A Note on $q$–Bernoulli Numbers and Polynomials

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Abstract

In this paper, we define a new $q$–analogy of the Bernoulli polynomials and the Bernoulli numbers and we deduced some important relations of them. Also, we deduced a $q$–analogy of the Euler-Maclaurin formulas. Finally, we present a relation between the $q$–gamma function and the $q$–Bernoulli polynomials.

1 $q$–Notations

Let $q \in (0, 1)$ and define the $q$–shifted factorials by

$$(a, q)_0 = 1,$$

$$(a_1, ..., a_r; q)_k = \prod_{i=1}^{r} \prod_{j=0}^{k-1} (1 - a_i q^j), \quad k = 0, 1, 2, ... ,$$

$$(a; q)_\infty = \prod_{i=0}^{\infty} (1 - a q^i).$$

The classical exponential function $e^z$ has two different natural $q$–extension [10] one of them denoted by $e_q(z)$ and given by

$$e_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{(q; q)_k} = \frac{1}{(z; q)_\infty},$$

where $z \in \mathbb{C}, |z| < 1$ and $0 < q < 1$. The function $e_q(z)$ can be considered as formal power series in the formal variable $z$ and satisfies that $\lim_{q \to 1} e_q((1-q)z) = e^z$. For the $q$-commuting variables $x$ and $y$ such that $xy = qyx$ [11],

$$e_q(x + y) = e_q(y)e_q(x).$$

The $q$–difference operator $D_q$ is defined by

$$D_q f(x) = \begin{cases} 
\frac{f(x) - f(qx)}{x(1-q)}, & x \neq 0 \\
\frac{df(0)}{dx}, & x = 0
\end{cases}$$

where

$$\lim_{q \to 1} D_q f(x) = \frac{df(x)}{dx}.$$
Thomae [1869-1870] defined the $q$–integral on the interval $[0,1]$ by
\[
\int_0^1 f(t) dq_t = (1-q) \sum_{n=0}^{\infty} f(q^n) q^n.
\]
Jackson [1910] extended this to the interval $[a,b]$ via
\[
\int_a^b f(t) dq_t = \int_0^b f(t) dq_t - \int_0^a f(t) dq_t,
\]
where
\[
\int_0^a f(t) dq_t = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n.
\]
The $q$–analogue of $n!$ is defined by
\[
[n]_q! = \begin{cases} 
1, & \text{if } n = 0 \\
[n]_q[n-1]_q...[1]_q, & \text{if } n = 1, 2, ...
\end{cases}
\]
where $[n]_q$ is the quantum number and is given by
\[
[n]_q = \frac{1-q^n}{1-q}.
\]
The $q$–binomial coefficient $\binom{n}{k}_q$ is defined by
\[
\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = \frac{[n]_q!}{[k]_q! [n-k]_q!} \quad k = 0, 1, ..., n.
\]

2 $q$–Bernoulli polynomials

The classical Bernoulli polynomials $B_n(x)$ are defined by the generating function
\[
\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n = \frac{z}{e^z - 1} e^{xz}.
\]
The Bernoulli numbers are defined through the relation $B_n = B_n(0)$.
The $q$–Bernoulli polynomials $B_n(x, h|q)$ [3]-[8] are defined by $q$–generating function
\[
e^{\frac{t}{1-q} - \frac{t}{q}} \sum_{j=0}^{\infty} \frac{j + h}{[j+h]|q} q^{jx} (-1)^j \frac{1}{(1-q)^j j!} = \sum_{n=0}^{\infty} \frac{B_n(x, h|q)}{n!} t^n \quad h \in \mathbb{Z}, x \in \mathbb{C}.
\]
Note that
\[
\lim_{q \to 1} B_n(x, h|q) = B_n(x).
\]
The $q$–Bernoulli numbers are defined through the relation
\[
B_n(0, h|q) = B_n(h|q).
\]
A Note on $q$–Bernoulli Numbers and Polynomials

In this paper we suggest a new approach to study the $q$–Bernoulli polynomials. Let $\hat{B}(t)$ be the generating function of the classical Bernoulli numbers $[12]$:

$$\hat{B}(t) = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = \frac{z}{e^z - 1}. $$

Then we get

$$\hat{B} \left( \frac{\partial}{\partial x} \right) x^k = \sum_{n=0}^{\infty} \frac{B_n}{n!} \left( \frac{\partial}{\partial x} \right)^n x^k = \sum_{n=0}^{k} \binom{k}{n} B_n x^{k-n}. $$

Also, on exponent

$$\hat{B} \left( \frac{\partial}{\partial x} \right) e^{tx} = \hat{B}(t) e^{tx} = B(x; t).$$

Now we will define a $q$–analogy of the generating function $\hat{B}(t)$ as

$$\hat{B}_q(t) = \sum_{n=0}^{\infty} \frac{b_n(q)}{n!} t^n,$$

where $b_n(q)$ is a $q$–analogy of the Bernoulli numbers. By using the $q$–difference operator $D_q$ we get

$$\hat{B}_q(D_q) x^k = \sum_{n=0}^{\infty} \frac{b_n(q)}{[n]_q!} D_q^n x^k$$

$$= \sum_{n=0}^{k} \binom{k}{n} \frac{[k]_q!}{[n]_q! [k-n]_q!} x^{k-n}$$

$$= \sum_{n=0}^{k} \binom{k}{n} q^n b_n(q) x^{k-n}.$$ 

This procedure will suggest the following $q$–analogy of Bernoulli polynomials

$$B_k(x, q) = \sum_{n=0}^{k} \binom{k}{n} q^n b_n(q) x^{k-n}.$$ 

Also,

$$\hat{B}_q(D_q) e_q(x t) = \sum_{n=0}^{\infty} \frac{b_n(q)}{[n]_q!} D_q^n \left( \sum_{k=0}^{\infty} \frac{x^k}{(q, q)_k^t} \right)$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{(q, q)_k} \sum_{n=0}^{\infty} \frac{b_n(q)}{[n]_q!} D_q^n x^k$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{(q, q)_k} B_k(x, q) = B(x, t, q).$$

From this point of view we can define the $q$–Bernoulli polynomials.
Definition 1. The $q$–Bernoulli polynomials $B_n(x, q)$ are defined by

$$
\sum_{n=0}^{\infty} B_n(x, q) \frac{z^n}{(q; q)_n} = \frac{z}{(1-q)(e^{\frac{z}{1-q}} - 1)} e_q(zx),
$$

(2.1)

where $\lim_{q \to 1} B_n(x, q) = B_n(x)$, $B_n(x)$ are the ordinary Bernoulli polynomials.

Proposition 1.

$$D_q B_n(x, q) = [n]_q B_{n-1}(x, q).$$

(2.2)

Proof.

$$
\sum_{n=1}^{\infty} D_q B_n(x, q) \frac{z^n}{(q; q)_n} = \frac{z}{(1-q)(e^{\frac{z}{1-q}} - 1)} 1 - q
eq e_q(zx)
$$

$$= \frac{z}{1-q} \sum_{n=0}^{\infty} B_n(x, q) \frac{z^n}{(q; q)_n}
$$

$$= \frac{1}{1-q} \sum_{n=1}^{\infty} B_{n-1}(x, q) \frac{z^n}{(q; q)_{n-1}}
$$

$$= \sum_{n=1}^{\infty} [n]_q B_{n-1}(x, q) \frac{z^n}{(q; q)_n}.
$$

Proposition 2. For $q$–commuting variables $x$ and $y$ such that $xy = qyx$, we have

$$B_n(x + y, q) = \sum_{i=0}^{n} \binom{n}{i} q^{n-i} B_i(x, q).$$

(2.3)

Proof.

$$
\sum_{n=0}^{\infty} B_n(x + y, q) \frac{z^n}{(q; q)_n} = \frac{z}{(1-q)(e^{\frac{z}{1-q}} - 1)} e_q(z(x + y))
$$

$$= \frac{z}{(1-q)(e^{\frac{z}{1-q}} - 1)} e_q(zy)e_q(zx)
$$

$$= e_q(zy) \left( \frac{z}{(1-q)(e^{\frac{z}{1-q}} - 1)} e_q(zx) \right)
$$

$$= e_q(zy) \sum_{n=0}^{\infty} B_n(x, q) \frac{z^n}{(q; q)_n}.
$$

Also,

$$
\sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} q^{n-i} B_i(x, q) \frac{z^n}{(q; q)_n} = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{y^{n-i} B_i(x, q)}{(q, q)_i(q, q)_{n-i}} z^n
$$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{n} (zy)^{n-i} B_i(x, q) \frac{z^n}{(q, q)_i(q, q)_{n-i}}.
$$
\[
\begin{align*}
&= \sum_{l=0}^{\infty} \sum_{i=0}^{\infty} (zy)^l \frac{B_i(x, q)}{(q, q)_i} z^i \\
&= e_q(zy) \sum_{n=0}^{\infty} B_n(x, q) \frac{z^n}{(q; q)_n},
\end{align*}
\]
as desired.

In equation (2.3), if we take the limit as \(q \to 1\). Then we get
\[
B_n(x + y) = \sum_{i=0}^{n} \binom{n}{i} y^{n-i} B_i(x),
\]
where \(B_n(x)\) are the ordinary Bernoulli polynomials. And this relation satisfied for the ordinary Bernoulli polynomials [1].

### 3 \(q\)-Bernoulli numbers

**Definition 2.** For \(n \geq 0\), \(b_n(q) = B_n(0, q)\) are called \(q\)-Bernoulli numbers.

**Lemma 1.**
\[
b_n(q) = \frac{b_n(q; q)_n}{n! (1 - q)^n},
\]
(3.1)
\[ \text{where } \lim_{q \to 1} b_n(q) = b_n, b_n \text{ are the ordinary Bernoulli numbers}. \]

**Proof.** Putting \(x = 0\) in equation (2.1), we get
\[
\sum_{n=0}^{\infty} b_n(q) \frac{z^n}{(q; q)_n} = \frac{z}{(1 - q)(e^{z/q} - 1)}
\]
and replace \(z\) by \((1 - q)z\), then
\[
\sum_{n=0}^{\infty} b_n(q) \frac{(1 - q)z^n}{(q; q)_n} = \frac{z}{e^z - 1}.
\]
But the ordinary Bernoulli numbers \(b_n\) satisfy
\[
\sum_{n=0}^{\infty} b_n \frac{z^n}{n!} = \frac{z}{e^z - 1}.
\]
Then
\[
b_n(q) = \frac{b_n(q; q)_n}{n! (1 - q)^n}.
\]
Also,
\[
\lim_{q \to 1} b_n(q) = \lim_{q \to 1} \frac{b_n(q; q)_n}{n! (1 - q)^n} = \frac{b_n}{n!} (1) = b_n,
\]
where \((a)_n\) is the Pochhammer-symbol.
The knowledge of the Bernoulli numbers and the lemma (3.1) allows us to determine the \( q \)-Bernoulli numbers. The first five of them are:

\[
b_0(q) = 1, \quad b_1(q) = -\frac{1}{2}, \quad b_2(q) = \frac{[2]_q}{12}, \quad b_3(q) = 0, \quad b_4(q) = -\frac{[2]_q[3]_q[4]_q}{720}.
\]

By using the properties of the ordinary Bernoulli numbers \( b_n [6] \), we can prove that

1. \( 1 - b_n(q) = 0 \) \( \forall \ n \) odd and \( n \geq 3 \),
2. \( 2 - \sum_{j=0}^{n-1} P_j \frac{(1-q)^j}{(q;q)_j} \cdot b_j(q) = 0 \),
3. \( 3 - \sum_{j=1}^{n-1} (-1)^j \cdot n P_j \frac{(1-q)^j+1}{(q;q)_j+1} \cdot b_{j+1}(q) = \frac{1-n}{2(1+n)} \).

**Proposition 3.** For any \( n \geq 1 \)

\[
\sum_{j=0}^{n-1} n P_j \frac{(1-q)^j}{(q;q)_j} B_j(x, q) = \frac{n!}{(n-1)!} x^{n-1}.
\]

**Proof.** The case where \( n = 1 \) is obvious. If we assume that the relation is true for some \( k \geq 1 \), we have

\[
D_q \sum_{j=0}^{k} k+1 P_j \frac{(1-q)^j}{(q;q)_j} B_j(x, q) = \sum_{j=1}^{k} k+1 P_j \frac{(1-q)^j}{(q;q)_j} \frac{j}{q} B_{j-1}(x, q)
\]

\[
= (k+1) \sum_{j=0}^{k-1} k P_j \frac{(1-q)^j}{(q;q)_j} B_j(x, q)
\]

\[
= (k+1) \frac{k!}{[k-1]! q} x^{k-1} = \frac{(k+1)!}{[k-1]! q} x^{k-1}
\]

\[
= D_q \left( \frac{(k+1)!}{[k]! q} x^k \right).
\]

Then

\[
\sum_{j=0}^{k} k+1 P_j \frac{(1-q)^j}{(q;q)_j} B_j(x, q) = \frac{(k+1)!}{[k]! q} x^k + c.
\]

Put \( x = 0 \), then

\[
\sum_{j=0}^{k} k+1 P_j \frac{(1-q)^j}{(q;q)_j} b_j(q) = c.
\]

Using the second property of \( b_j(q) \), we get \( c = 0 \). Hence, by induction, relation is true for any positive integer. \( \blacksquare \)

**Proposition 4.**

\[
B_n(x, q) = \sum_{i=0}^{n} \binom{n}{i} q b_i(q)x^{n-i}.
\]
Proof. Let
\[ F_n(x, q) = \sum_{i=0}^{n} \binom{n}{i}_q b_i(q)x^{n-i}. \]

It suffices to show that (i) \( F_n(0, q) = b_n(q) \) for \( n \geq 0 \) and (ii) \( D_q F_n(x, q) = \lfloor n \rfloor_q F_{n-1}(x, q) \) for any \( n \geq 1 \), since these two properties uniquely characterize \( B_n(x, q) \). The first property is obvious. As for the second property,
\[
D_q F_n(x, q) = \frac{1}{(1-q)x} \sum_{i=0}^{n-1} \binom{n}{i}_q b_i(q)x^{n-i}(1-q^{n-i})
\]
\[
= \frac{1}{(1-q)x} \sum_{i=0}^{n-1} \frac{(q; q)_n}{(q; q)_i(q; q)_{n-i-1}} b_i(q)x^{n-i}
\]
\[
= \frac{q^n - 1}{(q-1)} \sum_{i=0}^{n-1} \frac{(q; q)_{n-i}}{(q; q)_i(q; q)_{n-i-1}} b_i(q)x^{n-i-1}
\]
\[
= \lfloor n \rfloor_q \sum_{i=0}^{n-1} \binom{n-i}{i}_q b_i(q)x^{n-i-1}
\]
\[
= \lfloor n \rfloor_q F_{n-1}(x; q),
\]
as desired. \( \square \)

The knowledge of \( q \)-Bernoulli numbers allow us to determine the \( q \)-Bernoulli polynomials. The five of them are listed below:

\[
B_0(x, q) = 1,
\]
\[
B_1(x, q) = x - \frac{1}{2!},
\]
\[
B_2(x, q) = x^2 - \frac{[2]_q}{2!} x + \frac{[2]_q}{2(3)!},
\]
\[
B_3(x, q) = x^3 - \frac{[3]_q}{2!} x^2 + \frac{[2]_q[3]_q}{2(3)!} x,
\]
\[
B_4(x, q) = x^4 - \frac{[4]_q}{2!} x^3 + \frac{[3]_q[4]_q}{2(3)!} x^2 + \frac{[2]_q[3]_q[4]_q}{30(4)!}.
\]

Lemma 2. The \( q \)-Bernoulli polynomials have the following symmetry property
\[
(-1)^n B_n(-x, q) = B_n(x, q) + \lfloor n \rfloor_q x^{n-1}, \quad \forall n \geq 1.
\]

Proof. The case where \( n = 1 \) is obvious. If we assume that relation is true for some \( k \geq 1 \), we get
\[
D_q \left( (-1)^{k+1} B_{k+1}(-x, q) \right) = (-1)^k[1+k]_q B_k(-x, q)
\]
\[
= [k+1]_q B_k(x, q) + [k+1]_q[k]_q x^{k-1}
\]
\[
= D_q \left( B_{k+1}(x, q) + [k+1]_q x^k \right),
\]
then
\[
(-1)^{k+1} B_{k+1}(-x, q) = B_{k+1}(x, q) + [k+1]_q x^k + c.
\]
Put $x = 0$, then
\[
\left( (-1)^{k+1} - 1 \right) b_{k+1}(q) = c
\]
but \((-1)^{k+1} - 1\) = 0 if $k$ is an odd number and $b_{k+1}(q) = 0$ if $k$ is an even number. Then $c = 0$ and hence, by induction, relation is true $\forall n \geq 1$. □

Lemma 3.
\[
\int_a^x B_n(t, q) d_q t = \frac{B_{n+1}(x, q) - B_{n+1}(a, q)}{[n + 1]_q}. \tag{3.4}
\]

Proof. By using $D_q B_n(t, q) = [n]_q B_{n-1}(t, q)$, then we get
\[
\int_a^x B_n(t, q) d_q t = \frac{1}{[n+1]_q} \int_a^x D_q B_{n+1}(t, q) d_q t
\]
\[
= \frac{1}{[n+1]_q} B_{n+1}(t, q) \bigg|_a^x
\]
\[
= \frac{B_{n+1}(x, q) - B_{n+1}(a, q)}{[n+1]_q}.
\]

\[\Box\]

4 A $q$–Euler-Maclaurin formulas

Let the function $P(x) = B_1(x - [x], q)$, in which $[x]$ means the greatest integer $\leq x$. The function $P(x)$ is periodic $P(x + 1) = P(x)$. Also,
\[
\int_0^1 P(x) d_q x = \int_t^{t+1} P(x) d_q x = 0 \quad \forall t \geq 0.
\]

We employed $P(x)$ in obtaining a $q$–analogy of the Euler-Maclaurin formulas \cite{13}.

Theorem 1.
\[
\sum_{k=0}^n f(k) = \frac{f(n) + f(a)}{2} + \int_a^n f(qx) d_q x + \int_a^n P(x) D_q f(x) d_q x,
\]

where $f(x)$ is differentiable.

Proof. First write
\[
\int_a^n P(x) D_q f(x) d_q x = \sum_{k=1}^n \int_{k-1}^k P(x) D_q f(x) d_q x.
\]

Now
\[
\int_{k-1}^k P(x) D_q f(x) d_q x = \int_{k-1}^k (x - k + 1/2) D_q f(x) d_q x
\]
and we integrate by parts to obtain
\[
\int_{k-1}^k P(x) D_q f(x) d_q x = (x - k + 1/2) f(x) \bigg|_{k-1}^k - \int_{k-1}^k f(qx) D_q P(x) d_q x
\]
then
\[ \int_{k-1}^k P(x)D_q f(x)d_q x = \frac{f(k) + f(k-1)}{2} - \int_{k-1}^k f(qx)d_q x \]
hence
\[ \int_{0}^n P(x)D_q f(x)d_q x = \sum_{k=0}^{n} f(k) - \frac{f(n) + f(o)}{2} - \int_{0}^n f(qx)d_q x \]
which is a simply rearrangement of the result in the theorem. ■

Also, by induction we can get the following lemma

**Lemma 4.** Let \( f(x) \) be a differentiable function. Then \( \forall r = 2, 3, 4, \ldots \)

\[
\sum_{k=m}^{n} f(q^{r-1}k) = \frac{f(q^{r-1}n) + f(q^{r-1}m)}{2} + \sum_{i=0}^{r-2} \frac{(-1)^{i+r}}{[r-i]q^i} b_{r-i}(q)[f(q^{r-i-1}n) - f(q^{r-i-1}m)] 
+ \int_{m}^{n} f(q^r x)d_q x + \frac{(-1)^{r+1}}{[r]q^i} \int_{m}^{n} B_r(x-[x],q)D_q f(x)d_q x.
\]

### 5 A relation between \( B_n(x, q) \) and \( \Gamma_q(x) \)

The \( q \)-gamma function \([5]-[2]\]

\[
\Gamma_q(x) = \frac{(q; q)_\infty}{(q^r; q)_\infty} (1 - q)^{1-x} \quad 0 < q < 1,
\]

was introduced by Thomae [1869] and later by Jackson [1904]. By using the definition of \( e_q \) we can see that

\[
\Gamma_q(x + 1) = (q; q)_\infty (1 - q)^{-x} e_q(q^{x+1}).
\]

Also, if we replace \( x \) by \( q^x \) and \( z \) by \( q \) in equation (2.1), then we have

\[
\sum_{n=0}^{\infty} B_n(q^x, q) \frac{q^n}{(q; q)_n} = \frac{q/(1 - q)}{e^{q/(1 - q)} - 1} e_q(q^{x+1}).
\]

Then we get the following relation between \( B_n(x, q) \) and \( \Gamma_q(x) \)

\[
\Gamma_q(x + 1) = (e^{q/(1 - q)} - 1)(q; q)_\infty (1 - q)^{1-x} \sum_{n=0}^{\infty} B_n(q^x, q) \frac{q^{n-1}}{(q; q)_n},
\]

and then \( q \)-gamma function is a generating function of the \( q \)-Bernoulli polynomials.

### References


