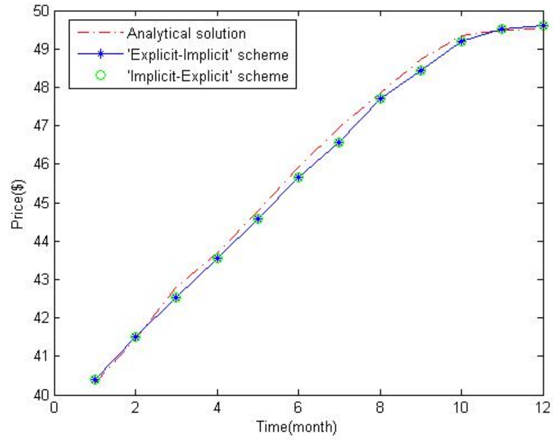

Figure 1. The compasion of analytical and numerical solution table.

From table 1 and figure 1, we can see that ’Explicit-Implicit’ and ’Implicit-Explicit’ scheme of AOS algorithm has higher calculation accuracy. With a longer deadline of the option, the advantage of the scheme is more obvious. The numerical result demonstrates the theoretic analysis that ’Explicit-Implicit’ and ’Implicit-Explicit’ scheme of AOS algorithm is effective.

VII. CONCLUSION

In order to solve Black-Scholes equation of basket option pricing model by numerical method, we construct the ’Explicit-Implicit’ and ’Implicit-Explicit’ scheme of additive operator splitting algorithm. The main idea of the scheme is to split the multi-dimensional Black-Scholes equation into one-dimensional equation set, and this method can avoid the complexity of using difference method directly on high dimensional equation. Then construct the 'explicit-implicit' and the 'implicit-explicit' schemes. Classical implicit scheme hides the potential stability, which is no use in the calculation, but when it is applied in the alternate scheme, this potential stability just cover the stability shortage of explicit scheme. Therefore those schemes are second-order accuracy, stable and convergent unconditionally. Finally, the total computation of these schemes is only a quarter of the traditional additive operator splitting scheme. Because the implicit scheme calculates the approximate value of the analytical solution from above, and the explicit scheme calculates it from below. Every two steps produce errors with the opposite symbol, which can counteract with each other, and then obtain the more accurate result.

From the theory analysis and numerical experiment, it can be seen that the 'explicit-implicit' and the 'implicit-explicit' schemes of additive operator splitting algorithm have practical significance in solving basket option pricing model.

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