

Properties of the Dominant Behaviour of Quadratic Systems

A MAHARAJ^a and P G L LEACH^b

*School of Mathematical Sciences, Howard College, University of KwaZulu-Natal,
Durban 4041, South Africa*

^a *E-mail: 981169617@ukzn.ac.za*

^b *E-mail: leachp@ukzn.ac.za*

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Abstract

We study the dominant terms of systems of Lotka-Volterra-type which arise in the the mathematical modelling of the evolution of many divers natural systems from the viewpoint of both symmetry and singularity analyses. The connections between an increase in the amount of symmetry possessed by the system and the possession of the Painlevé Property are noted. For specific values of the parameters of the system we see that possession of the Painlevé Property is characterised by a Left Painlevé Series rather than the more standard Right Painlevé Series.

1 Introduction

In the modelling of the time-evolution of many divers natural systems a system of first-order equations is obtained of the form

$$\dot{x}_i = a_{ij}x_j + b_{ijk}x_jx_k, \quad i, j, k = 1, n, \quad (1.1)$$

where the terms linear in the dependent variables represent rates of change of the ‘population’ proportional to the population and the terms quadratic in the dependent variables represent rates of change proportional to interactions between different components of the ‘population’. The parameters of proportionality, a_{ij} and b_{ijk} , may be time-dependent, although in applications they are frequently taken as constant to enable an easier analysis. Typically a_{ij} is diagonal and represents the most elementary model of population growth which was introduced by Malthus in 1798 [20]. The quadratic, interaction, terms include the self-interaction term introduced by Verhulst in 1838 [32], known as the logistic term, to counter the absurdity of the continued exponential growth displayed by the Malthusian model. The interaction with other ‘species’ was introduced in the context of chemical reactions by Lotka in 1925 [19] and in the context of the populations of competing species of fishes by Volterra in 1926 [33]. Since then this type of representation of the interaction of two components of a system has been extended to many areas in Ecology, Epidemiology, Economics and Medicine. As Murray observes [25] [p 327] in his discussion of a model

of the incidence of gonorrhoea, an interaction term of this type implicitly assumes free mixing of the elements of each component of the system. The possibility of explicit time-dependence in the parameters of the system under consideration is frequently admitted in the statement of the model to be investigated and then abandoned in the analysis since the time-dependence plays havoc with the standard methods of dynamical systems (see, for example, Murray's analysis of a model for HIV-AIDS [25]).

Lotka-Volterra and the related quadratic systems have received much attention, mainly from the viewpoint of dynamical systems for which they are textbook examples (see, for example, Murray [25] [Chapter3]) although there have been some investigations based on the Carleman method [3, 4] and a combination of Lie and Painlevé analyses [10]. These analyses have been devoted to systems of the type (1.1). Golubchik and Sokolov [12, 13] introduced a class of integrable quadratic systems in the context of the Yang-Baxter equations and Cotsakis *et al* [16] analysed exemplars of this class in terms of their singularity and symmetry properties.

In this paper we wish to examine a class of equations of the type of (1.1) in terms of the properties of the dominant terms in the system. Thus we consider the specific system

$$\dot{x} = x(x + by) \tag{1.2}$$

$$\dot{y} = y(ax + y) \tag{1.3}$$

in which a and b are constant parameters. The linear terms of (1.1) are omitted since they are not dominant terms. We allow for self-interaction terms and cross-species-interaction terms as factors affecting the rate of change of the two dependent variables. We do not consider the possibility of other-species-interaction terms as affecting the rate of change, *ie* we consider a specific type of quadratic system. The number of possible parameters in the system is reduced to two by means of rescalings of the variables. To keep the discussion compact we treat a system of just two equations.

The system (1.2,1.3) differs from the system (1.1) – apart from a possible difference in the number of constituent equations – in that the linear terms of the latter are missing from the former. The derivatives and quadratic terms of (1.1) are the dominant terms of the system in terms of singularity analysis. In this analysis the linear terms may cause a loss of integrability in the sense of Painlevé by the introduction of an inconsistency at the nongeneric resonance. The system (1.2,1.3) is automatically integrable in the sense of Lie since it possesses the two obvious point symmetries of invariance under time translation and rescaling. The additional linear terms of (1.1) can break this symmetry. Our aim is to investigate the conditions under which the quadratic system is integrable in the sense of Painlevé for it is from the integrability of the basic quadratic system that the possible integrability of the full system follows. In this there is a certain amusing point since the basis of these models originated in the linear terms.

We examine the system (1.2,1.3) in terms of its singular behaviour both as the system stands and as it can be rewritten in terms of a single second-order equation. We identify specific values of the parameters, a and b , for which the Painlevé criteria for integrability are satisfied. The system as written in terms of a single second-order equation is examined for its Lie point symmetries and again critical values of the parameters are identified. In our concluding remarks we compare the results of the two methods of analysis.

2 Singularity analysis of the two-dimensional system

We commence our investigation with the singularity analysis of the given system, (1.2,1.3). To obtain the leading-order behaviour we substitute

$$x = \alpha\tau^p \quad y = \beta\tau^q \quad (2.1)$$

into (1.2,1.3) to obtain

$$\begin{aligned} \alpha p\tau^{p-1} &= \alpha^2\tau^{2p} + b\alpha\beta\tau^{p+q} \\ \beta q\tau^{q-1} &= a\alpha\beta\tau^{p+q} + \beta^2\tau^{2q} \end{aligned}$$

from which it is evident that $p = q = -1$ and that the coefficients of the leading-order terms, α and β , are solutions of the system

$$\begin{bmatrix} 1 & b \\ a & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (2.2)$$

and so

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{1-b}{ab-1} \\ \frac{1-a}{ab-1} \end{bmatrix}, \quad ab \neq 1. \quad (2.3)$$

The case $ab = 1$ is treated below as one of the special cases.

We substitute

$$\begin{aligned} x &= \alpha\tau^{-1} + \mu\tau^{r-1} \\ y &= \beta\tau^{-1} + \nu\tau^{r-1} \end{aligned}$$

into (1.2,1.3) to determine the resonances which are the eigenvalues of the equation

$$\begin{bmatrix} r-1-2\alpha-b\beta & -\alpha b \\ -a\beta & r-1-a\alpha-2\beta \end{bmatrix} \begin{bmatrix} \mu \\ \nu \end{bmatrix} = 0. \quad (2.4)$$

From (2.4) it follows that

$$r = -1, \quad \frac{(1-b)(1-a)}{(ab-1)}, \quad ab \neq 1. \quad (2.5)$$

Since all terms are dominant in the system (1.2,1.3), the Painlevé Test is passed for all values of the parameters a and b for which the nongeneric resonance is an integer. We return to this aspect below.

In the case that $ab = 1$ the singularity analysis of the system (1.2,1.3) differs from the analysis given above. Now the coefficient matrix in (2.2) is singular. We require consistency in the augmented matrix. The only possibility is that $a = 1$ and $b = 1$. It is not possible for the system to be consistent if $ab = 1$ and a and b take other values. In terms of the singularity analysis two of the coefficients of the leading-order terms are

related according to $\alpha + \beta = -1$, *ie* the second arbitrary constant enters at the leading order terms, the resonances are $r = 0, -1$ and the system possesses the Painlevé Property for these particular values of the parameters.

The system

$$\begin{aligned}\dot{x} &= x(x + y) \\ \dot{y} &= y(x + y)\end{aligned}\tag{2.6}$$

is an example of a decomposed system [1, 2] in that it can be composed as

$$\dot{w} = w^2,\tag{2.7}$$

where $w = x + y$. Equation (2.7) has the form of a Riccati equation, a Bernoulli equation and a variables separable equation. The solution is simply

$$w = x + y = \frac{1}{t_0 - t}.$$

Equally we could write

$$\frac{d(x - y)}{dt} = (x - y)w$$

so that

$$x - y = \frac{K}{t_0 - t},$$

where K is a second constant of integration, and the solution

$$x = \frac{1 + K}{2(t_0 - t)}, \quad y = \frac{1 - K}{2(t_0 - t)}\tag{2.8}$$

follows.

We observe that this is an instance in which the solution of (1.1), given by (2.8), possesses both Left and Right Painlevé Series. The latter is trivial since it consists of just the single term of the solution. The former is a true Laurent series convergent in the open disc centred on t_0 defined by $|t| > |t_0|$ with the limits on the summation being -1 to $-\infty$.

3 Singularity analysis of the related second-order equation

From (1.2)

$$y = \frac{\dot{x}}{bx} - \frac{x}{b}\tag{3.1}$$

and we substitute this into (1.3) to obtain the second-order equation

$$x\ddot{x} - \dot{x}^2 \left(1 + \frac{1}{b}\right) - x^2\dot{x} \left(1 - \frac{2}{b} + a\right) + x^4 \left(a - \frac{1}{b}\right) = 0.\tag{3.2}$$

We begin the singularity analysis of (3.2) in the usual way by substituting $x = \alpha\tau^p$ to determine the leading order behaviour. We find that

$$p = -1 \quad \text{and} \quad \alpha = \begin{cases} -1 \\ \frac{1-b}{ab-1} \end{cases}\tag{3.3}$$

ie the coefficient of $x(\tau)$ obtained in the analysis of the system (1.2,1.3) and a second. The equation for the coefficients of the leading-order term is of the fourth degree, but two of the roots are zero and so are automatically excluded. In general nontrivial coefficients for the leading-order term are determined by a polynomial of degree given by the difference between the highest and lowest powers of x , including derivatives, in the equation and their number has nothing to do with the order of the equation.

In the case that $\alpha = (1 - b)/(ab - 1)$ we find that the resonances are given by

$$r = -1 \quad \text{and} \quad r = \frac{(1 - b)(1 - a)}{ab - 1}, \quad ab \neq 1, \tag{3.4}$$

which are the same as those obtained for the two-dimensional system, (1.2,1.3).

When $\alpha = -1$, we find that the resonances are given by

$$r = -1 \quad \text{and} \quad r = 1 - a. \tag{3.5}$$

For the second branch it is evident from (3.5) that a must be an integer for this branch to pass the Painlevé Test. Suppose that $r = m$. Then $a = 1 - m$. For the first branch to pass the Painlevé Test the nongeneric resonance in (3.4) must also be an integer, say n . Then $b = (m + n)/(m + n - mn)$. Note that b cannot be zero in this reduction. Apart from the restrictions $m + n \neq 0$ and $mn \neq m + n$ the integers m and n may take any positive or negative values and so the two principal branches can represent any two combinations of Left Painlevé Series and Right Painlevé Series [9].

The root $\alpha = -1$ is a valid option for (3.2), but must be discarded for the system (1.2,1.3) since it corresponds to $\beta = 0$ and consequently violates the assumption of leading-order behaviour. Were one concerned only with (3.2) the analysis would have to include this branch.

Equally we could have solved (1.3) for x and substituted this into (1.2) to obtain a second-order equation for y . The resulting equation is the same as (3.2) up to the interchanges $x \leftrightarrow y$ and $a \leftrightarrow b$. There is no further information to be gained.

4 Symmetry analysis of the related second-order equation

We use Program LIE's [14, 29] capacity to indicate divisors by factors of combinations of the parameters in the symmetry analysis of (3.2), *videlicet*

$$x\ddot{x} - \dot{x}^2 \left(1 + \frac{1}{b}\right) - x^2\dot{x} \left(1 - \frac{2}{b} + a\right) + x^4 \left(a - \frac{1}{b}\right) = 0, \tag{4.1}$$

to determine any special values of the parameters for which the symmetry of the equation is greater than generic. The Lie point symmetries for all values of a and b are

$$\begin{aligned} \Gamma_1 &= t\partial_t \\ \Gamma_2 &= t\partial_t - x\partial_x. \end{aligned} \tag{4.2}$$

The factor removed is $ab + b - 2$ and we set $a = (2 - b)/b$ in (4.1) to obtain

$$x\ddot{x} - \dot{x}^2 \left(1 + \frac{1}{b}\right) + x^4 \left(\frac{1 - b}{b}\right) = 0. \tag{4.3}$$

From a further analysis of (4.3) using the same facility we identify special values of b to be $1, 2, \pm\frac{1}{2}$.

Singularity analysis of (4.3) gives the nongeneric resonance as $2(b-1)/b$. For an integral nongeneric resonance, n , we have $b = -2/(n-2)$. Some values are (in pairs (b, n)): $(1, 0)$, $(2, 2)$, $(\frac{1}{2}, -2)$, $(-\frac{1}{2}, 6)$ and $(-1, 4)$. The specific second-order equations are

$$b = 1 : \quad x\ddot{x} - 2\dot{x}^2 = 0 \quad (4.4)$$

$$b = 2 : \quad x\ddot{x} - \frac{3}{2}\dot{x}^2 - \frac{1}{2}x^4 = 0 \quad (4.5)$$

$$b = \frac{1}{2} : \quad x\ddot{x} - 3\dot{x}^2 + x^4 = 0 \quad (4.6)$$

$$b = -\frac{1}{2} : \quad x\ddot{x} + \dot{x}^2 - 3x^4 = 0 \quad (4.7)$$

$$b = -1 : \quad x\ddot{x} - 2x^4 = 0. \quad (4.8)$$

We note that this sampling covers the potentially special values of b revealed by LIE.

4.1 Case $b = 1$ (4.4)

The Lie point symmetries of (4.4) are

$$\begin{aligned} \Gamma_1 &= x\partial_x & \Gamma_5 &= t\partial_t \\ \Gamma_2 &= \partial_t & \Gamma_6 &= t^2\partial_t - tx\partial_x \\ \Gamma_3 &= tx^2\partial_x & \Gamma_7 &= x^2\partial_x \\ \Gamma_4 &= \frac{1}{x}\partial_t & \Gamma_8 &= \frac{t}{x}\partial_t - \partial_x. \end{aligned} \quad (4.9)$$

Since (4.4) is a second-order equation and possesses eight Lie point symmetries, it is related to the generic second-order equation $d^2X/dT^2 = 0$ by means of a point transformation [18] [p 405]. The point transformation is $T = t$, $X = 1/x$. This is a specific instance of a Möbius transformation which preserves the Painlevé Property.

4.2 Case $b = 2$ (4.5)

We obtain the three Lie point symmetries

$$\begin{aligned} \Gamma_1 &= \partial_t \\ \Gamma_2 &= t\partial_t - x\partial_x \\ \Gamma_3 &= t^2\partial_t - 2tx\partial_x. \end{aligned}$$

The Lie algebra of these three symmetries is $A_{3,8}$ in the Mubarakzhanov classification scheme [21, 22, 23, 24] and has the common name $sl(2, R)$. Equation (4.5) is related to the most elementary form of the Ermakov-Pinney equation [8, 27], *videlicet*

$$\frac{d^2X}{dT^2} + \frac{1}{X^3} = 0, \quad (4.10)$$

by means of the point transformation

$$2T = t, \quad X = x^{-1/2}. \quad (4.11)$$

The solution of (4.5) follows from the solution of (4.11), which was given by Pinney in 1950 [27]. It is

$$x(t) = \frac{1}{A + 2Bt + Ct^2}, \quad B^2 - AC = \frac{1}{4}. \quad (4.12)$$

4.3 Case $b = \frac{1}{2}$ (4.6)

The Lie point symmetries of (4.6) are

$$\begin{aligned}
 \Gamma_1 &= \partial_t & \Gamma_5 &= (x - t^2 x^3) \partial_x \\
 \Gamma_2 &= x^3 \partial_x & \Gamma_6 &= t \partial_t - x \partial_x \\
 \Gamma_3 &= t x^3 \partial_x & \Gamma_7 &= (t x + t^3 x^3 - 2x^2) \partial_x \\
 \Gamma_4 &= \left(t^3 - \frac{t}{x^2} \right) \partial_t - \frac{1}{2} \left(t^4 x^3 - \frac{1}{x} \right) \partial_x & \Gamma_8 &= \frac{2}{x^2} \partial_t - (3t x - t^3 x^3) \partial_x
 \end{aligned} \tag{4.13}$$

which must be a representation of $sl(3, R)$ as was the case for (4.4). Equation (4.6) is linearisable to

$$\frac{d^2 X}{dT^2} = 2 \tag{4.14}$$

by means of the point transformation

$$T = t \quad \text{and} \quad X = x^{-2}. \tag{4.15}$$

From the solution of (4.14) by means of the transformation (4.15) we obtain the solution of (4.6) as

$$x(t) = \frac{1}{(A + 2Bt + t^2)^{\frac{1}{2}}}, \tag{4.16}$$

where A and B are the two constants of integration.

4.4 Case $b = -\frac{1}{2}$ (4.7)

We find the two Lie point symmetries

$$\begin{aligned}
 \Gamma_1 &= \partial_t \\
 \Gamma_2 &= t \partial_t - x \partial_x,
 \end{aligned} \tag{4.17}$$

ie, there are no Lie point symmetries additional to those for the generic form of the equation, (3.2). Under the point transformation $T = t$ and $X = x^2$ (4.7) becomes a particular instance of the Emden-Fowler equation

$$\frac{d^2 X}{dT^2} = 6X^2 \tag{4.18}$$

which is well-known to be integrable in terms of elliptic functions.

4.5 Case $b = -1$ (4.8)

We obtain the two Lie point symmetries

$$\begin{aligned}
 \Gamma_1 &= \partial_t \\
 \Gamma_2 &= t \partial_t - x \partial_x,
 \end{aligned} \tag{4.19}$$

ie, again there are no Lie point symmetries additional to that for the generic form of the equation, (3.2). The same comments apply as for Subsection 4.3.

We observe that the particular values revealed by LIE ($b = 1, 2, \pm\frac{1}{2}$) lead to additional symmetry for $b = 1, 2, \frac{1}{2}$. These three values are a subset of the values which give an integral nongeneric resonance and of which we have been examining the properties of a sampling. For these three cases the solutions of the equations, (4.4,4.5,4.6) respectively, are expressed in terms of elementary functions. The two other values ($r = -\frac{1}{2}, -1$; equations (4.7) and (4.8) respectively) have solutions which are certainly not expressed in terms of elementary functions. The original second-order differential equation, (4.1), is integrable in the sense of Lie. For specific values of the parameters the nongeneric resonance is integral and so (4.1) possesses the Painlevé Property. However, this does not mean that there exists a closed-form solution. As ever [17] the relationship between Lie and Painlevé tantalises.

We compare the results of the analysis for the specific values of the parameters with the analysis of the general equation, (4.1), in terms of the standard Lie treatment. The Lie Bracket of the two symmetries of (4.2) is $[\Gamma_1, \Gamma_2]_{LB} = \Gamma_1$ and so Γ_1 is indicated as the symmetry for the reduction of order of (4.1). We set $u = x$ and $v = \dot{x}$ so that the first-order equation for v is

$$uvv' - v^2 \left(1 + \frac{1}{b}\right) - u^2v \left(1 - \frac{2}{b} + a\right) + u^4 \left(a - \frac{1}{b}\right) = 0. \quad (4.20)$$

In the new variables Γ_2 becomes $u\partial_u + 2v\partial_v$ and (4.20) may be rendered autonomous by the change of variables

$$U = \log u, \quad V = \frac{v}{u^2}. \quad (4.21)$$

Then (4.20) is

$$\frac{dV}{dU} = \left(\frac{1}{b} - 1\right)V + \left(1 - \frac{2}{b} + a\right) - \left(\frac{1}{b} - a\right)\frac{1}{V} \quad (4.22)$$

which is variables separable and may be written as

$$\frac{VdV}{\left[\left(\frac{1}{b} - 1\right)V - \left(\frac{1}{b} - a\right)\right](V - 1)} = dU. \quad (4.23)$$

Equation (4.23) may be integrated to give

$$(V - 1) \left[\left(\frac{1}{b} - 1\right)V - \left(\frac{1}{b} - a\right) \right]^{-\left(\frac{1}{b}-a\right)/\left(\frac{1}{b}-1\right)} = K \exp [(a - 1)U], \quad (4.24)$$

where K is the constant of integration.

In general (4.24) cannot be solved for V (and so $(-1/x)$) in closed form.

The special cases with additional symmetry were trivially integrable. The two cases without additional symmetry could also be reduced to expressions in terms of elliptic integrals. The easy further progress possible when the additional symmetry is available does not surprise. The not so elementary further progress in the two other cases does suggest the presence of a nonlocal symmetry of the useful variety [11].

5 Painlevé Analysis of the Special Cases

5.1 Case $b = 1$ (4.4)

When we make the usual substitution for the leading order behaviour, we obtain $p = -1$ and $\alpha = \pm 1$. Both branches are principal and in each case we obtain the resonances $r = \pm 1$. As was already presaged by the symmetry analysis and the connection to the generic equation of the second order by means of a Möbius transformation, (4.4) possesses the Painlevé Property.

5.2 Case $b = 2$ (4.5)

The leading order behaviour is $\pm\tau^{-1}$. Each branch is a principal branch with resonances $r = \pm 1$. Only the Right Painlevé Series exists with the series commencing at τ^{-1} . This is a consequence of the constraint upon the three constants in the solution (4.12) which precludes the possibility of a double root. If we set $A = (4B^2 - 1)/(4C)$ in accordance with the constraint in (4.12), the solution has singularities at

$$t = -\frac{B}{C} - \frac{1}{2C} \quad \text{and} \quad t = -\frac{B}{C} + \frac{1}{2C}. \quad (5.1)$$

From this we see the impossibility of the existence of a Left Painlevé Series. The Right Painlevé Series can be about either of the singularities and has a radius of convergence of C^{-1} . We note that the case $C = 0$ requires that $B = \pm\frac{1}{2}$ so that we are looking at a one-parameter solution. That solution can have both Left and Right Series, but this represents a particular solution and not the general solution.

5.3 Case $b = \frac{1}{2}$ (4.6)

The singularity analysis of (4.6) gives $p = -1$, $\alpha = \pm 1$ and for both branches $r = -1, -2$. This indicates the existence of a Left Painlevé Series.

5.4 Case $b = -\frac{1}{2}$ (4.7)

We obtain the leading order behaviour $\pm\tau^{-1}$ and for each branch the resonances are $r = -1, 6$. These results are to be expected from the relationship with the Emden-Fowler equation (4.18). Equation (4.7) possesses the Painlevé Property.

5.5 Case $b = -1$ (4.8)

The singularity analysis of (4.8) gives $p = -1$, $\alpha = \pm 1$ and for both branches $r = -1, 4$. The same comments apply as for Subsection 5.4 and (4.8) possesses the Painlevé Property.

6 The System (1.2,1.3) in the Special Cases

The case $a = 1$ and $b = 1$, *ie* the system leading to (4.4), was treated above under the general heading of $ab = 1$.

Equation (4.5) corresponds to $a = 0$ and $b = 2$. The system, (1.2,1.3), takes the specific form

$$\begin{aligned}\dot{x} &= x(x + 2y) \\ \dot{y} &= y^2.\end{aligned}\tag{6.1}$$

We solve for $y(t)$ from the second of (6.1) and substitute this solution into the first to obtain a nonautonomous equation for $x(t)$. We obtain in turn

$$y = \frac{1}{t_0 - t} \quad \text{and} \quad x = \frac{1}{(t - t_0)[A(t - t_0) + 1]}.\tag{6.2}$$

We note that, as in the case of $a = 1$ and $b = 1$, the system (6.1) is a decomposition of (2.7) with the same definition for w .

When we make a singularity analysis of (6.1), we obtain the leading order behaviours τ^{-1} and $-\tau^{-1}$ for x and y respectively. The resonances are $r = \pm 1$ and the equation for the arbitrary coefficients at the resonances is

$$\begin{bmatrix} r - 1 & -3 \\ 0 & r + 1 \end{bmatrix} \begin{bmatrix} \mu \\ \nu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.\tag{6.3}$$

At $r = -1$ both μ and ν are zero since this value of the resonance is related to the arbitrary location of the singularity (in the absence of specific initial conditions) [28, 31]. At $r = 1$ we see that μ is arbitrary and ν is zero which is already seen in the solution given in (6.2).

When $a = 3$ and $b = \frac{1}{2}$, the leading order behaviours are τ^{-1} and the resonances are $r = -1, -2$. Equation (4.6) results when $a = 3$ and $b = \frac{1}{2}$ is substituted into (4.1). That equation (4.6) is linearisable has already been noted.

In the case that $a = -5$ and $b = -\frac{1}{2}$ we find that the leading order behaviour is $\pm\tau^{-1}$. The resonances are $r = -1, 6$. We gave the connection to the Emden-Fowler equation above.

Equation (4.7) corresponds to $a = -3$ and $b = -1$ and the same connection to the Emden-Fowler equation exists.

The singularity analysis of (4.8) gives $p = -1$, $\alpha = \pm 1$ and for both branches $r = -1, 4$. These results are to be expected from the relationship with the Emden-Fowler equation and (4.8). Equation (4.8) possesses the Painlevé Property.

7 Some Instances of the General Case

We revisit the original system of equations (1.2,1.3) and also recall that the resonance for the general case is given by (2.5).

7.1 Case $r = 1$

When $r = 1$, we have from (2.6)

$$a = 2 - b, \quad a \neq 1, b \neq 1.\tag{7.1}$$

Hence for the case $r = 1$ the system (1.2,1.3) is an example of a decomposed system in that it can be composed as

$$\dot{w} = w^2, \quad (7.2)$$

where $w = x + y$ which is the same definition as was used in Section 2. The solution is simply

$$w = x + y = \frac{1}{t_0 - t}. \quad (7.3)$$

This is an extension of the case for which $ab = 1$ treated in Section 2. We could also write (1.2) as

$$\frac{\dot{x}}{x^2} = \frac{by}{x} - 1 \quad (7.4)$$

and substitute (7.3) into (7.4) to obtain

$$\frac{1}{x} = A(t - t_0)^b - (t - t_0), \quad b \neq 1, \quad (7.5)$$

where A is the second constant of integration. The solution is not analytic unless b is an integer. In the case that b is rational t_0 is a branch point singularity. When we make the usual substitution for the leading-order behaviour, we obtain $p = -1$ and $\alpha = -1$. When $\alpha = -1$, we obtain the resonance $r = -1$ and thus (7.2) possesses the Painlevé Property. We recall that (7.2) is the only first-order equation of the form

$$\dot{w} = f(w, t) \quad (7.6)$$

to possess the Painlevé Property.

7.2 Case $r = 2$

When $r = 2$, we have from (2.6) that

$$a = 3 - b - ab, \quad a \neq 1, b \neq 1. \quad (7.7)$$

By way of example, if we let $a = b$ in (7.7), we have $b = 1, -3$ and the corresponding a values are $a = 1, -3$. The former pair is not admissible. For the latter the system (1.2,1.3) is as follows

$$\dot{x} = x(x - 3y) \quad (7.8)$$

$$\dot{y} = y(-3x + y). \quad (7.9)$$

The above system is not a decomposition. By adding (7.8) to (7.9) and also subtracting (7.9) from (7.8) we have

$$\begin{aligned} \frac{d(x+y)}{dt} &= (x+y)^2 - 8xy \\ &= 2(x-y)^2 - (x+y)^2 \end{aligned} \quad (7.10)$$

$$\frac{d(x-y)}{dt} = (x-y)(x+y) \quad (7.11)$$

respectively. Equations (7.10) and (7.11) can be written as

$$\begin{aligned} \dot{v} &= 2u^2 - v^2 \\ \dot{u} &= uv, \end{aligned} \quad (7.12)$$

where $v = x + y$ and $u = x - y$. From (7.2)

$$v = \frac{\dot{u}}{u}$$

and we substitute this into (7.12) to obtain the second-order Emden-Fowler equation

$$\ddot{u} - 2u^3 = 0 \quad (7.13)$$

which is well-known to be integrable in terms of elliptic functions. Multiplying (7.13) by $2\dot{u}$ and integrating we have

$$\begin{aligned} \dot{u}^2 &= u^4 + I \\ t - t_0 &= \int \frac{du}{\sqrt{1+u^4}} \end{aligned} \quad (7.14)$$

which can be written as

$$\int_a^\infty \frac{du}{\sqrt{1+u^4}} = \frac{1}{2}F \left[\arccos \frac{a^2-1}{a^2+1}, \frac{1}{2}\sqrt{2} \right], \quad (7.15)$$

where $F(.,.)$ is an incomplete elliptical integral of the first kind. This can be inverted to give u in terms of elliptic functions. The singularity analysis of (7.13) gives $p = -1$, $\alpha = \pm 1$ and for both branches $r = -1, 4$. These results are to be expected from the relationship with the Emden-Fowler equation (7.13). The system (7.8,7.9) possesses the Painlevé Property.

7.3 Case $r = -2$

When $r = -2$, we have from (2.6)

$$a + b + ab = 3. \quad (7.16)$$

By way of example we let $a = b$ in (7.16). We have $b = 1, -1/3$ with the corresponding a values $a = 1, -1/3$. The former pair is not admissible. For the latter the system (1.2,1.3) is

$$\dot{x} = x(x - \frac{1}{3}y) \quad (7.17)$$

$$\dot{y} = y(-\frac{1}{3}x + y). \quad (7.18)$$

We add (7.17) to (7.18) and also subtract (7.18) from (7.17) to obtain

$$\begin{aligned}\frac{d(x+y)}{dt} &= (x+y)^2 - \frac{8}{3}xy \\ &= \frac{1}{3}(x+y)^2 + \frac{2}{3}(x-y)^2 \\ \frac{d(x-y)}{dt} &= (x-y)(x+y)\end{aligned}\tag{7.19}$$

Equations (7.19) and (7.3) can be written as

$$\begin{aligned}\dot{v} &= \frac{1}{3}v^2 + \frac{2}{3}u^2 \\ \dot{u} &= uv,\end{aligned}\tag{7.20}$$

where u and v have the same definition as was used in Section (7.2). From (7.3)

$$v = \frac{\dot{u}}{u}\tag{7.21}$$

and we substitute this into (7.20) to obtain the second-order equation

$$u\ddot{u} - \frac{4}{3}\dot{u}^2 - \frac{2}{3}u^4 = 0.\tag{7.22}$$

After further manipulation of (7.22) by use of the transformation $\tau = u^{-1/3}$ (7.22) becomes a particular instance of the Emden-Fowler equation

$$\ddot{\tau} = -\frac{2}{9}\tau^{-5}\tag{7.23}$$

which also is integrable in terms of elliptic functions. Equation (7.23) can be reduced to the quadrature

$$t - t_0 = \int^{\tau} \frac{u^2 du}{\sqrt{9u^4 I + 1}}.\tag{7.24}$$

The singularity analysis of (7.22) gives $p = -1$, $\alpha = \pm 1$ and for both branches $r = -1, \frac{4}{3}$. Equation (7.22) possesses the weak Painlevé Property. The transformation of (7.22) to (7.23) is not homeographic. The Painlevé Property of (7.23) becomes the Weak Painlevé Property for (7.22).

Equation (7.20) can equally be written as

$$u^2 = \frac{3}{2}\dot{v} - \frac{1}{2}v^2.\tag{7.25}$$

We differentiate (7.25) with respect to t and substitute (7.3) into (7.25) to obtain the second-order equation

$$\ddot{v} - \frac{8}{3}v\dot{v} + \frac{2}{3}v^3 = 0\tag{7.26}$$

The singularity analysis of (7.25) gives $p = -1$, $\alpha = -1, -3$. For $\alpha = -1$ the resonances are $r = -1, \frac{4}{3}$ and for $\alpha = -3$ the resonances are $r = -1, -4$. The latter resonance implies a Left Painlevé Series and the former implies a Right Painlevé Series in fractional powers.

8 Conclusion

In this paper we have studied the dominant terms of systems of Lotka-Volterra-type which arise in the mathematical modelling of the evolution of many diverse natural systems from the viewpoint of both symmetry and singularity analysis. We carried out a singularity analysis of the two-dimensional system and calculated the resonance which is given by (2.6). Since all terms are dominant in the system (1.2,1.3), the Painlevé Test is passed for all values of the parameters a and b for which the nongeneric resonance is an integer. The system (1.2,1.3) was solved for y and then substituted into (1.3) to obtain a second-order equation for x , (3.2). We applied the singularity analysis to (3.2) in the usual way by substituting $x = \alpha\tau^p$ to determine the leading order-behaviour. We found that

$$p = -1 \quad \text{and} \quad \alpha = \begin{cases} -1 \\ \frac{1-b}{ab-1} \end{cases} \quad (8.1)$$

In the case $\alpha = (1-b)/(ab-1)$ we found the resonance for equation (3.2) to be the same as for the system (1.2,1.3). Hence we have some consistency. Equally we could have solved (1.3) for x and substituted into (1.2) to obtain a second-order equation in y . The resulting equation is the same as (3.2) up to the exchanges $x \leftrightarrow y$ and $a \leftrightarrow b$. There is no further information to be gained. The Case $\alpha = -1$ was not relevant to the singularity analysis of the system (1.2,1.3) and is a timely reminder that the transition from the system to second-order differential equation and vice versa has implications for the singularity analysis just as it has for the symmetry analysis [26].

Equation (4.1) only has the two Lie Point Symmetries (4.3), but for special values of a and b more point symmetries were obtained. The Painlevé Property is maintained for each of the special cases. Equation (4.4) can be expressed in the form of the Emden-Fowler equation,

$$\ddot{\tau} = \frac{1-b}{b^2} \tau^{-2b+1} \quad (8.2)$$

which is well known to be integrable in terms of elliptic functions for some very specific values of the parameter, b .

This paper demonstrates that the system (1.2,1.3) which at first glance looks fairly simple has some interesting properties from both the viewpoint of singularity analysis and symmetry analysis. The generic case highlights the distinction between the analytic requirement of Painlevé and the lesser demand of Lie. One also notes an increase in the **ease** of integrability under the presence of greater symmetry.

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