Proceedings of the 2012 2nd International Conference on Computer and Information Application (ICCIA 2012)

# Sliding Mode $H_\infty$ Control for Active Vehicle Suspension Systems

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***Abstract*—This paper is concerned with the sliding mode $H_\infty$ control for an active vehicle suspension system. The suspension system is first decomposed into two virtual subsystems via a linear transformation. By considering the tire deflection of the suspension system as a virtual control for the first subsystem, the virtual state feedback $H_\infty$ controller is derived. Then, based on the virtual $H_\infty$ controller, a novel sliding mode surface is proposed. Third, a sliding mode $H_\infty$ controller is designed to ensure that the state trajectories can reach the sliding surface in finite time and maintain on it thereafter. Simulation results show that the designed controller can achieve the specified $H_\infty$ performance for the active suspension system and preserve the asymptotic stability of the closed-loop system.**

***Keywords-Sliding mode control; $H_\infty$ controller; Suspension system; Active control***

## I. INTRODUCTION

Suspension system plays an import role in modern vehicles to provide vehicle support, safety, ride comfort, road holding, and suspension deflection. Vehicle suspensions systems including passive [1], semi-active [2] and active suspensions [3], [4] have gained wide concern over the last few decades. A considerable amount of theoretical and experimental research has been carried out to improve the control performance of the suspension systems. Due to the fact that the active suspensions can continuously change the vibration energy of the vehicle body induced by the road excitation, the active suspensions have a great potential to improve both ride comfort and handling performance. Various schemes have been developed to improve the performance of active vehicle suspension systems, such as linear quadratic control [5], [6], adaptive control [7], $H_\infty$ control [8], [9], and preview control [10], etc.

As is well known, sliding mode control is effective to achieve high-performance robust control against external disturbances and unpredictable parameter variations, one can see [11], [12] and the references therein. Sliding mode control for the active suspension systems are intensively investigated in the context of robustness and disturbance attenuation. For example, by combining the optimal control scheme and the sliding mode control scheme, a feedforward and feedback optimal sliding mode control scheme has been developed to improve the control performance of the suspension system [6]. Based on two-time scale singularly perturbed dynamic model, a sliding mode control strategy has been proposed to deal with the design of active vehicle suspension control systems [13]. In [14] and [15], fuzzy sliding mode control schemes have been presented to control the active suspension systems. These control schemes are capable of improving the control performance of the suspension systems to some acceptable level. However, it should be pointed out that in [6], the optimal sliding mode controller design is based on the fact that the road surface disturbance acting on the suspension system is formulated as the output of an exogenous linear system. This indicates that the road surface disturbance is regarded as a deterministic input signal. In fact, on the one hand, the road surface disturbance is random and irregular. On the other hand, as the dimension of the exogenous system increases, the calculated quantity rapidly does. Therefore, from a point of implementation view, the optimal sliding mode control scheme is not always work.

In this paper, for a quarter-car model with active suspension systems, a sliding mode $H_\infty$ control scheme will be proposed to improve the control performance of the suspension system. Compared with the design of the optimal sliding mode controller in [6], the design of the sliding mode $H_\infty$ controller is based on the assumption that the random road surface disturbance is unknown but bounded. The suspension system is decomposed into two subsystems such that the road surface disturbance and the active control input are in the different subsystems. By considering the tire deflection of the suspension system as a virtual control force for the first subsystem, the virtual $H_\infty$ controller is designed. Then, a novel sliding mode surface is proposed via the obtained virtual $H_\infty$ controller. Third, a sliding mode $H_\infty$ controller is developed to guarantee that the state trajectories are reachable in finite time and maintain on the sliding surface thereafter. Simulation results are given to show the effectiveness of the proposed control scheme.

Throughout this paper, all the matrices are real matrices. The superscripts '−1' and 'T' mean the inverse and transpose of a matrix, respectively; $P>0$ means that the matrix $P$ is a positive-definite symmetric matrix; $I$ is the identity matrix of appropriate dimensions.

## II. SYSTEM MODELING

A two-degree-of freedom quarter-car suspension system considered in this paper is shown in Fig. 1, where $m_s$ is the sprung mass, $m_u$ is the unsprung mass, $k_s$ is the stiffness of the suspension, $c_s$ is the damping of the suspension, $k_t$ is the stiffness of the tire, $z_s$ is the displacements of the sprung, $z_u$ is the displacements of unsprung masses, $z_r$ is the road displacement input, and $u$ is the active control.

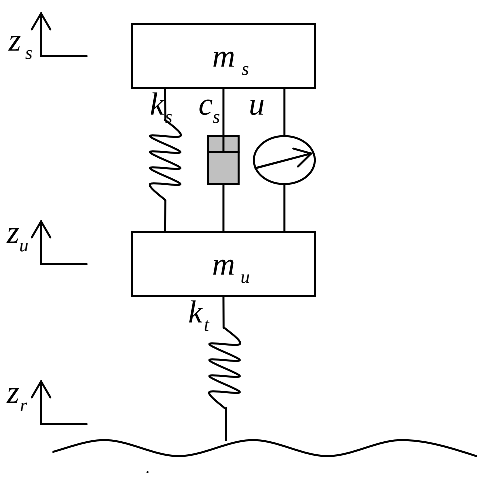


Fig. 1 Quarter-car suspension system.

Assume that the characteristics of all passive suspension elements are linear, the tire does not leave the ground, and $z_s$ and $z_u$ are measured from the static equilibrium point. The model can be represented by the equations:

$$\begin{aligned} &m_s\ddot{z}_s-k_s(z_u-z_s)-c_s(\dot{z}_u-\dot{z}_s)-u=0,\\ &m_u\ddot{z}_u+k_s(z_u-z_s)+c_s(\dot{z}_u-\dot{z}_s)+u+k_t(z_u-z_r)=0. \end{aligned} \tag{1}$$

Choosing the state variables as

$$\begin{aligned} &\bar{x}_1(t)=z_s(t)-z_u(t), \quad \bar{x}_3(t)=\dot{z}_s(t),\\ &\bar{x}_2(t)=z_u(t)-z_r(t), \quad \bar{x}_4(t)=\dot{z}_u(t); \end{aligned} \tag{2}$$

where $\bar{x}_1(t)$ is the suspension deflection, $\bar{x}_2(t)$ is the tire deflection, $\bar{x}_3(t)$ is the velocity of car body and $\bar{x}_4(t)$ is the velocity of tire.

Denote $\bar{x}=\begin{bmatrix}\bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4\end{bmatrix}^T$ and $v(t)=\dot{z}_r(t)$. Then, system (1) can be rewritten as the state space form

$$\dot{\bar{x}}(t)=\bar{A}\bar{x}(t)+\bar{B}u(t)+\bar{D}v(t), \tag{3}$$

where

$$\begin{aligned} &\bar{A}=\begin{bmatrix} 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1\\ -k_s/m_s & 0 & -c_s/m_s & c_s/m_s\\ k_s/m_u & -k_t/m_u & c_s/m_u & -c_s/m_u \end{bmatrix},\\ &\bar{B}=\begin{bmatrix}0 & 0 & 1/m_s & -1/m_u\end{bmatrix}^T,\\ &\bar{D}=\begin{bmatrix}0 & -1 & 0 & 0\end{bmatrix}^T. \end{aligned} \tag{4}$$

It is assumed that the road surface disturbance term $v(t)\in L_2[0,\ \infty)$, and $\|v(t)\|\le v^*$, where $v^*>0$ is a known constant. .In what follows, we will design a sliding mode $H_\infty$ control scheme to improve the performance of the suspension system.

## III. SLIDING MODE $H_\infty$ CONTROL DESIGN

### A. System Decomposition

In this subsection, based on a linear transformation, we decompose the system (3) into two subsystems.

Let

$$x=M\bar{x} \tag{6}$$

where

$$M=\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & m_u/m_s\\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

From (3) and (6), one yields the following two subsystems. The first subsystem is in the form

$$\dot{x}_1(t)=A_{11}x_1(t)+A_{12}x_2(t)+Dv(t), \tag{8}$$

where $x_1\in R^3$ is the state variables and $x_2\in R$ is the virtual control variable, and

$$A_{11} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -k_t/m_s & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} -(m_u + m_s)/m_s \\ 1 \\ 0 \end{bmatrix},$$
$$D = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}^T. \tag{9}$$

The second subsystem is as

$$\dot{x}_2(t) = A_{22}x_2(t) + Bu(t) + A_{21}x_1(t) \tag{10}$$

where

$$\begin{aligned} A_{21} &= \begin{bmatrix} k_s/m_u & -k_t/m_u & c_s/m_u \end{bmatrix}, \\ A_{22} &= -c_s(m_u + m_s)/m_u m_s, \\ B &= -1/m_u. \end{aligned} \tag{11}$$

*B. Sliding Surface Design*

To design a sliding surface, for the first subsystem (8), we introduce the control output equation as

$$z(t) = C_1\bar{x}(t) + D_1 v(t), \tag{12}$$

where $C_1$ and $D_1$ are problem-dependent constant matrices with appropriate dimensions.

Now, we intend to design a virtual state feedback $H_\infty$ control law

$$x_2(t) = Kx_1(t), \tag{13}$$

such that under the designed control law, the subsystem (8) with $v(t) = 0$ is asymptotically stable; and under the zero initial condition, the $H_\infty$ performance

$$\|z(t)\| \le \gamma \|v(t)\|, \tag{14}$$

of the resulting closed-loop subsystem is guaranteed for nonzero $v(t)$ and a prescribed $\gamma > 0$, where $K \in R^{1\times 3}$ is the gain matrix to be determined.

The matrix $D_1$ in output equation (12) and the $H_\infty$ performance level $\gamma$ are assumed to satisfy the constraint as

$$D_1^T D_1 < \gamma^2 I \tag{15}$$

Substituting (14) into (8) yields

$$\dot{x}_1(t) = (A_{11} + A_{12}K)x_1(t) + Dv(t). \tag{16}$$

Choose a Lyapunov function as

$$V(x_1) = x_1^T(t)Px_1(t), \tag{17}$$

where $P \in R^{3\times 3}$ and $P > 0$.

Taking the derivative of $V(x_1)$ with respect to $t$ along the state trajectory of the subsystem (16), after simple manipulation, it can be obtained that under the constraint (15), for given scalar $\gamma > 0$, the closed-loop subsystem (16) with $v(t) = 0$ is asymptotically stable, and the $H_\infty$ performance (14) is guaranteed, if there exist $\bar{K} \in R^{1\times 3}$ and $\bar{P} \in R^{3\times 3}$, $\bar{P} > 0$ such that

$$\begin{bmatrix} A_{11}\bar{P} + \bar{P}A_{11}^T + A_{12}\bar{K} + \bar{K}^T A_{12}^T & D & \bar{P}C_1^T \\ D^T & -\gamma^2 I & D_1^T \\ C_1\bar{P} & D_1 & -I \end{bmatrix} < 0 \tag{18}$$

If the linear matrix inequality (18) is feasible, then the gain matrix $K$ in (13) can be obtained as $K = \overline{KP}^{-1}$. Then, the virtual state feedback $H_\infty$ control law (13) is available.

Now, we design a sliding surface function as:

$$s(t) = x_2(t) - Kx_1(t), \tag{19}$$

It should be pointed out that on the above sliding surface, the subsystem (8) with $v(t) = 0$ is asymptotically stable and the $H_\infty$ performance is guaranteed.

Noticing $\dot{s}(t) = 0$, combining (10) and (16), one yields the equivalent control law as

$$\begin{aligned} u_{eq}(t) = B^{-1}(KA_{11} + KA_{12}K - A_{21})x_1(t) \\ - B^{-1}A_{22}x_2(t) + B^{-1}KDv(t) \end{aligned} \tag{20}$$

In this situation, the sliding motion can be obtained as

$$\dot{x}_2(t) = K\dot{x}_1(t) \tag{21}$$

which indicates that the sliding motion is asymptotically stable.

*C. Sliding mode $H_\infty$ controller design and the reachability condition*

The sliding mode $H_\infty$ controller is designed as

$$u(t) = B^{-1}[(KA_{11} + KA_{12}K - A_{21})x_1(t) \\ - B^{-1}A_{22}x_2(t) + \rho(t)\operatorname{sgn}(s(t))], \tag{22}$$

where $\operatorname{sgn}(\bullet)$ is the sign function, and $\rho(t)$ is the switching function as

$$\rho(t) = \|KD\| v^* + \eta \tag{23}$$

where $\eta > 0$.

Choose a Lyapunov function as

$$V_S(s(t)) = s^2(t)/2 \tag{24}$$

Notice that

$$\dot{x}_2(t) = K(A_{11} + A_{12}K)x_1(t) + \rho(t)\operatorname{sgn}(s(t)). \tag{25}$$

Then, from (19), one can obtain

$$\dot{s}(t) = KDv(t) - \rho(t)\operatorname{sgn}(s(t)). \tag{26}$$

Further, from (23) , (24) and (26), we have

$$\dot{V}_S(s(t)) < -\eta s(t), \tag{27}$$

which indicates that under the sliding mode $H_\infty$ control law (22), the state trajectory of the second subsystem (10) can be driven into the sliding surface $s(t) = 0$ in finite time and maintain on it thereafter.

**Remark 1.** To design the sliding mode $H_\infty$ controller (22), only the upper bound $v^*$ of the road surface disturbance is needed, and the deterministic dynamic model of the road surface disturbance is not necessary. Therefore, compared with the feedforward and feedback optimal sliding mode controller [6], the sliding mode $H_\infty$ controller is more practical than the optimal sliding mode controller.

## IV. SIMULATION EXAMPLE

In this section, a simulation example is given to illustrate the effectiveness of the proposed sliding mode $H_\infty$ control scheme. In Fig. 1, the parameters of the suspension system are from [9], where

$$m_s = 972.2 \text{ kg},\ m_u = 113.6 \text{ kg},\ c_s = 1095 \text{ Ns/m}, \\ k_s = 42719.6 \text{ N/m},\ k_t = 101115 \text{ N/m},$$

The vehicle is assumed to travel at a constant speed on a given horizontal road, the road surface disturbances can be approximated by the following series [9]

$$z_r(t) = \sum_{i=1}^{N} s_i \sin(i\omega_0 t + \varphi_i) \tag{28}$$

where $s_i = \sqrt{2s_g(i\Delta\Omega)\Delta\Omega}$ , $\Delta\Omega = 2\pi/l$ , $l$ is the length of the road segment, $\omega_0 = (2\pi/l)v_0$ , $v_0$ is the horizontal speed of the vehicle, and $\varphi_n$ is random frequency, $\varphi_n \in [0,\ 2\pi)$ , $N$ limits the considered frequency range, and

$$s_g(\Omega) = \begin{cases} t_g(\Omega_0)(\Omega/\Omega_0)^{-n_1},\ \Omega \le \Omega_0 \\ t_g(\Omega_0)(\Omega/\Omega_0)^{-n_2},\ \Omega \ge \Omega_0 \end{cases} \tag{29}$$

where $t_g(\Omega_0)$ provides a measure for the roughness of the road, $n_1$ and $n_2$ are road roughness constant.

For the random road profile, the road roughness is chosen as $t_g(\Omega_0) = 64\times 10^{-6} m^3$ [9], and let $v_0 = 20$ m/s, $n_1 = 2$ , $n_2 = 1.5$ , $l = 100$ , $N = 5$ . In this case, the response of the road surface disturbance is presented by Figure 2, it can be computed that the upper bound of the road surface disturbance $v^* = 0.0746$ .

Let the initial state $x(0) = [0.5\ 0.4\ 0.5\ 0.3]^T$ , $\eta = 0.01$, and $\gamma = 0.2$. The matrices $C_1$ and $D_1$ in (12) is given as

$$C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ D_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}. \tag{30}$$

By solving the linear matrix inequality (18), one yields the matrix $K$ in (13) as

$$K = [134.97 \quad \text{-}20.2771 \quad 18.0791]. \tag{31}$$

When the sliding mode $H_\infty$ controller (22) is used to control the suspension system, the variation of the sliding surface is shown in Figure 3, the responses of the deflection and the velocity of the suspension, the deflection and the velocity of the tire are depicted in Figs. 4-7, respectively, where, the controlled responses are compared with the ones when no controller is applied the system. Figure 8 presents the response of the control force required.

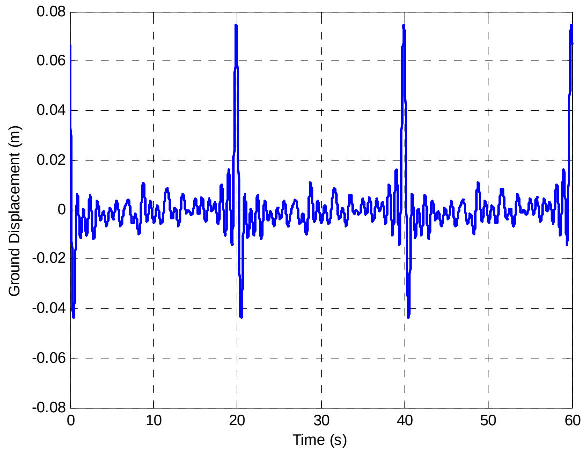


Figure 2. The random road surface disturbance.

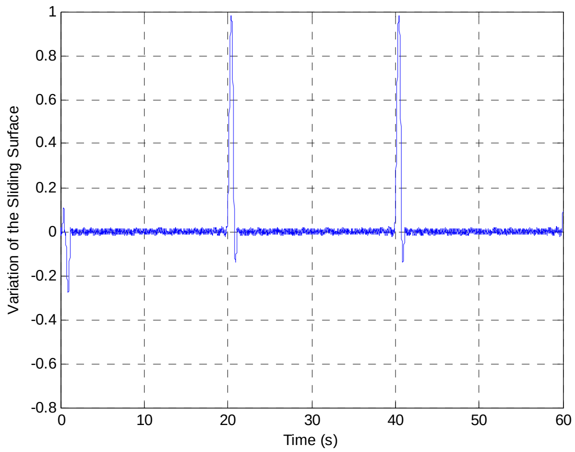


Figure 3. Variation of the sliding surface function.

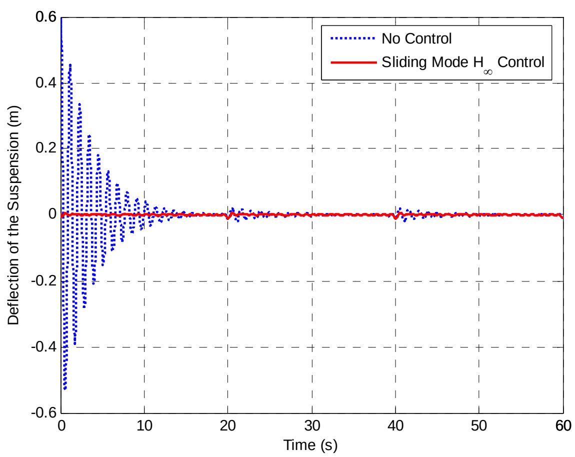


Figure 4. Deflection of the suspension.

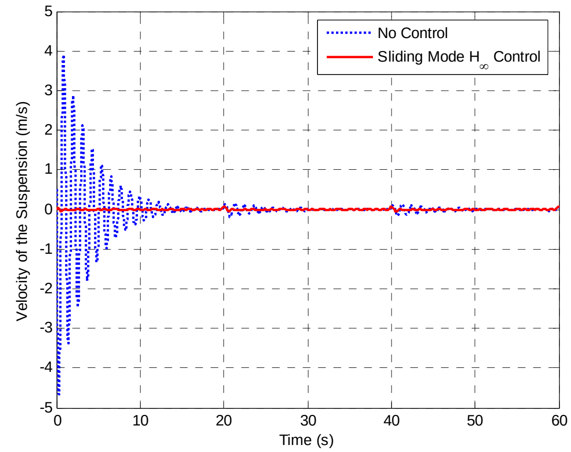


Figure 5. Velocity of the suspension.

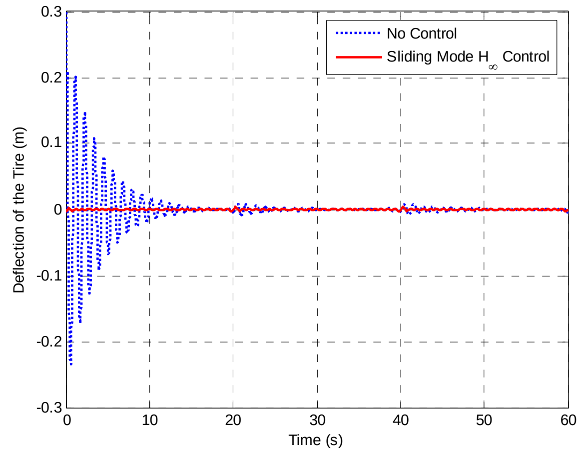


Figure 6. Deflection of the tire.

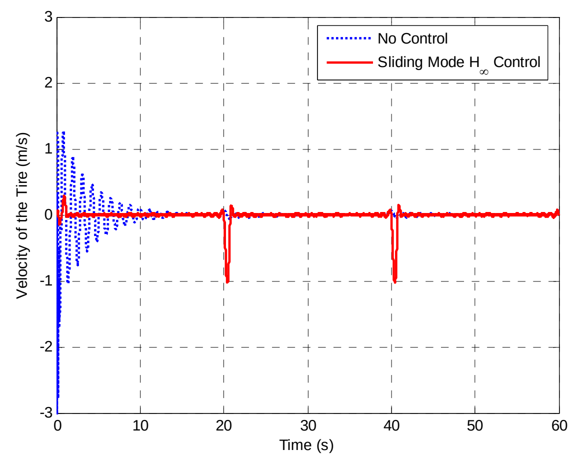


Figure 7. Velocity of the tire.

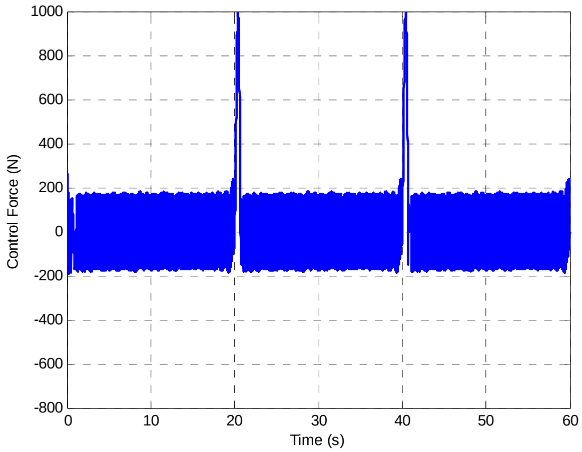


Figure 8. Control force.

It can be seen from Figs. 4-7 that under the designed sliding mode $H_\infty$ controller, the oscillation amplitudes of the deflection and the velocity of the suspension, and the deflection and the velocity of the tire are reduced significantly, which indicates that the proposed sliding mode $H_\infty$ control scheme is effective to improve the performance of the suspension systems.

## V. Conclusion

By combining the sliding mode control scheme and the $H_\infty$ control scheme, the sliding mode $H_\infty$ control scheme has been developed to improve the performance of the suspension system. It has been demonstrated from the simulation results that the designed sliding mode $H_\infty$ controller is capable of attenuating the vibration caused by the random road surface disturbance effectively.

## Acknowledgements

This work was supported by the Science and Technology Planning Project of Zhejiang Province under Grant 2012C21022.

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