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Deforming the Lie Superalgebra of Contact Vector Fields on $S^{1|1}$ Inside the Lie Superalgebra of Superpseudodifferential Operators on $S^{1|1}$

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Abstract

We classify nontrivial deformations of the standard embedding of the Lie superalgebra K(1) of contact vector fields on the (1,1)-dimensional supercircle into the Lie superalgebra of superpseudodifferential operators on the supercircle. This approach leads to the deformations of the central charge induced on K(1) by the canonical central extension of $S\Psi DO$.

1 Introduction

The study of multi-parameter deformations of the standard embedding of the Lie algebra $\mathrm{Vect}(S^1)$ of vector fields on the circle S^1 inside the Lie algebra $\Psi\mathcal{D}O$ of pseudodifferential operators on S^1 was carried out in [10, 11]. In this paper we address the computation of the integrability conditions of infinitesimal deformations of the standard embedding of the Lie superalgebra $\mathcal{K}(1)$ of contact vector fields on the supercircle $S^{1|1}$ inside the Lie superalgebra $\mathcal{S}\Psi\mathcal{D}\mathcal{O}$ of superpseudodifferential operators on $S^{1|1}$. The infinitesimal deformations of this embedding are classified by $H^1(\mathcal{K}(1), \mathcal{S}\Psi\mathcal{D}\mathcal{O})$. This space is four dimensional and it was calculated in [5]. The obstructions for integrability of infinitesimal deformations lie in $H^2(\mathcal{K}(1), \mathcal{S}\Psi\mathcal{D}\mathcal{O})$. Our goal is to study these obstructions.

It turns out that there exist four even one-parameter families of nontrivial deformations. We will compute explicit formulas describing these families. A contraction procedure of those deformations leads to four one-parameter deformations of the standard embedding of $\mathcal{K}(1)$ into the Poisson Lie superalgebra $\mathcal{SP}$ of superpseudodifferential symbols on $S^{1|1}$. Each parameter describes an interesting algebraic curve in the space of parameters.

The well-known nontrivial central extension of $\mathcal{S}\Psi\mathcal{D}\mathcal{O}$ induces a central extension of the subsuperalgebra $\mathcal{K}(1)$(see [3]). The restriction of the 2-cocycle generating this extension is the 2-cocycle defining the central extension of $\mathcal{K}(1)$ known as the Neveu-Schwartz Lie superalgebra (see [3], [12]). As an application of our results, we obtain a "deformed" expression for the central charge induced by the deformations of the standard embedding we have constructed.

2 The main definitions

2.1 Superpseudodifferential operators on $S^{1|1}$

We first recall the definition of $\mathcal{S}\Psi\mathcal{D}\mathcal{O}$ (cf. [8, 12]). The supercircle $S^{1|1}$ is the superextension of the circle S^1 with local coordinates (x,θ), where $x \in S^1$ and θ is odd. A C^∞-function on $S^{1|1}$ has the form $F = f(x) + 2g(x)\theta$, with $f, g \in C^\infty(S^1)$. The vector field $\eta = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}$ on $S^{1|1}$ sends F to $\eta(F) = 2g(x) + f'(x)\theta$, so that $\eta^2 = \frac{1}{2}[\eta,\eta] = \frac{\partial}{\partial x}$. The usual Leibniz rule: $\frac{\partial}{\partial x}\circ f = f'(x) + f(x)\frac{\partial}{\partial x}$ on $C^\infty(S^1)$, is replaced on $C^\infty(S^{1|1})$ by (p is for parity):

$$\eta\circ F = \eta(F) + \sigma(F)\eta, \quad \text{where } \sigma(F) = (-1)^{p(F)}F. \tag{2.1}$$

Formula (2.1) generalizes by induction on m to the graded Leibniz formula:

$$\eta^m\circ F = \sum_{k=0}^{\infty}\binom{m}{k}_s\eta^k(\sigma^{m-k}(F))\eta^{m-k} \tag{2.2}$$

for all integer m, where the super binomial coefficients $\binom{m}{k}_s$ are defined by:

$$\binom{m}{k}_s = \begin{cases} \binom{[\frac{m}{2}]}{[\frac{k}{2}]} & \text{if either } k \text{ is even or } m \text{ is odd}\\ 0 & \text{otherwise,}\end{cases}$$

$[x]$ is the integer part of a real number x, and $\binom{x}{l} = \frac{x(x-1)\cdots(x-l+1)}{\ell!}$ for $l\in\mathbb{Z}_{\geq 0}$. Set:

$$\mathcal{S}\Psi\mathcal{D}\mathcal{O} = \left\{\sum_{k\in\mathbb{Z}_{\geq 0}} F_k\,\eta^{\omega-k} \;\mid\; w\in\mathbb{Z},\; F_k\in C^\infty(S^{1|1})\right\},$$

where the composition of superpseudodifferential operators is given by (2.2):

$$F\eta^m\circ G\eta^n = \sum_{k=0}^{\infty}\binom{m}{k}_s F\eta^k(\sigma^{m-k}(G))\eta^{m+n-k} \;\text{ for any }\; m,n\in\mathbb{Z} \text{ and } F,\,G\in C^\infty(S^{1|1}).$$

Denote by $\mathcal{S}\Psi\mathcal{D}\mathcal{O}_{SL}$ the Lie superalgebra with the same superspace as $\mathcal{S}\Psi\mathcal{D}\mathcal{O}$ and the supercommutator defined on homogeneous elements by:

$$[A,B] = A\circ B - (-1)^{p(A)p(B)}\,B\circ A. \tag{2.3}$$

The space $\mathcal{SP}$ of superpseudodifferential symbols on $S^{1|1}$ has the following form:

$$\mathcal{SP} = C^\infty(S^{1|1})\otimes(\mathbb{C}[\xi][[\xi^{-1}]]\oplus\mathbb{C}[\xi][[\xi^{-1}]]\zeta),$$

where $\mathbb{C}[\xi][[\xi^{-1}]]$ is the space of (formal) Laurent series of finite order in ξ.

Any element of $\mathcal{SP}$ can be expressed in the following form:

$$S(x,\xi,\zeta) = \sum_{-\infty}^{n} F_k(x)\xi^k + \Big(\sum_{-\infty}^{n} G_k(x)\xi^k\Big)\zeta,$$

where $F_k, G_k \in C^\infty(S^{1|1})$, the symbol $\zeta = \overline{\theta} + \theta\xi$ corresponds to η, ξ corresponds to $\frac{\partial}{\partial x}$ and $\overline{\theta}$ corresponds to $\frac{\partial}{\partial \theta}$ (hence $\overline{\theta}^2 = \zeta^2 = 0$).

For $F \in C^\infty(S^{1|1})$, one has $\zeta F \xi^m = \sigma(F)\xi^m\zeta$, so then, the multiplication in $\mathcal{SP}$ is obvious. On $\mathcal{SP}$, there is a super Poisson bracket given by (cf. [9]):

$$\{S,T\} = \frac{\partial S}{\partial \xi}\frac{\partial T}{\partial x} - \frac{\partial S}{\partial x}\frac{\partial T}{\partial \xi} - (-1)^{p(S)}\Big(\frac{\partial S}{\partial \theta}\frac{\partial T}{\partial \overline{\theta}} + \frac{\partial S}{\partial \overline{\theta}}\frac{\partial T}{\partial \theta}\Big) \text{ for any } S,\ T \in \mathcal{SP}. \quad (2.4)$$

Consider a family of associative laws on $\mathcal{S}\Psi\mathcal{DO}$ depending on one parameter $h \in]0,1]$ by:

$$F\,\eta^{\,m} \circ_h G\,\eta^{\,n} = \begin{cases} \displaystyle\sum_{k=0}^{\infty} \tbinom{m}{k}_s F\, h^{\,[\frac{k}{2}]}\, \eta^{\,k}(\sigma^{m-k}(G))\,\eta^{\,m+n-k} & \text{if } m \text{ and } n \text{ are odd },\\ \displaystyle\sum_{k=0}^{\infty} \tbinom{m}{k}_s F\, h^{\,[\frac{k-1}{2}]}\, \eta^{\,k}(\sigma^{m-k}(G))\,\eta^{\,m+n-k} & \text{otherwise.}\end{cases}$$

Denote by $\mathcal{S}\Psi\mathcal{DO}_h$ the associative superalgebra of superpseudodifferential operators on $S^{1|1}$ equipped with the multiplication $\circ_h$. It is clear that all the associative superalgebras $\mathcal{S}\Psi\mathcal{DO}_h$ are isomorphic to each other.

For the supercommutator $[A,B]_h := \frac{1}{h}(A \circ_h B - (-1)^{p(A)p(B)} B \circ_h A)$, one has:

$$[A,B]_h = \{A,B\} + O(h),$$

and therefore $\lim_{h\to 0}[A,B]_h = \{A,B\}$, where we identify $\mathcal{SP}$ with $\mathcal{S}\Psi\mathcal{DO}$ as vector spaces. Hence the Lie superalgebra $\mathcal{S}\Psi\mathcal{DO}_{SL}$ contracts to the Poisson superalgebra $\mathcal{SP}$ (cf. [7]).

Furthermore, $\mathcal{S}\Psi\mathcal{DO}_{SL}$ admits an analogue of the Adler trace defined on the Lie algebra $\Psi\mathcal{DO}$ of pseudodifferential operators on S^1 (cf. [12, 3]): let $A = \sum_{k\in\mathbb{Z}} F_k\eta^k$ be a superpseudodifferential operator. Its super residue $\mathrm{Sres}(A)$ is the coefficient $F_{-1} \in C^\infty(S^{1|1})$ and the *Adler supertrace* (which vanishes on the brackets) is

$$\mathrm{Str}(A) = \int_{S^{1|1}} \mathrm{Sres}(A)\ \mathrm{vol}(x,\theta) = \int_{S^1} \frac{\partial F_{-1}}{\partial_\theta} dx. \quad (2.5)$$

Recall that the Lie superalgebra $\mathcal{K}(1)$ (also known as the Neveu-Schwartz superalgebra without central charge, cf. [3, 5]) consists of vector fields on $S^{1|1}$ preserving the Pfaff equation given by the contact 1-form $\alpha = dx + \theta d\theta$. Explicitly $\mathcal{K}(1)$ consists of vector fields of the form:

$$v_F = F\,\eta^2 + \frac{1}{2}\,\eta(F)\,\bar{\eta}, \qquad \text{where } \ \bar{\eta} = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial x}\ .$$

3 Statement of the problem

The main purpose of this paper is to study deformations of the canonical embedding $\rho : \mathcal{K}(1) \to \mathcal{S}\Psi\mathcal{DO}_{SL}$ defined by

$$\rho(v_F) = F\,\eta^2 + \frac{1}{2}\,\eta(F)\,\bar{\eta} \quad (3.1)$$

into a one-parameter family of Lie superalgebra homomorphisms.

3.1 Formal deformations

Let $\rho : \mathcal{K}(1) \to \mathcal{S}\Psi\mathcal{DO}_{SL}$ be an embedding of Lie superalgebras,

$$\widetilde{\rho}_t = \rho + \sum_{k=1}^{\infty} t^k \rho_k \ : \mathcal{K}(1) \to \mathcal{S}\Psi\mathcal{DO}_{SL}, \quad \text{satisfying } \ \widetilde{\rho}_t([X,Y]) = [\widetilde{\rho}_t(X), \widetilde{\rho}_t(Y)], \tag{3.2}$$

where $\rho_k : \mathcal{K}(1) \to \mathcal{S}\Psi\mathcal{DO}$ are even linear maps, a *formal deformation* of ρ.

The bracket in the right hand side in (3.2) is a natural extension of the Lie bracket in $\mathcal{S}\Psi\mathcal{DO}_{SL}$ to $\mathcal{S}\Psi\mathcal{DO}_{SL}[[t]]$. Two formal deformations $\widetilde{\rho}_t$ and $\widetilde{\rho}'_t$ are said to be *equivalent* if there exists an inner automorphism $I_t : \mathcal{S}\Psi\mathcal{DO}[[t]] \to \mathcal{S}\Psi\mathcal{DO}[[t]]$

$$I_t = \exp(t \ \mathrm{ad}F_1 + t^2 \ \mathrm{ad}F_2 + \cdots), \tag{3.3}$$

where $F_i \in \mathcal{S}\Psi\mathcal{DO}$ such that $p(F_i) = p(t^i)$, satisfying

$$\widetilde{\rho}'_t = I_t \circ \widetilde{\rho}_t. \tag{3.4}$$

3.2 Polynomial deformations

Observe that a polynomial deformation defined in this section is NOT a particular case of a formal definition. Recall that a deformation $\widetilde{\pi}$ of a homomorphism $\pi : \mathrm{Vect}(S^1) \to \Psi\mathcal{DO}$ defined by

$$\pi(f(x)\,\partial_x) = f(x)\,\xi$$

is (after [10]) said to be **polynomial** if it is an homomorphism of the following form

$$\widetilde{\pi}(c) = \pi + \sum_{k\in\mathbb{Z}} \widetilde{\pi}_k(c)\,\xi^k,$$

where $c \in \mathbb{R}^n$ are parameters of deformation, each linear map $\widetilde{\pi}_k(c) : \mathrm{Vect}(S^1) \to C^\infty(S^1)$ being polynomial in c, $\widetilde{\pi}_k \equiv 0$ for sufficiently large k and $\widetilde{\pi}_k(0) = 0$.

Now, consider a Lie superalgebra homomorphism $\widetilde{\rho}(c) : \mathcal{K}(1) \to \mathcal{S}\Psi\mathcal{DO}_{SL}$ of the following form:

$$\widetilde{\rho}(c) = \rho + \sum_{k\in\mathbb{Z}} \widetilde{\rho}_k(c), \tag{3.5}$$

where $\widetilde{\rho}_k(c) : \mathcal{K}(1) \to \mathcal{SP}_k$ are even linear maps, polynomial in $c \in \mathbb{R}^n$ and such that $\widetilde{\rho}_k \equiv 0$ for sufficiently large k and $\widetilde{\rho}_k(0) = 0$.

To define the notion of equivalence in the case of polynomial deformations, one simply replaces the formal automorphism I_t in (3.3) by an automorphism

$$I(c) : \mathcal{S}\Psi\mathcal{DO}_{SL} \longrightarrow \mathcal{S}\Psi\mathcal{DO}_{SL} \tag{3.6}$$

depending on $c \in \mathbb{R}^n$ in the following way :

$$I(c) = \exp(\sum_{i=1}^{n} c_i \ \mathrm{ad}F_i + \sum_{i,j=1}^{n} c_i c_j \ \mathrm{ad}F_{i,j} + \cdots), \tag{3.7}$$

where $F_i, F_{i,j}, \cdots F_{i_1\cdots i_k}$ are even elements of $\mathcal{S}\Psi\mathcal{DO}$.

Remark 1. Theory of polynomial deformations seems to be richer than that of formal ones. The equivalence problem for polynomial deformations has additional interesting aspects related to parameter transformations.

4 Deformations and cohomology

In this section, we will give a relationship between formal and polynomial deformations of Lie superalgebra homomorphisms and cohomology, cf. Nijenhuis and Richardson [2].

4.1 Infinitesimal deformations and the first cohomology

If $\rho : \mathfrak{g} \to \mathfrak{b}$ is a Lie superalgebra homomorphism, then $\mathfrak{b}$ is naturally a $\mathfrak{g}$-module. A map $\rho + t\rho_1 : \mathfrak{g} \to \mathfrak{b}$, where $\rho_1 \in Z^1(\mathfrak{g}, \mathfrak{b})$ is a Lie superalgebra homomorphism up to quadratic terms in t, it is said to be an **infinitesimal deformation**.

The problem is now to find higher order prolongations of these infinitesimal deformations. Setting $\varphi_t = \widetilde{\rho}_t - \rho$, one can rewrite the relation (3.2) in the following way:

$$[\varphi_t(X), \rho(Y)] + [\rho(X), \varphi_t(Y)] - \varphi_t([X,Y]) + \sum_{i,j>0} [\rho_i(X), \rho_j(Y)]t^{i+j} = 0 \ . \tag{4.1}$$

The first three terms are $(\delta\varphi_t)(X,Y)$, where δ stands for the coboundary. For arbitrary linear maps $\varphi, \varphi' : \mathfrak{g} \longrightarrow \mathfrak{b}$, define:

$$\begin{aligned} &[[\varphi, \varphi']] : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{b} \\ &[[\varphi, \varphi']](X,Y) = [\varphi(X), \varphi'(Y)] + [\varphi'(X), \varphi(Y)]. \end{aligned} \tag{4.2}$$

The relation (4.1) becomes now equivalent to:

$$\delta\varphi_t + \frac{1}{2}[[\varphi_t, \varphi_t]] = 0. \tag{4.3}$$

Expanding (4.3) in power series in t, we obtain the following equation for ρ_k:

$$\delta\rho_k + \frac{1}{2}\sum_{i+j=k} [[\rho_i, \rho_j]] = 0. \tag{4.4}$$

The first nontrivial relation is $\delta\rho_2 + \frac{1}{2}[[\rho_1, \rho_1]] = 0$ gives the first obstruction to integration of an infinitesimal deformation. Indeed, it is easy to check that for any two 1-cocycles γ_1 and $\gamma_2 \in Z^1(\mathfrak{g}, \mathfrak{b})$, the bilinear map $[[\gamma_1, \gamma_2]]$ is a 2-cocycle. The first nontrivial relation (4.4) is precisely the condition for this cocycle to be a coboundary. Moreover, if one of the cocycles γ_1 or γ_2 is a coboundary, then $[[\gamma_1, \gamma_2]]$ is a 2-coboundary. We therefore, naturally deduce that the operation (4.2) defines a bilinear map:

$$H^1(\mathfrak{g}, \mathfrak{b}) \otimes H^1(\mathfrak{g}, \mathfrak{b}) \longrightarrow H^2(\mathfrak{g}, \mathfrak{b}), \tag{4.5}$$

called the *cup-product*.

All the obstructions lie in $H^2(\mathfrak{g}, \mathfrak{b})$ and they are in the image of $H^1(\mathfrak{g}, \mathfrak{b})$ under the cup-product. So, in our case, we have to compute $H^1(\mathcal{K}(1), \mathcal{S}\Psi\mathcal{DO})$ and the product classes in $H^2(\mathcal{K}(1), \mathcal{S}\Psi\mathcal{DO})$.

5 The space $H^1(\mathcal{K}(1), \mathcal{S}\Psi\mathcal{DO})$

5.1 A filtration on $\mathcal{S}\Psi\mathcal{DO}$

The natural embedding of $\mathcal{K}(1)$ into $\mathcal{S}\Psi\mathcal{DO}$ given by the expression (3.1) induces a $\mathcal{K}(1)$-module structure on $\mathcal{S}\Psi\mathcal{DO}$. Analogously, we have a $\mathcal{K}(1)$-module structure on $\mathcal{SP}$ given by the natural embedding of $\mathcal{K}(1)$:

$$\overline{\pi} \ : \ v_F \mapsto F\,\xi + \frac{1}{2}\,\eta(F)\,\bar{\zeta}, \tag{5.1}$$

where $\bar{\zeta} = \overline{\theta} - \theta\xi$.

Setting the degree of x, θ be zero and the degree of ξ, ζ be 1 we introduce a $\mathbb{Z}$-grading in the Poisson superalgebra $\mathcal{SP}$. Then we have

$$\mathcal{SP} = \widetilde{\bigoplus}_{n\in\mathbb{Z}} \mathcal{SP}_n := (\bigoplus_{n<0} \mathcal{SP}_n) \bigoplus (\prod_{n\geq 0} \mathcal{SP}_n), \tag{5.2}$$

where $\mathcal{SP}_n = \{F\xi^{-n} + G\xi^{-n-1}\zeta \ | \ F, G \in C^\infty(S^{1|1})\}$ is the homogeneous subspace of degree $-n$. Each element of $\mathcal{S}\Psi\mathcal{DO}$ can be expressed as

$$A = \sum_{k\in\mathbb{Z}} (F_k + G_k\eta^{-1})\eta^{2k}, \quad \text{where} \quad F_k, \, G_k \in C^\infty(S^{1|1}).$$

We define the order of A to be

$$\mathrm{ord}(A) = \sup\{k \ | \ F_k \neq 0 \text{ or } G_k \neq 0\}.$$

This definition of order equips $\mathcal{S}\Psi\mathcal{DO}$ with a decreasing filtration as follows: set

$$\mathbf{F}_n = \{A \in \mathcal{S}\Psi\mathcal{DO} \ | \ \mathrm{ord}(A) \leq -n\}, \quad \text{where } n \in \mathbb{Z}.$$

So one has

$$\ldots \subset \mathbf{F}_{n+1} \subset \mathbf{F}_n \subset \ldots. \tag{5.3}$$

This filtration is compatible with the multiplication and the super Poisson bracket, that is, for $A \in \mathbf{F}_n$ and $B \in \mathbf{F}_m$, one has $A \circ B \in \mathbf{F}_{n+m}$ and $\{A, B\} \in \mathbf{F}_{n+m-1}$, after we identify $\mathcal{SP}$ with $\mathcal{S}\Psi\mathcal{DO}$. This filtration makes $\mathcal{S}\Psi\mathcal{DO}$ an associative filtered superalgebra. Moreover, this filtration is compatible with the natural action of $\mathcal{K}(1)$ on $\mathcal{S}\Psi\mathcal{DO}$. Indeed, if $v_F \in \mathcal{K}(1)$ and $A \in \mathbf{F}_n$, then

$$v_F \cdot A = [v_F, A] \in \mathbf{F}_n.$$

The induced $\mathcal{K}(1)$-action on the quotient $\mathbf{F}_n/\mathbf{F}_{n+1}$ is isomorphic to the $\mathcal{K}(1)$-action on $\mathcal{SP}_n$. Therefore, the $\mathcal{K}(1)$-action on the associated graded space of the filtration (5.3), is isomorphic to the graded $\mathcal{K}(1)$-module $\mathcal{SP}$, that is

$$\mathcal{SP} \simeq \widetilde{\bigoplus}_{n\in\mathbb{Z}} \mathbf{F}_n/\mathbf{F}_{n+1}.$$

Now we can deduce the cohomology of the filtered module from the cohomology of the associated graded module.

5.2 $H^1(\mathcal{K}(1), \mathcal{SP})$

Observe that $H^1(\mathcal{K}(1), \mathcal{SP}) = \bigoplus_{n\in\mathbb{Z}} H^1(\mathcal{K}(1), \mathcal{SP}_n)$. These spaces are known (see [5]). They are nontrivial if and only if $n = 0, 1$ and the corresponding dimensions are 3 and 1, respectively. Therefore, $H^1(\mathcal{K}(1), \mathcal{SP}) \cong \mathbb{R}^4$. The nontrivial cocycles generating the space $H^1(\mathcal{K}(1), \mathcal{SP})$ are $(\mathrm{ad}_\zeta(\overline{\pi}(v_F)) = \{\zeta, \overline{\pi}(v_F)\}$ with $\overline{\pi}$ as in (5.1)):

$$C_0(v_F) = \frac{1}{4}(F + \sigma(F)) + \frac{1}{2}F, \tag{5.4}$$

$$C_1(v_F) = \eta^2(F), \tag{5.5}$$

$$C_2(v_F) = \mathrm{ad}^3_\zeta(\overline{\pi}(v_F))\xi^{-2}\overline{\zeta} \tag{5.6}$$

with values in $\mathcal{SP}_0$, and

$$C_3(v_F) = \mathrm{ad}^5_\zeta(\overline{\pi}(v_F))\xi^{-3}\overline{\zeta} \tag{5.7}$$

with values in $\mathcal{SP}_1$.

5.3 $H^1(\mathcal{K}(1), \mathcal{S}\Psi\mathcal{DO})$ ([5])

The result of [5] is a specialization at $h = 1$ of the following theorem obtained as in [5]:

Theorem 1. *The space $H^1(\mathcal{K}(1), \mathcal{S}\Psi\mathcal{DO}_h)$ is purely even. It is spanned by the classes of the following nontrivial* 1*-cocycles*

$$\begin{aligned}
\Theta_0(v_F) = {} & \frac{1}{4}(F + \sigma(F)) + \frac{1}{2}F, \\
\Theta_1(v_F) = {} & \eta^2(F), \\
\Theta_{2_h}(v_F) = {} & \sum_{n=1}^{\infty}(-1)^n h^{n-1}\,\frac{n-2}{n}\sigma(\bar{\eta}^{\,2n+1}(F))\bar{\eta}^{\,-2n+1} + \\
& \textstyle\sum_{n=1}^{\infty}(-1)^n\, h^n\, \frac{n-3}{n+1}\bar{\eta}^{\,2n+2}(F)\bar{\eta}^{\,-2n}, \\
\Theta_{3_h}(v_F) = {} & \sum_{n=2}^{\infty}(-1)^n\, h^{n-2}\,\frac{n-1}{n}\sigma(\bar{\eta}^{\,2n+1}(F))\bar{\eta}^{\,-2n+1} + \\
& \textstyle\sum_{n=2}^{\infty}(-1)^n h^{n-1}\,\frac{n-1}{n+1}\bar{\eta}^{\,2n+2}(F)\bar{\eta}^{\,-2n}\ .
\end{aligned}$$

6 Integrability of infinitesimal deformations

The space $H^1(\mathcal{K}(1), \mathcal{S}\Psi\mathcal{DO})$ classifies infinitesimal deformations of the standard embedding $\mathcal{K}(1) \longrightarrow \mathcal{S}\Psi\mathcal{DO}_{SL}$ given by (3.1). In this section we will calculate the integrability conditions of infinitesimal deformations into polynomial ones. Any nontrivial infinitesimal deformation can be expressed in the following form:

$$\rho_1 = \rho + \sum_{0\le i\le 3} c_i\,\Theta_i\ ,\ \text{where } c_0,\ c_1,\ c_2,\ c_3 \in \mathbb{R}. \tag{6.1}$$

The integrability condition (below) imply that either $c_0 = 0$ or $c_2 = c_3 = 0$.

6.1 Deformations generated by Θ_0 and Θ_1

Since zero-order operators commute in $\mathcal{S}\Psi\mathcal{DO}$, it is evident that the cup-products $[[\Theta_0, \Theta_0]]$, $[[\Theta_0, \Theta_1]]$ and $[[\Theta_1, \Theta_1]]$ vanish identically, and therefore the map

$$\rho_{\nu,\lambda} : \mathcal{K}(1) \to \mathcal{S}\Psi\mathcal{DO}, \; v_F \mapsto \rho_{\nu,\lambda}(v_F) = \rho(v_F) + \nu\, \Theta_0(v_F) + \lambda\, \Theta_1(v_F) \tag{6.2}$$

is indeed, a nontrivial deformation of the standard embedding. This deformation is polynomial since it is of order 1.

Proposition 1. *Any nontrivial formal deformation of the embedding (3.1) generated by Θ_0 and Θ_1 is equivalent to a deformation of order 1, that is, to a deformation given by (6.2).*

Proof. Consider a formal deformation of the embedding (3.1) generated by Θ_0 and Θ_1:

$$\widetilde{\rho}_t = \rho + t_0\, \Theta_0 + t_1\, \Theta_1 + \sum_{m\geq 2} \sum_{i+j=m} t_0^i\, t_1^j\, \rho_{i,j}^{(m)}, \tag{6.3}$$

where the highest-order terms $\rho_{ij}^{(m)}$ are even linear maps from $\mathcal{K}(1)$ to $\mathcal{S}\Psi\mathcal{DO}$. The solution $\rho_{ij}^{(2)}$ of (4.4) is defined up to a 1-cocycle and it has been shown in [4, 1] that different choices of solutions of (4.4) correspond to equivalent deformations. Thus, one can always kill $\rho_{ij}^{(2)}$. Then, by recurrence, the highest-order terms satisfy the equation $\delta\rho_{ij}^{(m)} = 0$ and can also be killed. ■

Remark 2. Recall that, in the classical case (cf. [10]), there exists an analogous deformation $\pi_{\nu,\lambda}$ of π. Then, one can easily check that the following diagram commutes:

$$\begin{array}{l}
\mathcal{K}(1) \stackrel{\rho_{\nu,\lambda}}{\longrightarrow} \mathcal{S}\Psi\mathcal{DO} \\
\uparrow i \qquad\qquad \uparrow j \\
\mathrm{Vect}(S^1) \stackrel{\pi_{\nu,\lambda}}{\longrightarrow} \Psi\mathcal{DO}
\end{array}$$

where

$$\begin{aligned}
&i(f(x)\partial_x) = v_{f(x)}, \\
&\pi_{\nu,\lambda}(f(x)\partial_x) = f(x)\xi + \nu f(x) + \lambda f'(x), \\
&j(A) = A + \frac{1}{2}\eta^2(\partial_\xi A)\theta\eta\ .
\end{aligned}$$

Real difficulties begin when we deal with polynomial or formal integrability of the infinitesimal deformations corresponding to the cocycles Θ_1, Θ_2 and Θ_3.

6.2 Deformations generated by Θ_1, Θ_2 and Θ_3

Consider an infinitesimal deformation of the standard embedding of $\mathcal{K}(1)$ into $\mathcal{S}\Psi\mathcal{DO}_{SL}$ defined by the cocycles $\Theta_1, \Theta_2, \Theta_3$ and depending on the real parameters c_1, c_2, c_3

$$\widetilde{\rho}(c)(v_F) = \rho(v_F) + c_1\Theta_1(v_F) + c_2\Theta_2(v_F) + c_3\Theta_3(v_F). \tag{6.4}$$

Theorem 2. *The infinitesimal deformation (6.4) corresponds to a polynomial deformation, if and only if the following relations are satisfied :*

$$\begin{cases} 3c_1c_3 - 2c_1^3 - 2c_1^2c_3 + c_1^2 + 2c_3^2 = 0 \\ c_1 = c_2 \end{cases} \tag{6.5}$$

or

$$\begin{cases} c_3c_1 - 2c_3c_1^2 - 2c_3^2 = 0 \\ c_2 = 0 \end{cases} \tag{6.6}$$

To prove Theorem 2, we first introduce the notion of homogeneity for a deformation given by differentiable maps, then we will prove that the conditions (6.5–6.6) are necessary for integrability of infinitesimal deformations. In the end of this section we will show that these relations are sufficient by exhibiting explicit deformations.

6.2.1 Homogeneous deformation

Consider an arbitrary **polynomial** deformation of the standard embedding, corresponding to the infinitesimal deformation (6.4):

$$\begin{aligned} \widetilde{\rho}(c)(v_F) &= \rho(v_F) + c_1\eta^2(F) + c_2(\sigma(\bar\eta^3(F))\bar\eta^{-1} + \bar\eta^4(F)\bar\eta^{-2}) \\ &\quad + c_3(\sigma(\bar\eta^5(F)\bar\eta^{-3}) + \sum_{k\in\mathbb{Z}} P_k(c)\rho_k(v_F)\ , \end{aligned} \tag{6.7}$$

where $c = (c_1, c_2, c_3) \in \mathbb{R}^3$, P_k are polynomial functions of degree ≥ 2 and ρ_k are some differentiable even linear maps from $\mathcal{K}(1)$ to $\mathcal{SP}_k$.

Note that, since the cocycles $\Theta_1, \Theta_2, \Theta_3$ are defined by differentiable maps, an arbitrary solution of the deformation problem is also defined via differentiable maps. This follows from the Gelfand-Fuchs formalism of differentiable (or local) cohomology (see [6]).

Now, let us introduce a notion of homogeneity for deformation given by differentiable maps. A deformation (6.7) is said to be *homogeneous of degree* m if $\widetilde{\rho}(c)(v_F)$ is of the form:

$$\widetilde{\rho}(c)(v_F) = \sum_{k\in\mathbb{Z}} P_k(c)(\sigma^k(\bar\eta^{k+m}(F)))\bar\eta^{-k}.$$

Since the cocycles Θ_1, Θ_2 and Θ_3 are of degree 2, every homogeneous deformation (6.7) corresponding to a nontrivial infinitesimal deformation is of degree 2:

$$\begin{aligned} \widetilde{\rho}(c)(v_F) &= \rho(v_F) + c_1\eta^2(F) + c_2(\sigma(\bar\eta^3(F))\bar\eta^{-1} + \bar\eta^4(F)\bar\eta^{-2}) \\ &\quad + c_3(\sigma(\bar\eta^5(F)\bar\eta^{-3}) + \sum_{k\geq 4} P_k(c)(\sigma^k(\bar\eta^{k+2}(F)))\bar\eta^{-k}. \end{aligned} \tag{6.8}$$

Proposition 2. *Every deformation (6.7) is equivalent to a homogeneous deformation (6.8).*

Proof. It is easy to see that any homomorphism preserves homogeneity. This means that the first term in (6.7) (the term of the lowest degree in c) which is not homogeneous of degree 2 must lie in $H^1(\mathcal{K}(1), \mathcal{SP})$. Such a 1-cocycle is cohomologous to a linear combination of the 1-cocycles C_1, C_2 and C_3, see (5.5)–(5.7) which are homogeneous of degree 2. Thus, one can add (or remove) a coboundary in the term of the polynomial deformation (6.7) to obtain an equivalent one. ∎

6.2.2 Integrability conditions are necessary

The infinitesimal deformation (6.4) is clearly of the form

$$\begin{aligned}\widetilde{\rho}(c)(v_F) = \rho(v_F) &+ c_1\eta^2(F)\\ &+ c_2(\sigma(\bar{\eta}^3(F))\bar{\eta}^{-1} + \bar{\eta}^4(F)\bar{\eta}^{-2})\\ &+ c_3\sigma(\bar{\eta}^5(F))\bar{\eta}^{-3} + \cdots\ ,\end{aligned} \tag{6.9}$$

where "$\cdots$" means the terms in $\bar{\eta}^{-4}, \bar{\eta}^{-5}$. To compute the obstructions for integrability of the infinitesimal deformation (6.4), one has to add the first nontrivial terms and impose the homomorphism condition. So, put

$$\bar{\rho}(c)(v_F) = \widetilde{\rho}(c)(v_F) + P_4(c)\bar{\eta}^6(F)\bar{\eta}^{-4} + P_5(c)\sigma(\bar{\eta}^7(F))\bar{\eta}^{-5}, \tag{6.10}$$

where $P_4(c)$ and $P_5(c)$ are some polynomials in $c = (c_1, c_2, c_3)$ and compute the difference

$$[\bar{\rho}(c)(v_F), \bar{\rho}(c)(v_G)] - \bar{\rho}(c)([v_F, v_G]).$$

A straightforward but boring computation leads to the following equations:

$$\begin{aligned}&c_2c_1 = c_2^2\,,\\ &3P_4 = 2c_3 - c_2 - 2c_3c_1 + 4c_3c_2 + c_2c_1\,,\\ &3P_5 = -c_2 - 4c_3 + 2c_2^2 - 2c_3c_2 + 4c_3c_1.\end{aligned} \tag{6.11}$$

Let us go one step further, expand our deformation up to $\bar{\eta}^{-7}$, that is, put

$$\bar{\bar{\rho}}(c)(v_F) = \bar{\rho}(c)(v_F) + P_6(c)\bar{\eta}^8(F)\bar{\eta}^{-6} + P_7(c)\sigma(\bar{\eta}^9(F))\bar{\eta}^{-7}.$$

The homomorphism condition leads to a nontrivial relations for the parameters. For $c_1 = c_2$, the relations are:

$$\begin{aligned}&2P_6 = -c_1 + c_1^2 + P_4(-3 + 2c_1)\,,\\ &2P_7 = 5c_3 - 2c_3^2 - 6c_3c_1 - 3P_4 + 4c_1P_4\,,\end{aligned} \tag{6.12}$$

$$5P_7 = c_1 - 2c_1^2 + c_1P_4 + (3c_1 - \frac{9}{2})P_5 - \frac{3}{2}P_6. \tag{6.13}$$

Substituting expressions (6.11), and (6.12) for P_4, P_5, P_6 and P_7 in (6.13), one gets formula (6.5).

For $c_2 = 0$, the relations are:

$$\begin{aligned}&P_6 = (\tfrac{3}{2} - c_1)(P_4 + P_5)\,,\\ &2P_7 = 3c_3 - 4c_3c_1 - 2c_3^2\,,\end{aligned} \tag{6.14}$$

$$4(P_6 + P_7) = (-3 + 2c_1)(P_4 + P_5). \tag{6.15}$$

Substituting expressions (6.11), and (6.14) for P_4, P_5, P_6 and P_7 in (6.15), we get formula (6.6). We have thus shown that the conditions (6.5)are necessary for integrability of infinitesimal deformations.

Remark 3. The obstructions to integrability of an infinitesimal deformation (6.4) which does not satisfy the conditions (6.5)corresponds to a nontrivial class of $H^2(\mathcal{K}(1), \mathcal{SP}_3)$.

6.3 Introducing the parameter h

One can now modify the relations in order to get a deformation in $\mathcal{S}\Psi\mathcal{D}\mathcal{O}_h$, the scalar h then appears with different powers according to the "weight" of the respective terms in formulas (6.5–6.6). One finally gets

$$\begin{cases} h^2c_1^2 + h(3c_1c_3 - 2c_1^3) + 2c_3^2 - 2c_1^2c_3 = 0 \\ c_1 = c_2 \end{cases} \quad \text{or} \quad \begin{cases} hc_3c_1 - 2c_3c_1^2 - 2c_3^2 = 0 \\ c_2 = 0\cdot \end{cases} \tag{6.16}$$

These relations are necessary for the integrability of the infinitesimal deformation (6.4) in $\mathcal{S}\Psi\mathcal{D}\mathcal{O}_h$.

Remark 4. By setting weights: $wht(c_1) = wht(h) = 1$ and $wht(c_3) = 2$, we make the polynomials (6.16) homogeneous of weight 4. Moreover, setting $h = 0$, one gets the necessary conditions to have a polynomial deformation of the standard embedding (5.1) corresponding to a given infinitesimal one generated by the cocycles C_1, C_2 and C_3 given by (5.5)–(5.7).

Now, we will give a natural description of the curves defined by equations (6.16) in order to unveil their algebraic nature.

6.3.1 A rational parameterization

There exists a rational parameterization of the curves (6.16):

Proposition 3. *i) For all $\lambda \in \mathbb{R}$, the constants*

$$\begin{cases} c_1 &= -\lambda \\ c_2 &= -\lambda \\ c_3 &= h\lambda \end{cases} \quad or \quad \begin{cases} c_1 &= -\lambda \\ c_2 &= -\lambda \\ c_3 &= \lambda^2 + \frac{1}{2}h\lambda \end{cases} \tag{6.17}$$

satisfy the first of relations (6.16).

ii) For all $\lambda \in \mathbb{R}$, the constants

$$\begin{cases} c_1 &= -\lambda \\ c_2 &= c_3 = 0 \end{cases} \quad or \quad \begin{cases} c_1 &= -\lambda \\ c_2 &= 0 \\ c_3 &= -\lambda^2 - \frac{1}{2}h\lambda \end{cases} \tag{6.18}$$

satisfy the second of relations (6.16).

iii) Any triple $c_1, c_2, c_3 \in \mathbb{R}$ satisfying (6.16) is of the form (6.17) or(6.18) for same λ.

Proof. By direct computations. ∎

Remark 5. Geometrically, the curves (6.5) and (6.6) are just lines and parabolas.

Now, the analogue of Richardson-Nijenhuis theory in supergeometry (cf. (4.5)) prescribes us to compute $H^2(\mathcal{K}(1), \mathcal{S}\Psi\mathcal{D}\mathcal{O})$ in order to obtain the complete information concerning the cohomological obstructions. This, however, seems to be a quite difficult problem. We shall not do that; an explicit construction of deformations will allow to avoid the standard obstruction framework.

6.3.2 Construction of deformations

Now, to complete the proof of our main result (Theorem 2), we construct a polynomial deformation corresponding to any infinitesimal deformation (6.4) satisfying the condition (6.16). This implies that these conditions are not only necessary, but also sufficient for integrability of infinitesimal deformations (6.4).

The space $H^1_0(\mathcal{S}\Psi\mathcal{DO}, \mathcal{S}\Psi\mathcal{DO})$ of even outer superderivations of the Lie superalgebra $\mathcal{S}\Psi\mathcal{DO}$ contains the linear operator $\operatorname{ad}\log\xi$ on $\mathcal{S}\Psi\mathcal{DO}$ (cf. [3]). This outer superderivation can be integrated to a one-parameter family of outer automorphisms denoted by Ψ_ν and defined by

$$\Psi_\nu(F) = \xi^\nu \circ F \circ \xi^{-\nu}, \tag{6.19}$$

which should be understood as a Laurent series in $\overline{\eta}$ (depending on the parameter ν).

Let us apply the automorphism (6.19) to the elementary deformation $\rho_{0,\lambda}$ (6.2):

$$\begin{aligned}\widetilde{\rho}^{\lambda}_1(v_F) &= \Psi_{\frac{-2\lambda}{h}}(\rho(v_F) + \lambda\eta^2(F)) \\ &= \rho(v_F) - \lambda\eta^2(F) \\ &\quad - \lambda(\sigma(\overline{\eta}^3(F))\overline{\eta}^{-1} + \overline{\eta}^4(F)\overline{\eta}^{-2}) \\ &\quad + (\lambda^2 + \frac{1}{2}\lambda h)(\sigma(\overline{\eta}^5(F)\overline{\eta}^{-3}) + \cdots\end{aligned} \tag{6.20}$$

Since $\Psi_{\frac{-2\lambda}{h}}$ is an automorphism, it is, indeed, a polynomial deformation of embedding (3.1) for any $\lambda \in \mathbb{R}$, corresponding to any infinitesimal deformation (6.4) satisfying the second of conditions (6.17).

Proposition 4. *The map*

$$\widetilde{\rho}^{\lambda}_2 : v_F \rightarrow \rho(v_F) + \lambda\,\widetilde{\Theta}_h(v_F), \tag{6.21}$$

where $\widetilde{\Theta}_h = 2h\Theta_{3_h} - \Theta_{2_h} - \Theta_1$, is both a polynomial and a formal deformation of embedding (3.1) for any $\lambda \in \mathbb{R}$, corresponding to any infinitesimal deformation (6.4) satisfying the first of conditions (6.17).

Proof. Since $\widetilde{\Theta}_h$ is an even 1-cocycle, the map $\widetilde{\rho}^{\lambda}_2$ is a polynomial deformation if the supercommutator $[\widetilde{\Theta}_h,\ \widetilde{\Theta}_h]$ vanishes. So put

$$\begin{aligned}\widetilde{\Theta}_h(v_F) = &\sum_{n\geqslant 1}(-1)^n h^{n-1} a^{(n+1)}\xi^{-n}\theta\frac{\partial}{\partial\theta} + \sum_{n\geqslant 0}(-1)^{n+1}h^n a^{(n+1)}\xi^{-n} \\ &+ 2\sum_{n\geqslant 1}(-1)^n h^{n-1} b^{(n)}\xi^{-n}\frac{\partial}{\partial\theta}\ ,\end{aligned}$$

where $F = a+2b\theta$ with $a,\ b \in C^\infty(S^1)$ and compute $[\widetilde{\Theta}_h(v_F),\ \widetilde{\Theta}_h(v_G)]$, where $G = c+2d\theta$ with $c,\ d \in C^\infty(S^1)$. Collect the terms with $a^{\,(\alpha+1)}\ c^{(\beta+1)}\ \xi^{-\alpha-\beta}$ for $\alpha,\ \beta\ \in\ \mathbb{N}$:

$$\begin{cases} H(\alpha,\beta) & \text{if } \alpha \geq 2 \ \text{ and } \ \beta \geq 2, \\ 0 & \text{otherwise}\ , \end{cases} \tag{6.22}$$

where

$$H(\alpha,\beta) = -(-h)^{\alpha+\beta-1} \times \left(\binom{\alpha+\beta-1}{\alpha-1} - \binom{\alpha+\beta-1}{\beta-1} + \sum_{n=1}^{\beta-1} \binom{\alpha+n-1}{\alpha-1} - \sum_{k=1}^{\alpha-1} \binom{\beta+k-1}{\beta-1} \right). \quad (6.23)$$

It is now easy to check that the expression (6.23) vanishes for $\beta = 2$ and $\alpha \geq 2$. We will prove by recurrence that this expression vanishes for $\alpha \geq 2$ and $\beta \geq 2$.

Assume that, for $\beta \geq 2$, one has

$$\binom{\alpha+\beta-1}{\beta-1} - \binom{\alpha+\beta-1}{\alpha-1} - \sum_{n=1}^{\beta-1} \binom{\alpha+n-1}{\alpha-1} + \sum_{n=1}^{\alpha-1} \binom{\beta+n-1}{\beta-1} = 0. \quad (6.24)$$

Using that

$$\sum_{k=1}^{\alpha} \binom{\beta+k-1}{\beta} = \binom{\alpha+\beta}{\alpha-1}, \qquad \binom{\alpha+\beta}{\beta} = \frac{\alpha+\beta}{\beta} \binom{\alpha+\beta-1}{\beta-1},$$
$$\binom{\alpha+\beta}{\alpha-1} = \frac{\alpha+\beta}{\beta+1} \binom{\alpha+\beta-1}{\alpha-1},$$

and equation (6.24), one obtains

$$\binom{\alpha+\beta}{\alpha} - \binom{\alpha+\beta}{\alpha-1} - \sum_{n=1}^{\beta} \binom{\alpha+n-1}{\alpha-1} + \sum_{n=1}^{\alpha-1} \binom{\beta+n}{\beta} = 0$$

which implies that the expression (6.22) vanishes.

Note that the term with $a^{(\alpha+1)}\, c^{(\beta+1)}\, \xi^{-\alpha-\beta} \theta \frac{\partial}{\partial\theta}$, where $\alpha,\ \beta \in \mathbb{N}$, vanishes since it has the same expression as (6.22). Finally, one can easily see that the coefficients of $a^{(\alpha+1)}\, d^{(\beta)}\, \xi^{-\alpha-\beta}\, \partial_\theta$ and of $c^{(\alpha+1)}\, b^{(\beta)}\, \xi^{-\alpha-\beta}\, \theta$ are the same, and hence

$$\begin{cases} L(\alpha,\beta) & \text{if } \alpha \geq 1 \text{ and } \beta \geq 2, \\ 0 & \text{otherwise,} \end{cases} \quad (6.25)$$

where

$$L(\alpha,\beta) = -(-h)^{\alpha+\beta-2} \left(1 - \binom{\alpha+\beta-1}{\beta-1} + \sum_{n=1}^{\beta-1} \binom{\alpha+n-1}{\alpha-1} \right). \quad (6.26)$$

The same arguments show that expression (6.26) vanishes. We have proved thus that $[\widetilde{\Theta}_h, \widetilde{\Theta}_h] = 0$. Hence, the map $\widetilde{\rho}_2^{\lambda}$ is both a polynomial and formal deformation. To complete the proof of Proposition (4), observe that every infinitesimal deformation (6.4) satisfying the first of conditions (6.17) can be realized as the infinitesimal part of the polynomial deformation $\widetilde{\rho}_2^{\lambda}$. ■

Finally, we will construct a polynomial deformation corresponding to any infinitesimal deformation (6.4) satisfying the second of conditions (6.18). We apply the automorphism (6.19) to the polynomial deformation (6.21):

$$\widetilde{\rho}_3^{\lambda}(v_F) = \Psi_{\frac{-2\lambda}{h}} \circ \widetilde{\rho}_2^{-\lambda}(v_F) = \rho(v_F) - \lambda\eta^2(F) - (\lambda^2 + \frac{1}{2}\lambda h)(\sigma(\bar\eta^5(F)\bar\eta^{-3}) + \cdots, \quad (6.27)$$

so we obtain a polynomial deformation corresponding to any infinitesimal deformation (6.4) satisfying the second of conditions (6.18).

Remark 6. Under the first of conditions (6.18), any infinitesimal deformation (6.4) becomes a polynomial deformation.

Applying the contraction procedure as $h \to 0$ to the deformations (6.20), (6.21) and (6.27), we get polynomial deformations of the infinitesimal deformation of the standard embedding (5.1) generated by the 1-cocycles C_1, C_2 and C_3 corresponding to the conditions (6.16) at $h = 0$. More precisely, we get:

Theorem 3. *Every nontrivial polynomial deformation of the standard embedding (5.1) is equivalent to one of the four following deformations:*

$$
\begin{aligned}
\rho_1^\lambda(v_F) &= \tfrac{1}{2}\left((F + \sigma(F))\,\xi + \eta(F)\,\zeta\,\right) + \lambda\,\eta^2(F),\\
\rho_2^\lambda(v_F) &= \tfrac{1}{2}\left((F + \sigma(F))\,\xi + \eta(F)\,\zeta\,\right) + \lambda\left(\eta^2(F) - \sigma(\bar\eta^3(F))\,\xi^{-1}\,\bar\zeta\,\right),\\
\rho_3^\lambda(v_F) &= \tfrac{1}{2}\left((F(x - \tfrac{2\lambda}{\xi}) + \sigma(F(x - \tfrac{2\lambda}{\xi})))\,\xi + \eta(F(x - \tfrac{2\lambda}{\xi}))\,\zeta\,\right) + \lambda\,\eta^2(F(x - \tfrac{2\lambda}{\xi})),\\
\rho_4^\lambda(v_F) &= \tfrac{1}{2}\left((F(x - \tfrac{2\lambda}{\xi}) + \sigma(F(x - \tfrac{2\lambda}{\xi})))\,\xi + \eta(F(x - \tfrac{2\lambda}{\xi}))\,\zeta\,\right) + \lambda\left(\eta^2(F(x - \tfrac{2\lambda}{\xi}))\right.\\
&\quad \left.- \sigma(\bar\eta^3(F(x - \tfrac{2\lambda}{\xi})))\,\xi^{-1}\,\bar\zeta\,\right),
\end{aligned}
$$

where $\lambda \in \mathbb{R}$ is parameter of the deformation.

7 A variation of the central charge

The outer superderivation $\operatorname{ad}\log\xi \in H_0^1(\mathcal{S}\Psi\mathcal{DO}, \mathcal{S}\Psi\mathcal{DO})$ defines a nontrivial 2-cocycle with scalar values by the formula ([3])

$$\widetilde{C}_1(A, B) = Str([\log\xi, A] \circ B). \tag{7.1}$$

It is known that $\dim H^2(\mathcal{K}(1), \mathbb{C}) = 1$ ([9]) and $H^2(\mathcal{K}(1), \mathbb{C})$ is spanned by the Neveu-Schwarz cocycle:

$$C(v_F, v_G) = \frac{-1}{4}\int_{S^{1|1}} F\eta^5(G)\,\mathrm{vol}(x, \theta) = \frac{-1}{4}\int_{S^1}(4bd'' + ac''')dx, \tag{7.2}$$

where $F = a + 2b\theta$ and $G = c + 2d\theta$ with $a, b, c, d \in C^\infty(S^1)$.

Remark 7. The restriction of the 2-cocycle (7.1) to the Lie superalgebra $\mathcal{K}(1)$ coincides with the Neveu-Schwarz cocycle.

Proposition 5. *The restriction of the cocycle $\widetilde{C}_1$ to $\mathcal{K}(1) \hookrightarrow \mathcal{S}\Psi\mathcal{DO}_h$ with respect to the embedding (6.20, 6.21 or 6.27) is*

$$\widetilde{\rho^\lambda}^*(\widetilde{C}_1) = (h - 4\lambda)C. \tag{7.3}$$

Proof. By direct computations. ■

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