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A Note on $q-$Bernoulli Numbers and Polynomials

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Abstract

In this paper, we define a new $q-$analogy of the Bernoulli polynomials and the Bernoulli numbers and we deduced some important relations of them. Also, we deduced a $q-$analogy of the Euler-Maclaurin formulas. Finally, we present a relation between the $q-$gamma function and the $q-$Bernoulli polynomials.

1 $q-$Notations

Let $q \in (0,1)$ and define the $q-$shifted factorials by
$(a,q)_0 = 1$,
$(a_1,...,a_r;q)_k = \prod_{i=1}^{r}\prod_{j=o}^{k-1}(1-a_iq^j), \quad k = 0,1,2,...$,
$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)$.
The classical exponential function e^z has two different natural $q-$extension [10] one of them denoted by $e_q(z)$ and given by

$$e_q(z) = \sum_{k=0}^{\infty}\frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty},$$

where $z \in \mathbb{C}, \mid z \mid < 1$ and $0 < q < 1$. The function $e_q(z)$ can be considered as formal power series in the formal variable z and satisfies that $\lim_{q\to1} e_q((1-q)z) = e^z$. For the q-commuting variables x and y such that $xy = qyx$ [11],

$$e_q(x+y) = e_q(y)e_q(x).$$

The $q-$difference operator D_q is defined by

$$D_qf(x) = \begin{cases} \frac{f(x)-f(qx)}{x(1-q)}, & x \neq 0 \\ \frac{df(0)}{dx}, & x = 0 \end{cases}$$

where

$$\lim_{q\to1} D_qf(x) = \frac{df(x)}{dx}.$$

Thomae [1869-1870] defined the $q-$integral on the interval $[0,1]$ [4]-[5] by

$$\int_0^1 f(t)d_qt = (1-q)\sum_{n=0}^{\infty} f(q^n)q^n.$$

Jackson [1910] extended this to the interval $[a,b]$ [4]-[5] via

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt,$$

where

$$\int_0^a f(t)d_qt = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n.$$

The $q-$analogue of $n!$ is defined by

$$[n]_q! = \begin{cases} 1, & if\ n=0 \\ [n]_q[n-1]_q...[1]_q, & if\ n=1,2,... \end{cases}$$

where $[n]_q$ is the quantum number and is given by

$$[n]_q = \frac{1-q^n}{1-q}.$$

The $q-$binomial coefficient $\binom{n}{k}_q$ is defined by

$$\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = \frac{[n]_q!}{[k]_q![n-k]_q!} \quad k=0,1,...,n.$$

2 $q-$Bernoulli polynomials

The classical Bernoulli polynomials $B_n(x)$ are defined by the generating function

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!}z^n = \frac{z}{e^z-1}e^{zx}.$$

The Bernoulli numbers are defined through the relation $B_n = B_n(0)$.
The $q-$Bernoulli polynomials $B_n(x,h|q)$ [3]- [8] are defined by $q-$generating function

$$e^{\frac{t}{1-q}}\sum_{j=0}^{\infty}\frac{j+h}{[j+h]_q}q^{jx}(-1)^j\frac{1}{(1-q)^j}\frac{t^j}{j!} = \sum_{n=0}^{\infty}\frac{B_n(x,h|q)}{n!}t^n \qquad h\in\mathbb{Z}, x\in\mathbb{C}.$$

Note that

$$\lim_{q\to 1} B_n(x,h|q) = B_n(x).$$

The $q-$Bernoulli numbers are defined through the relation

$$B_n(0,h|q) = B_n(h|q).$$

In this paper we suggest a new approach to study the $q-$Bernoulli polynomials. Let $\widehat{B}(t)$ be the generating function of the classical Bernoulli numbers [12]

$$\widehat{B}(t)=\sum_{n=0}^{\infty}\frac{B_n}{n!}z^n=\frac{z}{e^z-1}.$$

Then we get

$$\widehat{B}\left(\frac{\partial}{\partial x}\right)x^k=\sum_{n=0}^{\infty}\frac{B_n}{n!}\left(\frac{\partial}{\partial x}\right)^n x^k=\sum_{n=0}^{k}\binom{k}{n}B_n x^{k-n}.$$

Also, on exponent

$$\widehat{B}\left(\frac{\partial}{\partial x}\right)e^{tx}=\widehat{B}(t)e^{tx}=B(x;t).$$

Now we will define a $q-$analogy of the generating function $\widehat{B}(t)$ as

$$\widehat{B}_q(t)=\sum_{n=0}^{\infty}\frac{\mathbf{b}_n(q)}{[n]_q!}t^n,$$

where $\mathbf{b}_n(q)$ is a $q-$analogy of the Bernoulli numbers. By using the $q-$difference operator D_q we get

$$\begin{aligned}\widehat{B}_q\left(D_q\right)x^k &= \sum_{n=0}^{\infty}\frac{\mathbf{b}_n(q)}{[n]_q!}D_q^n x^k\\ &= \sum_{n=0}^{k}\frac{\mathbf{b}_n(q)}{[n]_q!}\frac{[k]_q!}{[k-n]_q!}x^{k-n}\\ &= \sum_{n=0}^{k}\binom{k}{n}_q\mathbf{b}_n(q)x^{k-n}.\end{aligned}$$

This procedure will suggest the following $q-$analogy of Bernoulli polynomials

$$\mathcal{B}_k(x,q)=\sum_{n=0}^{k}\binom{k}{n}_q\mathbf{b}_n(q)x^{k-n}.$$

Also,

$$\begin{aligned}\widehat{B}_q\left(D_q\right)e_q(xt) &= \sum_{n=0}^{\infty}\frac{\mathbf{b}_n(q)}{[n]_q!}D_q^n\left(\sum_{k=0}^{\infty}\frac{x^k}{(q,q)_k}t^k\right)\\ &= \sum_{k=0}^{\infty}\frac{t^k}{(q,q)_k}\sum_{n=0}^{\infty}\frac{\mathbf{b}_n(q)}{[n]_q!}D_q^n x^k\\ &= \sum_{k=0}^{\infty}\frac{t^k}{(q,q)_k}\mathcal{B}_k(x,q)=\mathcal{B}(x,t,q).\end{aligned}$$

From this point of view we can define the $q-$Bernoulli polynomials.

Definition 1. The $q-$Bernoulli polynomials $\mathcal{B}_n(x,q)$ are defined by

$$\sum_{n=0}^{\infty}\mathcal{B}_n(x,q)\frac{z^n}{(q;q)_n}=\frac{z}{(1-q)(e^{\frac{z}{1-q}}-1)}e_q(zx), \tag{2.1}$$

where $\lim_{q\longrightarrow 1}\mathcal{B}_n(x,q)=B_n(x)$, $B_n(x)$ are the ordinary Bernoulli polynomials.

Proposition 1.

$$D_q\mathcal{B}_n(x,q)=[n]_q\mathcal{B}_{n-1}(x,q). \tag{2.2}$$

Proof.

$$\begin{aligned}\sum_{n=1}^{\infty}D_q\mathcal{B}_n(x,q)\frac{z^n}{(q;q)_n} &= \frac{z}{(1-q)(e^{\frac{z}{1-q}}-1)}\frac{z}{1-q}e_q(zx)\\ &= \frac{z}{1-q}\sum_{n=0}^{\infty}\mathcal{B}_n(x,q)\frac{z^n}{(q;q)_n}\\ &= \frac{1}{1-q}\sum_{n=1}^{\infty}\mathcal{B}_{n-1}(x,q)\frac{z^n}{(q;q)_{n-1}}\\ &= \sum_{n=1}^{\infty}[n]_q\mathcal{B}_{n-1}(x,q)\frac{z^n}{(q;q)_n}.\end{aligned}$$

■

Proposition 2. *For q-commuting variables x and y such that $xy=qyx$, we have*

$$\mathcal{B}_n(x+y,q)=\sum_{i=0}^{n}\binom{n}{i}_q y^{n-i}\mathcal{B}_i(x,q). \tag{2.3}$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{B}_n(x+y,q)\frac{z^n}{(q;q)_n} &= \frac{z}{(1-q)(e^{\frac{z}{1-q}}-1)}e_q(z(x+y))\\ &= \frac{z}{(1-q)(e^{\frac{z}{1-q}}-1)}e_q(zy)e_q(zx)\\ &= e_q(zy)\left(\frac{z}{(1-q)(e^{\frac{z}{1-q}}-1)}e_q(zx)\right)\\ &= e_q(zy)\sum_{n=0}^{\infty}\mathcal{B}_n(x,q)\frac{z^n}{(q;q)_n}.\end{aligned}$$

Also,

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}_q y^{n-i}\mathcal{B}_i(x,q)\frac{z^n}{(q;q)_n} &= \sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{y^{n-i}\mathcal{B}_i(x,q)}{(q,q)_i(q,q)_{n-i}}z^n\\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{(zy)^{n-i}}{(q,q)_{n-i}}\frac{\mathcal{B}_i(x,q)}{(q,q)_i}z^i\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{\infty}\sum_{i=0}^{\infty}\frac{(zy)^l}{(q,q)_l}\frac{\mathcal{B}_i(x,q)}{(q,q)_i}z^i \\
&= e_q(zy)\sum_{n=0}^{\infty}\mathcal{B}_n(x,q)\frac{z^n}{(q;q)_n}.
\end{aligned}
$$

as desired ■

In equation (2.3), if we take the limit as $q \longrightarrow 1$. Then we get

$$B_n(x+y) = \sum_{i=0}^{n}\binom{n}{i}y^{n-i}B_i(x),$$

where $B_n(x)$ are the ordinary Bernoulli polynomials. And this relation satisfied for the ordinary Bernoulli polynomials [1].

3 $q-$Bernoulli numbers

Definition 2. For $n \geq 0$, $\mathbf{b}_n(q) = \mathcal{B}_n(0,q)$ are called $q-$Bernoulli numbers.

Lemma 1.

$$\mathbf{b}_n(q) = \frac{b_n}{n!}\frac{(q;q)_n}{(1-q)^n}, \tag{3.1}$$

where $\lim_{q\longrightarrow 1}\mathbf{b}_n(q) = b_n$, b_n *are the ordinary Bernoulli numbers .*

Proof. Putting $x = 0$ in equation (2.1), we get

$$\sum_{n=0}^{\infty}\mathbf{b}_n(q)\frac{z^n}{(q;q)_n} = \frac{z}{(1-q)(e^{\frac{z}{1-q}}-1)}$$

and replace z by $(1-q)z$, then

$$\sum_{n=0}^{\infty}\mathbf{b}_n(q)\frac{((1-q)z)^n}{(q;q)_n} = \frac{z}{e^z-1}.$$

But the ordinary Bernoulli numbers b_n satisfy

$$\sum_{n=0}^{\infty}b_n\frac{z^n}{n!} = \frac{z}{e^z-1}.$$

Then

$$\mathbf{b}_n(q) = \frac{b_n}{n!}\frac{(q;q)_n}{(1-q)^n}.$$

Also,

$$
\begin{aligned}
\lim_{q\longrightarrow 1}\mathbf{b}_n(q) &= \lim_{q\longrightarrow 1}\frac{b_n}{n!}\frac{(q;q)_n}{(1-q)^n} \\
&= \frac{b_n}{n!}(1)_n = b_n,
\end{aligned}
$$

where $(a)_n$ is the Pochhammer-symbol. ■

The knowledge of the Bernoulli numbers and the lemma (3.1) allows us to determine the $q-$Bernoulli numbers. The first five of them are:

$$\mathbf{b}_0(q) = 1, \quad \mathbf{b}_1(q) = -\frac{1}{2}, \quad \mathbf{b}_2(q) = \frac{[2]_q}{12}, \quad \mathbf{b}_3(q) = 0, \quad \mathbf{b}_4(q) = -\frac{[2]_q[3]_q[4]_q}{720}.$$

By using the properties of the ordinary Bernoulli numbers b_n [6], we can prove that
1− $\mathbf{b}_n(q) = 0 \; \forall \; n$ odd and $n \geq 3$,
2− $\sum_{j=0}^{n-1} {}^nP_j \frac{(1-q)^j}{(q;q)_j} \mathbf{b}_j(q) = 0$,
3− $\sum_{j=1}^{n-1} (-1)^j \, {}^nP_j \frac{(1-q)^{j+1}}{(q;q)_{j+1}} \mathbf{b}_{j+1}(q) = \frac{1-n}{2(1+n)}$.

Proposition 3. *For any* $n \geq 1$

$$\sum_{j=0}^{n-1} {}^nP_j \frac{(1-q)^j}{(q;q)_j} \mathcal{B}_j(x,q) = \frac{n!}{[n-1]_q!} x^{n-1}. \tag{3.2}$$

Proof. The case where $n = 1$ is obvious. If we assume that the relation is true for some $k \geq 1$, we have

$$\begin{aligned} D_q \sum_{j=0}^{k} {}^{k+1}P_j \frac{(1-q)^j}{(q;q)_j} \mathcal{B}_j(x,q) &= \sum_{j=1}^{k} {}^{k+1}P_j \frac{(1-q)^j}{(q;q)_j} [j]_q \mathcal{B}_{j-1}(x,q) \\ &= (k+1) \sum_{j=0}^{k-1} {}^kP_j \frac{(1-q)^j}{(q;q)_j} \mathcal{B}_j(x,q) \\ &= (k+1) \frac{k!}{[k-1]!_q} x^{k-1} = \frac{(k+1)!}{[k-1]!_q} x^{k-1} \\ &= D_q \left(\frac{(k+1)!}{[k]!_q} x^k \right). \end{aligned}$$

Then

$$\sum_{j=0}^{k} {}^{k+1}P_j \frac{(1-q)^j}{(q;q)_j} \mathcal{B}_j(x,q) = \frac{(k+1)!}{[k]_q!} x^k + c.$$

Put $x = 0$, then

$$\sum_{j=0}^{k} {}^{k+1}P_j \frac{(1-q)^j}{(q;q)_j} \mathbf{b}_j(q) = c.$$

Using the second property of $\mathbf{b}_j(q)$, we get $c = 0$. Hence, by induction, relation is true for any positive integer. ■

Proposition 4.

$$\mathcal{B}_n(x,q) = \sum_{i=0}^{n} \binom{n}{i}_q \mathbf{b}_i(q) x^{n-i}. \tag{3.3}$$

Proof. Let

$$F_n(x,q)=\sum_{i=0}^{n}\binom{n}{i}_q \mathbf{b}_i(q)x^{n-i}.$$

It suffices to show that (i) $F_n(0,q)=\mathbf{b}_n(q)$ for $n\geq 0$ and (ii) $D_qF_n(x,q)=[n]_qF_{n-1}(x,q)$ for any $n\geq 1$, since these two properties uniquely characterize $\mathcal{B}_n(x,q)$. The first property is obvious. As for the second property,

$$\begin{aligned}
D_qF_n(x,q) &= \frac{1}{(1-q)x}\sum_{i=0}^{n-1}\binom{n}{i}_q \mathbf{b}_i(q)x^{n-i}(1-q^{n-i})\\
&= \frac{1}{(1-q)x}\sum_{i=0}^{n-1}\frac{(q;q)_n}{(q;q)_i(q;q)_{n-i-1}}\mathbf{b}_i(q)x^{n-i}\\
&= \frac{q^n-1}{(q-1)}\sum_{i=0}^{n-1}\frac{(q;q)_{n-1}}{(q;q)_i(q;q)_{n-i-1}}\mathbf{b}_i(q)x^{n-i-1}\\
&= [n]_q\sum_{i=0}^{n-1}\binom{n-1}{i}_q \mathbf{b}_i(q)x^{n-i-1}\\
&= [n]_qF_{n-1}(x;q),
\end{aligned}$$

as desired. ■

The knowledge of $q-$Bernoulli numbers allow us to determine the $q-$Bernoulli polynomials. The five of them are listed below:

$$\begin{aligned}
\mathcal{B}_0(x,q) &= 1 \quad ,\\
\mathcal{B}_1(x,q) &= x-\frac{1}{2!},\\
\mathcal{B}_2(x,q) &= x^2-\frac{[2]_q}{2!}x+\frac{[2]_q}{2(3!)},\\
\mathcal{B}_3(x,q) &= x^3-\frac{[3]_q}{2!}x^2+\frac{[2]_q[3]_q}{2(3!)}x,\\
\mathcal{B}_4(x,q) &= x^4-\frac{[4]_q}{2!}x^3+\frac{[3]_q[4]_q}{2(3!)}x^2+\frac{[2]_q[3]_q[4]_q}{30(4!)}.
\end{aligned}$$

Lemma 2. *The $q-$Bernoulli polynomials have the following symmetry property*

$$(-1)^n\mathcal{B}_n(-x,q)=\mathcal{B}_n(x,q)+[n]_qx^{n-1},\qquad \forall n\geq 1.$$

Proof. The case where $n=1$ is obvious. If we assume that relation is true for some $k\geq 1$, we get

$$\begin{aligned}
D_q\left((-1)^{k+1}\mathcal{B}_{k+1}(-x,q)\right) &= (-1)^k[k+1]_q\mathcal{B}_k(-x,q)\\
&= [k+1]_q\mathcal{B}_k(x,q)+[k+1]_q[k]_qx^{k-1}\\
&= D_q\left(\mathcal{B}_{k+1}(x,q)+[k+1]_qx^k\right),
\end{aligned}$$

then

$$(-1)^{k+1}\mathcal{B}_{k+1}(-x,q)=\mathcal{B}_{k+1}(x,q)+[k+1]_qx^k+c.$$

Put $x = 0$, then

$$\left((-1)^{k+1} - 1\right) \mathbf{b}_{k+1}(q) = c$$

but $\left((-1)^{k+1} - 1\right) = 0$ if k is an odd number and $\mathbf{b}_{k+1}(q) = 0$ if k is an even number. Then $c = 0$ and hence, by induction, relation is true $\forall n \geq 1$. ■

Lemma 3.

$$\int_a^x \mathcal{B}_n(t,q)d_qt = \frac{\mathcal{B}_{n+1}(x,q) - \mathcal{B}_{n+1}(a,q)}{[n+1]_q}. \tag{3.4}$$

Proof. By using $D_q\mathcal{B}_n(t,q) = [n]_q\mathcal{B}_{n-1}(t,q)$, then we get

$$\begin{aligned}\int_a^x \mathcal{B}_n(t,q)d_qt &= \tfrac{1}{[n+1]_q} \int_a^x D_q\mathcal{B}_{n+1}(t,q)d_qt \\ &= \frac{1}{[n+1]_q}\mathcal{B}_{n+1}(t,q) \mid_a^x \\ &= \frac{\mathcal{B}_{n+1}(x,q) - \mathcal{B}_{n+1}(a,q)}{[n+1]_q}.\end{aligned}$$

■

4 A $q-$Euler-Maclaurin formulas

Let the function $P(x) = \mathcal{B}_1(x - [x], q)$, in which $[x]$ means the greatest integer $\leq x$. The function $P(x)$ is periodic $P(x+1) = P(x)$. Also,

$$\int_0^1 P(x)d_qx = \int_t^{t+1} P(x)d_qx = 0 \quad \forall t \geq 0.$$

We employed $P(x)$ in obtaining a $q-$analogy of the Euler-Maclaurin formulas [13].

Theorem 1.

$$\sum_{k=o}^{n} f(k) = \frac{f(n) + f(o)}{2} + \int_o^n f(qx)d_qx + \int_o^n P(x)D_qf(x)d_qx,$$

where $f(x)$ is differentiable.

Proof. First write

$$\int_o^n P(x)D_qf(x)d_qx = \sum_{k=1}^{n} \int_{k-1}^{k} P(x)D_qf(x)d_qx.$$

Now

$$\int_{k-1}^{k} P(x)D_qf(x)d_qx = \int_{k-1}^{k} (x - k + 1/2)D_qf(x)d_qx$$

and we integrate by parts to obtain

$$\int_{k-1}^{k} P(x)D_qf(x)d_qx = (x - k + 1/2)f(x) \mid_{k-1}^{k} - \int_{k-1}^{k} f(qx)D_qP(x)d_qx$$

then

$$\int_{k-1}^{k} P(x)D_q f(x)d_q x = \frac{f(k)+f(k-1)}{2} - \int_{k-1}^{k} f(qx)d_q x$$

hence

$$\int_{o}^{n} P(x)D_q f(x)d_q x = \sum_{k=o}^{n} f(k) - \frac{f(n)+f(o)}{2} - \int_{o}^{n} f(qx)d_q x$$

which is a simply rearrangement of the result in the theorem. ∎

Also, by induction we can get the following lemma

Lemma 4. *Let $f(x)$ be a differentiable function. Then $\forall r = 2, 3, 4, ...$*

$$\begin{aligned}\sum_{k=m}^{n} f(q^{r-1}k) &= \frac{f(q^{r-1}n)+f(q^{r-1}m)}{2} + \sum_{i=o}^{r-2} \frac{(-1)^{i+r}}{[r-i]_q!}\mathbf{b}_{r-i}(q)[f(q^{r-i-1}n) - f(q^{r-i-1}m)] \\ &+ \int_{m}^{n} f(q^r x)d_q x + \frac{(-1)^{r+1}}{[r]_q!}\int_{m}^{n} \mathcal{B}_r(x-[x],q)D_q^r f(x)d_q x.\end{aligned}$$

5 A relation between $\mathcal{B}_n(x, q)$ and $\Gamma_q(x)$

The $q-$gamma function [5]-[2]

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} \qquad 0 < q < 1,$$

was introduced by Thomae [1869] and later by Jackson [1904].
By using the definition of e_q we can see that

$$\Gamma_q(x+1) = (q;q)_\infty(1-q)^{-x}e_q(q^{x+1}).$$

Also, if we replace x by q^x and z by q in equation (2.1), then we have

$$\sum_{n=0}^{\infty} \mathcal{B}_n(q^x, q)\frac{q^n}{(q;q)_n} = \frac{q/(1-q)}{e^{q/(1-q)}-1}e_q(q^{x+1}).$$

Then we get the following relation between $\mathcal{B}_n(x, q)$ and $\Gamma_q(x)$

$$\Gamma_q(x+1) = (e^{q/(1-q)}-1)(q;q)_\infty(1-q)^{1-x}\sum_{n=0}^{\infty} \mathcal{B}_n(q^x, q)\frac{q^{n-1}}{(q;q)_n},$$

and then $q-$gamma function is a generating function of the $q-$Bernoulli polynomials.

References

[1] Andrews G E, Askey R and Roy R, Special Functions, Cambridge University Press, Cambridge (1999).

[2] Askey R, A Brief Introduction to the World of q, Centre de Recherches Mathématiques CRM Proceedings and Lecture Notes Vol. 9, (1996).

[3] Carlitz L, $q-$Bernoulli numbers and polynomials, *Duke Math. J.* **15** (1948), 987–1000.

[4] Gasper G, Lecture notes for an introductory minicourse on $q-$series, `arXiv:math.CA/9509223`.

[5] Gasper G and Rahman M, Basic Hypergeometric Series, Cambridge University Press (1990).

[6] Kac V and Cheung P, Quantum Calculus, Springer-Verlag, New-York (2002).

[7] Kim T, Non-Arichimedean $q-$integrals associated with multipe Changhee $q-$Bernoulli polynomials, *Russ. J. Math. Phys.* **10** (2003), 91–98.

[8] Kim T and Rim S H, Generalized Carlitz's $q-$Bernoulli Numbers in the $p-$adic number field, *Adv. Stud. Contemp. Math.* **2** (2000), 9–19.

[9] Kim T, Jang L C , Rim S H and Pak H K, On the twisted $q-$Zeta functions and $q-$Bernoulli polynomials, *Far East J. Applied Math.* **13** (2003), 13–21.

[10] Koekoek R and Swarttouw R F, The Askey-scheme of hypergeometric orthogonal polynomials and its $q-$analogue, Report 94-05,Technical University Delft (1994) available from http://aw.twi.tudelft.nl/˜ koekoek/

[11] Koornwinder T H, Special Functions and $q-$Commuting Variables, *Fields Institute Communications* **14** (1997), 131–166.

[12] Kupershmidt B A, Reflection Symmetries of $q-$Bernoulli polynomials, *J. Nonlinear Math. Phys.* **12** (2005), 412–422.

[13] Rainville E D, Special functions, The Macmillan company, (1965).

[14] Richardson C H, An introduction to the Calculus of finite differences, D. Van Nostrand Company, (1963).