The Strong Convergence of a New Iterative Algorithm for Asymptotically Nonexpansive Mappings

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***Abstract*—In a real Banach space E with a uniformly $G\hat{a}teaux$ differentiable norm, we prove that a new iterative sequence converges strongly to a fixed point of an asymptotically nonexpansive mapping. The results in this paper improve and extend some recent results of other authors.**

Keywords- uniformly asymptotically regular; asymptotically nonexpansive; uniformly $G\hat{a}teaux$ differentiable norm; strong convergence; fixed point

I. INTRODUCTION

It is well known that Fixed Point Theory has emerged as an important tool in several branches of pure and applied nonlinear sciences to study a wide range of subjects including computing science, communication engineering, obstacle, unilateral problems, optimization, theoretical mechanics, and control theory, in a unified and general framework. This alternative formulation has been used to study the existence of a fixed point as well as to develop some numerical methods. Based on the idea, we can examine some iterative methods for fixed points and the convergence of their iterative sequences.

Let K be a closed convex subset of a real Banach space E. Recall that a mapping T is called a nonexpansive mapping if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

First of all, we note the iterative sequence as follows:

$$x_{n+1} = \lambda_n y + (1-\lambda_n)Tx_n, n \ge 0, \qquad (1.1)$$

where the sequence $\{\lambda_n\}$ chosen in [0,1] and $y, x_0 \in K$. In 1967, Halpern [1] fist introduced this sequence (1.1) and pointed out that this sequence converges to the fixed point of T under the condition

$$(C_1)\lim_{n\to\infty}\lambda_n = 0, (C_2)\sum_{n=1}^{\infty}\lambda_n = \infty.$$

Since then, many mathematicians and scientists have studied the iterative sequence (1.1) such as Xu[2], Shioji and Takahashi [3], Chidume[4] etc. In [5], Xu denoted $T: K \to K$ is a nonexpansive mapping with $F(T) \ne \varnothing$, $f: K \to K$ is a contractive mapping with the coefficient $\alpha \in [0,1]$. Assume that the iterative sequence is generated by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n \ . \qquad (1.2)$$

Then $\{x_n\}$ converge strongly to a fixed point of T under certain conditions. If we let $f(x_n) \equiv y \in K$ in (1.2), then we can get (1.1).

In the present paper, motivated and inspired by the methods of Xu [5], we introduce a new iterative sequence $\{x_n\}$ generated by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)S^n x_n, n \ge 0 \ , \qquad (1.3)$$

where $T: K \to K$ is an asymptotically nonexpansive mapping with $k_n \in [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \forall x, y \in K,$$

and $f: K \to K$ is a contractive mapping, *S*: $S^n x = (1-\delta)x + \delta T^n x, \forall x \in K$. Under some suitable conditions, we prove that the iterative sequence $\{x_n\}$ converges strongly to a fixed point of the asymptotically nonexpansive mapping T. Our results improve and extend some recent results of other authors, such as [2-5].

II. PRELIMINARIES

Let *K* be a nonempty closed convex subset of the real Banach space *E* with a uniformly $G\hat{a}teaux$ differentiable norm. and *E** be the dual of *E.* The normalized duality mapping $J: E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^*, \langle x, f\rangle = \|x\|\|f\|, \|x\| = \|f\|\}, x \in E.$$

We denote by $\langle\cdot,\cdot\rangle$ the duality product and *J* the normalized duality mapping with a single value and *F(T)* the set of fixed points of *T.*

Let $S(E) := \{x \in E; \|x\| = 1\}$ denote the unit sphere of a Banach space *E*. The space *E* is said to have $G\hat{a}teaux$

differentiable norm (we also say that E is smooth), if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$$

exits for each $x, y\in S(E)$; E is said to have a uniformly $G\hat{a}teaux$ differentiable norm, if for each y in $S(E)$, the limit is attained uniformly for $x\in S(E)$.

Lemma 2.1([6]) Let E be a real Banach space, and $J: E\to 2^{E^*}$ is the normalized duality mapping, then it holds the inequality

$$\|x+y\|^2\le\|x\|+2\langle y, j(x+y)\rangle,$$
$$\forall j(x+y)\in J(x+y), \forall x, y\in E.$$

Lemma 2.2（[7]）Let $\{x_n\}$ and $\{y_n\}$ be two bonded sequences in areal Banach space E. We denote sequence $\{\beta_n\}$ chosen in [0,1] and

$$0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1.$$

Assume

$$x_{n+1}=\beta_n y_n+(1-\beta_n)x_n, (n\ge 0),$$
$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le 0.$$

Then $\lim_{n\to\infty}\|y_n-x_n\|=0$.

Lemma 2.3 ([2]) Let $\{\alpha_n\}$ be a sequences of non-negative real satisfying the property

$$a_{n+1}\le(1-\alpha_n)a_n+\alpha_n\sigma_n+\gamma_n, n\ge 0.$$

If the sequences $\{\alpha_n\}, \{\gamma_n\}$, $\{\sigma_n\}$ satisfie

(i) $\{\alpha_n\}\subset[0,1], \lim\alpha_n\to 0, \sum\alpha_n=\infty$;

(ii) $\limsup\sigma_n\le 0$;

(iii) $\gamma_n\le 0 (n\ge 0), or \sum\gamma_n<\infty$.

Then $a_n\to 0$, $n\to\infty$.

Definition 2.4 ([8]) Let K be a closed convex subset of a real Banach space E, $T: K\to K$. T is said to be asymptotically regular, if for all $x\in K$, we have

$$\lim_{n\to\infty}\|T^{n+1}x-T^n x\|=0.$$

III. Main Results

Theorem 3.1 Let K be a nonempty closed convex subset of a real Banach space E with a uniformly $G\hat{a}teaux$ differentiable norm. $T: K\to K$ is a asymptotically nonexpansive mapping with $k_n\in[1,\infty]$, $\lim_{n\to\infty}k_n=1$ and asymptotically regular properties and $F(T)\ne\varnothing$. For every fixed $\delta\in(0,1)$, we define a self mapping $S: K\to K$ by

$$S^n x=(1-\delta)x+\delta T^n x, \forall x\in K.$$

Let $f: K\to K$ is a contractive mapping with the contractive constant $\alpha\in(0,1)$. Suppose that

$$z_t:=tf(z_t)+(1-t)S^n z_t$$

is uniqueness in K for each t and when $t\to 0$, the sequence $\{z_t\}$ converges strongly to the fixed point z of T. Let $\{\alpha_n\}$ be a real sequence in [0,1] satisfied the follow conditions :

$$C_1: \lim_{n\to\infty}\alpha_n=0; \qquad C_2: \sum\alpha_n=\infty;$$

$$C_3: \lim_{n\to\infty}\frac{k_n-1}{\alpha_n}=0.$$

For any $x_0\in K$, if the sequence $\{x_n\}$ is defined by (1.3), then sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof For all $x, y\in K$, we can obtain

$$\begin{aligned}\|S^n x-S^n y\|&=\|(1-\delta)(x-y)\|+\delta\|T^n x-T^n y\|\\ &\le\|1-\delta(x-y)\|+\delta k_n\|x-y\|\\ &=(1-\delta+\delta k_n)\|x-y\|\end{aligned}$$

Let $L_n=1-\delta+\delta k_n$, obviously $L_n\in[1,\infty)$ and $\lim_{n\to\infty}L_n=1, \lim_{n\to\infty}\frac{L_n-1}{\alpha_n}=0$. So $\boldsymbol{S}$ is an asymptotically nonexpansive mapping. What's more, S and $\boldsymbol{T}$ have the same set of fixed point, marked as $F(T)=F(S)$.

Next, we show that the sequence $\{x_n\}$ is bounded. If we take $u\in F(T)$, then

$$\begin{aligned}&\|x_{n+1}-u\|=\|\alpha_n(f(x_n)-u)+(1-\alpha_n)(S^n-u)\|\\ &\le\alpha_n\|f(x_n)-u\|+\alpha_n\|f(u)-u\|+(1-\alpha_n)L_n\|(x_n-u)\|\\ &=\alpha_n\alpha\|x_n-u\|+\alpha_n\|f(u)-u\|+(1-\alpha_n)L_n\|x_n-u\|\\ &=[(1-\alpha_n)L_n+\alpha\alpha_n]\|x_n-u\|+\alpha_n\|f(u)-u\|\\ &=[(1-\alpha_n)L_n+\alpha_n\alpha]\|x_n-u\|\\ &\quad+[1-(1-\alpha_n)L_n-\alpha_n\alpha]\frac{\alpha_n\|f(u)-u\|}{[1-(1-\alpha_n)L_n-\alpha_n\alpha]}.\end{aligned}$$

Since

$$\frac{\alpha_n}{1-(1-\alpha_n)L_n-\alpha_n\alpha}=\frac{\alpha_n}{1-L_n-\alpha_n(L_n-\alpha)}=\frac{1}{\frac{1-L_n}{\alpha_n}+L_n-\alpha}$$

and

$$\lim_{n\to\infty}\frac{1-L_n}{\alpha_n}=0,$$

we have

$$\lim_{n\to\infty}\frac{\alpha_n}{1-(1-\alpha_n)L_n-\alpha_n\alpha}=\frac{1}{1-\alpha}.$$

Therefore there exists some constant M_1, such that

$|\dfrac{\alpha_n}{1-(1-\alpha_n)L_n-\alpha_n\alpha}|\le M_1$, for all $n\ge 0$.

$$\|x_{n+1}-u\|\le \max\{\|x_0-u\|, M_1\|f(u)-u\|\}.$$

Therefore $\{x_n\}$ is bounded. So are $\{y_n\}$, $\{T^n x_n\}$, $\{S^n x_n\}$. And

$$\|x_{n+1}-S^n x_n\| = \alpha_n \|f(x_n)-S^n x_n\| \to 0, (n\to\infty) \tag{3.1}$$

Next, let

$$y_n=\frac{\alpha_n f(x_n)+(1-\alpha_n)\delta T^n x_n}{\beta_n},$$

where $0<\liminf_{n\to\infty}\beta_n\le \limsup_{n\to\infty}\beta_n<1$.

Then

$$\begin{aligned}
&\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|\\
&=\|\frac{\alpha_{n+1}f(x_{n+1})+(1-\alpha_{n+1})\delta T^{n+1}x_{n+1}}{\beta_{n+1}}\\
&\quad-\frac{\alpha_n f(x_n)+(1-\alpha_n)\delta T^n x_n}{\beta_n}\|-\|x_{n+1}-x_n\|\\
&\le\|\frac{\alpha_{n+1}}{\beta_{n+1}}f(x_{n+1})-\frac{\alpha_n}{\beta_n}f(x_n)\|\\
&\quad+\|\frac{(1-\alpha_{n+1})\delta}{\beta_{n+1}}T^{n+1}x_{n+1}-\frac{(1-\alpha_n)\delta}{\beta_n}T^n x_n\|\\
&\quad-\|x_{n+1}-x_n\|\\
&\le\frac{\alpha_{n+1}}{\beta_{n+1}}\|f(x_{n+1})\|+\frac{\alpha_n}{\beta_n}\|f(x_n)\|\\
&\quad+\|\frac{(1-\alpha_{n+1})\delta}{\beta_{n+1}}(T^{n+1}x_{n+1}-T^{n+1}x_n)\|\\
&\quad+\|\frac{(1-\alpha_{n+1})\delta}{\beta_{n+1}}T^{n+1}x_n-\frac{(1-\alpha_n)\delta}{\beta_n}T^n x_n\|\\
&\quad-\|x_{n+1}-x_n\|\\
&\le\frac{\alpha_{n+1}}{\beta_{n+1}}\|f(x_{n+1})\|+\frac{\alpha_n}{\beta_n}\|f(x_n)\|\\
&\quad+(\frac{1-\alpha_{n+1}}{\beta_{n+1}}\delta k_n-1)\|x_{n+1}-x_n\|\\
&\quad+\frac{1-\alpha_{n+1}}{\beta_{n+1}}\delta\|T^{n+1}x_n-T^n x_n\|\\
&\quad+|\frac{(1-\alpha_{n+1})\delta}{\beta_{n+1}}-\frac{(1-\alpha_n)\delta}{\beta_n}|\bullet\|T^n x_n\|.
\end{aligned}$$

Since $\boldsymbol{T}$ is a asymptotically regular mapping and the sequences $\{x_n\}$, $\{f(x_n)\}$ and $\{T^n x_n\}$ are bounded, so

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le 0.$$

By Lemma 2.2, we have $\|y_n-x_n\|\to 0 (n\to\infty)$.

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\beta_n\|y_n-x_n\|=0.$$

By (3.1), we obtain that

$$\|x_n-S^n x_n\|\to 0 (n\to\infty).$$

Next we will prove

$$\limsup_{n\to\infty}\langle f(z)-z, j(x_{n+1}-z)\rangle\le 0.$$

For every positive integer $n\ge 0$, let $t_n\in(0,1)$, such that

$$t_n\to 0, \frac{L_n^2-1}{t_n}\to 0, \frac{\|x_n-S^n x_n\|}{t_n}\to 0 (n\to\infty). \tag{3.2}$$

Then we can obtain

$$z_{t_n}-x_n=t_n(f(x_n)-x_n)+(1-t_n)(S^n z_{t_n}-x_n).$$

$$\begin{aligned}
&\|z_{t_n}-x_n\|^2\le(1-t_n)^2\|S^n z_{t_n}-x_n\|^2+2t_n\langle f(x_n)-x_n, j(z_{t_n}-x_n)\rangle\\
&\le(1-t_n)^2(\|S^n z_{t_n}-S^n x_n\|+\|S^n x_n-x_n\|)^2\\
&\quad+2t_n\langle f(x_n)-x_n, j(z_{t_n}-x_n)\rangle\\
&\le(1-t_n)^2(L_n\|z_{t_n}-x_n\|+\|S^n x_n-x_n\|)^2\\
&\quad+2t_n\langle f(x_n)-x_n, j(z_{t_n}-x_n)\rangle\\
&\le(1-t_n)^2 L_n^2\|z_{t_n}-x_n\|^2\\
&\quad+\|S^n x_n-x_n\|(1-t_n)^2(2L_n\|z_{t_n}-x_n\|\\
&\quad+\|S^n x_n-x_n\|)+2t_n\langle f(x_n)-x_n, j(z_{t_n}-x_n)\rangle.
\end{aligned}$$

So,

$$\begin{aligned}
&\langle f(x_n)-x_n, j(z_{t_n}-x_n)\rangle\le\frac{(1-t_n)^2L_n^2-1}{2t_n}\|z_{t_n}-x_n\|^2\\
&+\frac{(1-t_n)^2\|S^n x_n-x_n\|}{2t_n}(2L_n\|z_{t_n}-x_n\|+\|S^n x_n-x_n\|).
\end{aligned}$$

Because

$$\lim_{n\to\infty}\frac{(1-t_n)^2L_n^2-1}{2t_n}=\lim_{n\to\infty}\frac{L_n^2-1-2t_nL_n^2+t_n^2L_n^2}{2t_n}$$

$$=\lim_{n\to\infty}\frac{\frac{L_n^2-1}{t_n}-2L_n^2+t_n^2L_n^2}{2}<0,$$

and $\{x_n\}$, $\{z_{t_n}\}$ and $\{S^n x_n\}$ are bounded, from (3.2), we can obtain

$$\limsup_{n\to\infty}\langle f(z_{t_n})-z_{t_n}, j(x_n-z_{t_n})\rangle\le 0.$$

Since $z_t \to z(t \to 0), \{z_t - x_n\}$ is bounded, as we know, the normalized duality mapping ***J*** is single valued and ***J*** is strong-weak* uniformly continuous on the bounded set in Banach space ***E***, then we can obtain

$$|\langle f(z)-z, j(x_n-z)\rangle| - |\langle f(z_t)-z_t, j(x_n-z_t)\rangle|$$
$$=|\langle f(z)-z, j(x_n-z)-j(x_n-z_t)\rangle|$$
$$+|\langle f(z)-z-(f(z_t)-z_t), j(x_n-z_t)\rangle|$$
$$\le|\langle f(z)-z, j(x_n-z)-j(x_n-z_t)\rangle|$$
$$+\| f(z)-z-(f(z_t)-z_t)\|\bullet\| x_n-z_t\| \to 0.$$

Hence, for $\forall \varepsilon > 0,$ if $\exists \delta > 0,$ we can get $\forall t \in (0,\delta)$, then for all n, we can obtain

$$\langle f(z)-z, j(x_n-z)\rangle < \langle f(z_t)-z_t, j(x_n-z_t)\rangle + \varepsilon \le \varepsilon .$$

For any $\varepsilon > 0$, we can get

$$\limsup_{n\to\infty} \langle f(z)-z, j(x_{n+1}-z)\rangle \le 0 . \qquad (3.3)$$

By (1.3), we can obtain

$$x_{n+1}-z = \alpha_n(f(x_n)-z+(1-\alpha_n)(S^n x_n - z).$$

By Lemma 2.1, it follows that

$$\| x_{n+1}-z\|^2 \le (1-\alpha_n)^2 \| S^n x_n - z\|^2 + 2\alpha_n \langle f(x_n)-z, j(x_{n+1}-z)\rangle$$

$$\le (1-\alpha_n)^2 L_n^2 \| x_n-z\|^2$$
$$+2\alpha_n \langle f(x_n)-f(z)+f(z)-z, j(x_{n+1}-z)\rangle$$
$$=(1-\alpha_n)^2 L_n^2 \| x_n-z\|^2 + 2\alpha_n \langle f(x_n)-f(z), j(x_{n+1}-z)\rangle$$
$$+2\alpha_n \langle f(z)-z,\rangle j(x_{n+1}-z)$$
$$\le (1-\alpha_n)^2 L_n^2 \| x_n-z\|^2 + 2\alpha_n\alpha \| x_n-z\|\bullet\| x_{n+1}-z\|$$
$$+2\alpha_n \langle f(z)-z, j(x_{n+1}-z)\rangle$$
$$\le (1-\alpha_n)^2 L_n^2 \| x_n-z\|^2 + \alpha_n(\alpha\| x_n-z\|^2 + \| x_{n+1}-z\|^2)$$
$$+2\alpha_n \langle f(z)-z, j(x_{n+1}-z)\rangle .$$

So,

$$\| x_{n+1}-z\|^2 \le \frac{L_n^2 - 2L_n^2\alpha_n + \alpha_n^2 L_n^2 + \alpha\alpha_n}{1-\alpha_n} \| x_n - z\|^2$$
$$+\frac{2\alpha_n}{1-\alpha_n}\langle f(z)-z, j(x_{n+1}-z)\rangle$$
$$=(1-\alpha_n \frac{2L_n^2-1-\alpha-\alpha_n L_n^2}{1-\alpha_n}) \| x_n-z\|^2 + \frac{L_n^2-1}{1-\alpha_n}\| x_n-z\|^2$$
$$-\frac{2\alpha_n}{1-\alpha_n}\langle f(z)-z, j(x_{n+1}-z)\rangle .$$

Let

$$\gamma_n = \alpha_n \frac{2L_n^2-1-\alpha-\alpha_n L_n^2}{1-\alpha_n}, \quad \delta_n = \frac{L_n^2-1}{1-\alpha_n}\| x_n - z\|^2$$
$$+\frac{2\alpha_n}{1-\alpha_n}\langle f(z)-z, j(x_{n+1}-z)\rangle .$$

We can have $\lim_{n\to\infty} \gamma_n = 0, \ \sum_{n=1}^{\infty} \gamma_n = \infty$, and $\lim_{n\to\infty} \sup \frac{\delta_n}{\gamma_n} \le 0$. By lemma 2.3, we can get $x_n \to z$.

IV. CONCLUSION

Through our results in ***Theorem 3.1*** with comparison in the reference [2-5], it is not difficult to see that, if ***T*** is a nonexpansive mapping, then we can get the results in Chidume[4] and Xu[5]. Further more, If $f(x) \equiv u$ and T is a nonexpansive, then we can obtain the conclusion in Xu[2], Shioji and Takahashi [3]. So our results improve and extend some recent results of other authors, such as [2-5].

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