Journal of Nonlinear Mathematical Physics Volume 12, Supplement 2 (2005), 197–205 SIDE VI

Finite reductions of the two dimensional Toda chain

E V GUDKOVA

Chernyshevsky str. 112, Institute of Mathematics, Ufa Scientific Center, Ufa, RAS, 450077, Russia
E-mail: elena.gudkova79@mail.ru

This article is a part of the special issue titled "Symmetries and Integrability of Difference Equations (SIDE VI)"

Abstract

The problem of the classification of integrable truncations of the Toda chain is discussed. A new example of the cutting off constraint is found.

1 Introduction

Consider the two dimensional infinite Toda chain:

$$u_{xt}(n) = w(n-1) - w(n), \quad -\infty < n < \infty, \tag{1.1}$$

where $w(n) = \exp\{u(n) - u(n+1)\}$. Suppose that a cutting off constraint (boundary condition) of the form

$$F[u(-1), u(0), ..., u(k)] = 0 \tag{1.2}$$

is imposed upon the chain. Here the brackets mean that F depends not only on the variables $u(j)$, $j = -1, 0, 1...k$, but also on a finite number of their derivatives with respect to x and t. We require the constraint (1.2) to be consistent with the integrability property of the chain. A formal definition of consistency is given in Definition 1 below.

In the papers [1], [2], [3] it has been shown that the problem of looking for integrable boundary conditions for discrete chains is reduced to searching for differential constraints linking two linear differential equations. To be more precise we recall the Lax representation for the chain (1.1):

$$\phi\,(n+1) = (D_x + u_x\,(n))\,\phi\,(n)\,, \tag{1.3a}$$

$$\phi_t(n) = -w(n-1)\phi(n-1). \tag{1.3b}$$

The equations (1.3) are nothing else but the pair of mutually inverse Laplace transformations for the following linear hyperbolic equation

$$\phi_{xt}(n) + u_x(n)\phi_t(n) + w(n-1)\phi(n) = 0, \tag{1.4}$$

Copyright © 2005 by E V Gudkova

As a Lax pair for the chain (1.1) one can also take the pair of formally conjugate equations to the equations (1.3):

$$y(n-1) = (-D_x + u_x(n))y(n), \tag{1.5a}$$
$$y_t(n) = w(n)y(n+1). \tag{1.5b}$$

Here the conjugation is understood as follows $(a(n))^* = a(n)$, $(a(n)D_x)^* = -D_x a(n)$, $(a(n)T)^* = a(n-1)T^{-1}$, $(a(n)T^{-1})^* = a(n+1)T$ and so on, where T, T^{-1} are the shift operators: $Ta(n) = a(n+1)$, $T^{-1}a(n) = a(n-1)$. Excluding shifts of the variable $y(n)$ from the equations (1.5) one gets the hyperbolic type equation conjugate to the equation (1.4)

$$y_{xt}(n) - u_x(n)y_t(n) + w(n)y(n) = 0. \tag{1.6}$$

Suppose that we are given a set of functions

$$S = \{a_0, a_1...a_m, b_0, b_1...b_m\}, \tag{1.7}$$

depending upon x, t such that at least one of them doesn't identically vanish.

Definition 1. The boundary condition (1.2) is called consistent with the integrability property of the Toda chain (1.1) if there exists a set of the functions S of the form (1.7) and integers k, l such that the constraint consisting of two equations

$$F[u(-1), u(0), ..., u(k)] = 0, \tag{1.8a}$$
$$\sum_0^m a_j\phi(k+j) + b_j y(l+j) = 0, \tag{1.8b}$$

is consistent with both Lax pairs (1.3) and (1.5).

Note that the first equation in the constraint coincides with the boundary condition while the second is a linear condition imposed on the eigenfunctions.

For any semi-infinite chain obtained from (1.1) by imposing the boundary condition satisfying the Definition 1 one can easily construct the appropriate Lax pair. Similarly, the Lax pair can be found for finite reductions of (1.1) with two integrable boundary conditions.

By excluding shifts of the variables $\phi(n)$ and $y(n)$ from the equation (1.8) one gets a constraint of the form

$$(L_1(D_x) + L_2(D_t))\phi(n_0) = (L_3(D_x) + L_4(D_t))y(n_0), \tag{1.9}$$

where $L_j(D)$ are linear differential operators with coefficients depending on x, t. The constraint (1.9) is a differential connection between solutions of the hyperbolic equations (1.4) and (1.6) conjugate to each other, taken for some fixed point $n = n_0$.

This observation implies the following statement.

Lemma 1. *The boundary condition (1.2) is consistent with the integrability if and only if the constraint consisting of two equations (1.2) and (1.9) is consistent with the pair of hyperbolic equations (1.4) and (1.6).*

The Definition 1 is supported by several examples. For instance, the following theorem from [2] shows that the well-known integrable boundary conditions satisfy the Definition 1.

Theorem 1. *Suppose that the boundary condition (1.2) is consistent with integrability and the corresponding linear constraint (1.9) is given by*

$$\phi(0) = My(0), \tag{1.10}$$

where M is the differential operator $M = aD_x^2 + bD_x + c$, then the boundary condition is of one of the forms: 1) $e^{u(-1)} = 0$, 2) $u(-1) = 0$, 3) $u(-1) = -u(0)$, 4) $u_x(-1) = -u_t(0)e^{-u(0)-u(-1)}$. The corresponding operator M is

$$M_1 = (D_x + a_0)e^u D_x e^{-u}, \tag{1.11a}$$
$$M_2 = D_x e^u D_x e^{-u}, \tag{1.11b}$$
$$M_3 = e^u D_x e^{-u}, \tag{1.11c}$$
$$M_4 = D_x e^u D_x e^{-u} + e^{-2u}, \tag{1.11d}$$

where a_0 is an arbitrary constant and $u = u(0)$.

2 A new example

The classification of all boundary conditions satisfying Definition 1 is rather a hard problem. In this paper a very particular case is studied when the operator in (1.10) is of the third order. The following statement takes place.

Theorem 2. *Suppose that the constraint*

$$F[u(-1), u(0), ..., u(k)] = 0, \tag{2.1a}$$
$$\phi(0) = My(0), \tag{2.1b}$$

where M is a differential operator of the third order, is consistent with pair of differential equations (1.4), (1.6) then the boundary condition (1.2) is of one of the forms:

$$e^{u(-1)} = 0, \tag{2.2a}$$
$$\{u_x(-2) + u_x(-1)\}e^{u(-2)-u(-1)} = -\frac{d}{dt}\mu, \tag{2.2b}$$

where $\mu = \rho(t)e^{-u(-1)-u}$, and $\rho(t)$ is an arbitrary function of t. The corresponding differential operators are respectively of the form

$$M_1 = (D_x^2 + a_0 D_x + b_0)e^u D_x e^{-u}, \tag{2.3a}$$
$$M_2 = e^u\{D_x^2 + 2u_x D_x + u_{xx} - u_{xx}(-1) + u_x^2 - u_x^2(-1) - \mu\}D_x e^{-u}, \tag{2.3b}$$

where a_0, b_0 are arbitrary constants and $u = u(0)$.

Proof. Let us change the dependent variable in the equation (1.6) by setting $y(n) = \psi(n)e^{u(n)}$:

$$\psi_{xt}(n) + u_t(n)\psi_x(n) + w(n-1)\psi(n) = 0. \tag{2.4}$$

Now the constraint (2.1) takes the form

$$\phi = N\psi, \tag{2.5}$$

where the operator $N = aD_x^3 + bD_x^2 + cD_x + d$ is to be found. It follows from (2.5) that

$$\phi_{xt} = N_{xt}\psi + N_x\psi_t + N_t\psi_x + N\psi_{xt}. \tag{2.6}$$

Eliminate the variables ϕ_{xt}, ψ_{xt} by means of the equations (1.4), (2.4) and then express the variable ϕ and all its derivatives ϕ_x, ϕ_t, $\phi_{xx}, \ldots$ through ψ, ψ_x, ψ_t, $\psi_{xx}, \ldots$ by means of the constraint (2.5). After all these transformations collect in (2.6) coefficients of the independent dynamical variables $D_x^4\psi$, $D_x^3\psi$, $D_x^2\psi$, $D_x\psi$, ψ, $D_t\psi$. As a result one gets six equations for four unknowns

$$a_t + a\gamma = 0, \tag{2.7a}$$

$$a(3\gamma_x + \beta) + a_{xt} + b_t + (a_x + b)\gamma = \alpha a\gamma + \alpha a_t + \beta a, \tag{2.7b}$$

$$\begin{aligned} &a(3\gamma_{xx} + 3\beta_x) + (a_x + b)(2\gamma_x + \beta) + b_{xt} + c_t + (b_x + c)\gamma = \\ &\qquad = \alpha(a(2\gamma_x + \beta) + b\gamma + b_t) + \beta b, \end{aligned} \tag{2.7c}$$

$$\begin{aligned} &a(\gamma_{xxx} + 3\beta_{xx}) + (a_x + b)(\gamma_{xx} + 2\beta_x) + (b_x + c)(\gamma_x + \beta) + c_{xt} + d_t + \\ &\qquad + (c_x + d)\gamma = \alpha(a(\gamma_{xx} + 2\beta_x) + b(\gamma_x + \beta) + c\gamma + c_t) + \beta c, \end{aligned} \tag{2.7d}$$

$$\begin{aligned} &a\beta_{xxx} + (a_x + b)\beta_{xx} + (b_x + c)\beta_x + (c_x + d)\beta + d_{xt} = \\ &\qquad = \alpha(a\beta_{xx} + b\beta_x + c\beta + d_t) + \beta d, \end{aligned} \tag{2.7e}$$

$$d_x = \alpha d, \tag{2.7f}$$

where $\alpha = -u_x$, $\beta = -e^{u(-1)-u}$, $\gamma = -u_t$.

The first, second, and the last equations of the system (2.7) are easily solved. The answers are $a = a_0e^u$, $d = d_0e^{-u}$, $b = (b_0 + 2u_x)e^u$, where $a_0 = a_0(x)$, $b_0 = b_0(x)$, $d_0 = d_0(t)$ are arbitrary functions, but without losing generality one can consider a_0 and d_0 as constants because the Toda chain is invariant under the shift transformation $u(x,t) \to u(x,t) + f(x) + g(t)$. Let us choose $a_0 = 1$. Introduce a new variable z, $c = ze^u$ and rewrite the rest of the system (2.7) in the form

$$z_t = (u_{xx} + u_x^2)_t - 2\beta u_x - 3\beta_x + b_0u_{xt}, \tag{2.8a}$$

$$b_0(\beta u_x + \beta_x) = d_0u_te^{-2u}, \tag{2.8b}$$

$$\begin{aligned} &-\beta_{xxx} - 4u_x\beta_{xx} - b_0\beta_{xx} - 2b_0\beta_xu_x - 2\beta_xu_{xx} - 4\beta_xu_x^2 = \\ &\qquad = -d_0u_{xt}e^{-2u} + 2\beta zu_x + z\beta_x + \beta z_x. \end{aligned} \tag{2.8c}$$

Depending on the choice of the values of the parameters one gets four alternatives:

i) $b_0 = 0$, $d_0 = 0$;

ii) $b_0 = 0$, $d_0 \neq 0$;

iii) $b_0 \neq 0$, $d_0 = 0$;

iv) $b_0 \neq 0$, $d_0 \neq 0$;

which are to be studied separately. Begin with the first one. In this case the second equation of the system (2.8) is identically satisfied and the last one takes the form

$$(z\beta e^{2u})_x = -(\beta_x e^{2u})_{xx}.$$

After integration it implies

$$z = -\frac{\beta_{xx}}{\beta} - \frac{2\beta_x u_x}{\beta} + \rho(t)\frac{e^{-2u}}{\beta}, \tag{2.9}$$

where $\rho(t)$ is an arbitrary function of t. Comparison of the equation found with the equation (2.8a) which looks now as follows

$$z_t = u_{xxt} + 2u_x u_{xt} - 2\beta u_x - 3\beta_x,$$

leads to the equation

$$-\frac{\partial}{\partial t}\left(\frac{\beta_{xx}}{\beta} + \frac{2\beta_x u_x}{\beta} - \rho(t)\frac{e^{-2u}}{\beta}\right) = u_{xxt} + 2u_x u_{xt} - 2\beta u_x - 3\beta_x. \tag{2.10}$$

This is just the boundary condition we have been searching for, because it is a differential constraint between the variables $u = u(0)$ and $\beta = -e^{u(-1)-u}$. Eliminate β and rewrite it as

$$u_{xxt}(-1) + 2u_x(-1)u_{xt}(-1) + (\rho'(t) - \rho u_t(-1) - \rho u_t)e^{-u-u(-1)} = 0.$$

Express the mixed derivatives by means of the Toda chain (1.1) and write the constraint in the required form (see (1.2)):

$$\{u_x(-2) + u_x(-1)\}e^{u(-2)-u(-1)} = -\{\rho(t)u_t(-1) - \rho(t)u_t - \rho'(t)\}e^{-u-u(-1)}.$$

Evidently it coincides with the second boundary condition from the theorem. Substitute the β found into the equation (2.9) to find z: $z = u_{xx} - u_{xx}(-1) + u_x^2 - u_x^2(-1) + \rho(t)e^{-u(-1)-u}$. Now one can find the operator N,

$$N = e^u D_x^3 + 2u_x e^u D_x^2 + (u_{xx} - u_{xx}(-1) + u_x^2 - u_x^2(-1) - \rho(t)e^{-u(-1)-u})e^u D_x.$$

Find $M = Ne^{-u}$ which evidently coincides with (2.3b).

Consider now the case ii). It follows from the equation (2.8b) that $u_t e^{-2u} = 0$, or, evidently, $u_t = 0$. The system (2.8) turns into

$$y_t = -2\beta u_x - 3\beta_x, \tag{2.11a}$$

$$(y\beta e^{2u})_x = -(\beta_x e^{2u})_{xx} - b_0(\beta_x e^{2u})_x. \tag{2.11b}$$

Integrating the last equation leads to the equation

$$y = \frac{1}{\beta}(-b_0\beta_x - \beta_{xx} - 2\beta_x u_x) + \frac{c_0(t)}{\beta e^{2u}}. \tag{2.12}$$

The compatibility condition of the equations (2.11a) and the equation (2.12) gives rise to a new constraint which is not a consequence of the constraint $u_t = 0$ already found. So this case doesn't lead to any integrable boundary condition for the Toda chain (1.1). It is proved similarly that the case iv) doesn't produce any integrable boundary condition. In the third case the answer coincides with the already known boundary condition $e^{u(-1)} = 0$. The corresponding differential operator is

$$N = e^u D_x^3 + (2u_x + b_0)e^u D_x^2 + (u_{xx} + u_x^2 + c_0)e^u D_x.$$

Evidently the operator $M = Ne^{-u}$ coincides with M_1. ■

The differential constraints of the form (1.9) allow one to find the Lax pairs for the truncated chains. To illustrate this statement we will find the Lax pair for the Toda chain truncated by the cutting off constraint (2.2b) found in the Theorem 2. Exclude from the constraint $\phi(0) = M_2 y(0)$, where M_2 is defined by (2.3b), all x-derivatives replacing them by the shifts by means of the equations (1.3) and (1.5). The formula $D_x e^{-u} y(0) = -e^{-u} y(-1)$ helps one to shorten the computations. As a result one gets a rather simple formula

$$\phi(0) = -y(-3) + (u_x(-2) + u_x(-1))y(-2) + \mu y(-1). \tag{2.13}$$

Differentiate the last equation with respect to t and replace again the derivatives by shifts

$$\begin{aligned} -w(-1)\phi(-1) = (-w(-3) + u_{xt}(-2) + u_{xt}(-1))y(-2) + \\ + \{(u_x(-2) + u_x(-1))w(-2) + \mu'\}y(-1) + \mu w(-1)y(0). \end{aligned}$$

The coefficient before $y(-1)$ is simplified due to the boundary condition (2.2b). Finally one gets

$$\phi(-1) = y(-2) - \mu y(0). \tag{2.14}$$

Consider now the following system of equations

$$\phi_x(n) = \phi(n+1) - u_x(n)\phi(n) \qquad \text{for} \quad n \geq 0, \tag{2.15a}$$
$$\phi_t(n) = -w(n-1)\phi(n-1) \qquad \text{for} \quad n \geq 1, \tag{2.15b}$$
$$y_x(n) = -y(n-1) + u_x(n)y(n) \qquad \text{for} \quad n \geq -1, \tag{2.15c}$$
$$y_t(n) = w(n)y(n+1) \qquad \text{for} \quad n \geq -2, \tag{2.15d}$$

and two more equations (see (2.13), (2.14))

$$\phi_t(0) = -w(-1)y(-2) + w(-1)\mu y(0), \tag{2.16a}$$
$$y_x(-2) = \phi(0) - u_x(-1)y(-2) - \mu y(-1). \tag{2.16b}$$

Evidently the system of equations (2.15), (2.16) provides the Lax pair for the semi-infinite lattice (1.1), (2.2b).

3 Moutard transformation and boundary conditions

As discussed above, any differential constraint of the form (1.2), (1.9) between a linear hyperbolic equation and its conjugate generates a boundary condition for the Toda chain.

All examples considered so far were connected with differential constraints of a special form $\phi = My$, i.e. ϕ was always expressed through y and its derivatives. Below we represent a constraint of the more symmetric form. Note that in some cases it is more convenient to use the ψ-equation (2.4) instead of the y-equation (1.6) – some formulae become more symmetrical. Hence below we consider the constraint

$$\phi_x = a\phi + b\psi + c\psi_x \tag{3.1}$$

where a, b, c are functions of x, t to be determined. Take the t-derivative of both sides in (3.1) and replace the mixed derivatives by means of the equations (1.4), (2.4):

$$a_t\phi + a\phi_t + b_t\psi + b\psi_t + c_t\psi_x + c(-u_t\psi_x - w(-1)\psi) = -u_x\phi_t - w(-1)\phi. \tag{3.2}$$

Require that the differential consequence of the constraint is also of the form (3.1)

$$\psi_t = A\phi + B\psi + C\psi_t \tag{3.3}$$

so that the coefficient before ψ_x in (3.3) vanishes $c_t - u_t c = 0$. Solving this equation one gets $c = c_0(x)e^u$. Without losing generality one can take $c_0 = 1$, as the dependance of the coefficient upon t can be removed by the shift $u \to u + f(t) + g(x)$ which leaves the Toda chain unchanged.

It is remarkable that the pair of equations (3.1), (3.3) looks like the well-known Moutard transformation for hyperbolic equations [5].

Comparison of the equations (3.2) and (3.3) gives a connection between the coefficients

$$Ab + a_t + w(n-1) = 0, \tag{3.4a}$$
$$Bb + b_t - cw(n-1) = 0, \tag{3.4b}$$
$$Cb + a + u_x = 0. \tag{3.4c}$$

Take the derivative of (3.3) with respect to x and replace all mixed derivatives by means of the equations (1.4), (2.4)

$$A_x\phi + A\phi_x + B_x\psi + B\psi_x + C_x\phi_t + C(-u_x\phi_t - w(-1)\phi) = -u_t\psi_x - w(-1)\psi. \tag{3.5}$$

By comparing the equation obtained with (3.1) one immediately gets $C = C_0(t)e^u$ (choose $C_0(t) = 1$), and derives the following system of equations

$$aA + A_x - Cw(n-1) = 0, \tag{3.6a}$$
$$bA + B_x + w(n-1) = 0, \tag{3.6b}$$
$$cA + B + u_t = 0. \tag{3.6c}$$

Express the variables $B = -u_t - cA$ and $a = -u_x - Cb$ from the last equations of (3.4) and (3.6) and substitute into the others. This allows one to derive an ordinary differential equation for A

$$A_x(1 - cC) + Au_x(-1 - cC) = Cu_{xt} \tag{3.7}$$

which is easily integrated, recall that $c = C = e^u$,

$$A = -\frac{u_t + \kappa(t)}{2\sinh u}, \tag{3.8}$$

where $\kappa(t)$ is an unknown function of one variable. Then

$$B = \frac{u_t e^{-u} + \kappa(t)e^u}{2\sinh u}. \tag{3.9}$$

In a similar way one can find

$$b = -\frac{u_x + \tau(x)}{2\sinh u} \quad \text{and} \quad a = \frac{u_x e^{-u} + \tau(x)e^u}{2\sinh u}, \tag{3.10}$$

with unknown $\tau(x)$.

We have found all unknown coefficients satisfying some of equations, but the systems above are over-determined, so one has to check the validity of the other equations. Really the coefficients found satisfy all equations (3.4), (3.6) except may be (3.4a) and (3.6b). Subtract from one of them the other and get a constraint $a_t - B_x = 0$. Write it in an enlarged form

$$\frac{\kappa(t)u_t e^u}{2\sinh u} - \frac{\cosh u\kappa(t)u_t e^u}{2\sinh^2 u} = \frac{\tau(x)u_x e^u}{2\sinh u} - \frac{\cosh u\tau(x)u_x e^u}{2\sinh^2 u}. \tag{3.11}$$

The last equation shows that the equations (3.4a) and (3.6b) produce two independent constraints unless $\kappa(t) = \tau(x) \equiv 0$. But due to the Definition 1 one can impose only one constraint. So it is necessary to put both κ and τ equal to zero. Then one gets the only constraint

$$u_{xt} = -2w(n-1)e^u \sinh u + \frac{u_x u_t}{2\sinh u}e^u. \tag{3.12}$$

The constraint found corresponds to the well-known boundary condition for the chain (1.1)

$$e^{u(-1)} = e^{-u(1)} + \frac{u_x(0)u_t(0)}{2\sinh u(0)}, \tag{3.13}$$

which is connected with the finite Toda lattice of series D (see [4]).

Let us give the final form of the Moutard type constraints (3.1), (3.3) associated with the boundary conditions (3.13)

$$\phi_x = \frac{u_x e^{-u}}{2\sinh u}\phi - \frac{u_x}{2\sinh u}\psi + e^u\psi_x, \tag{3.14}$$

$$\psi_t = -\frac{u_t}{2\sinh u}\phi + \frac{u_t e^{-u}}{2\sinh u}\psi + e^u\psi_t. \tag{3.15}$$

Acknowledgments. *The author is grateful to Professor I. Habibullin for valuable discussions. The work was partially supported by RFBR grant 05-01-97910-r-agidel'-a.*

References

[1] Habibullin I T and Kazakova T G, Boundary conditions for integrable discrete chains, *J. Phys. A* **34** (2001), 10369–10376.

[2] Khabibullin I T and Gudkova E V, Boundary Conditions for Multidimensional Integrable Equations, *Func. Anal. Appl.* **38** (2004), 138–148.

[3] Kazakova T G, Finite dimensional discrete systems, integrated in quadratures. *Theor. and Math. Phys.* [in Russian] **138** (2004), 422–436.

[4] Leznov A N and Savel'ev, Group Methods for the Integration of Nonlinear Dynamical Systems [in Russian], Nauka, Moscow, 1985.

[5] Moutard T, Sur la construction des équations de la forme $\frac{1}{z}\frac{d^2z}{dxdy} = \lambda(x, y)$, qui admittent une intégrale générale explicite, *J.École Polytechnique* **45** (1878), 1–11.