

Volichenko Algebras as Algebras of Differential Operators

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1 Introduction

Throughout this paper, \mathbb{k} denotes a field of characteristic 0 and all tensor products are over \mathbb{k} . Further, $\mathbb{k}[X; n]$ is the polynomial algebra in n commuting indeterminates $X = (x_1, x_2, \dots, x_n)$ and $\Lambda[Y; m]$ is the Grassmann algebra in n anti-commuting indeterminates $Y = (y_1, y_2, \dots, y_m)$.

Supersymmetries are symmetries of supervarieties, i.e., objects, functions on which depend on both usual commuting (even) variables and on anticommuting (odd) ones. For numerous applications of supersymmetry and for basics, see [3], [2] and [10]. Supersymmetries widened the notion of group in order to be able to mix Bose and Fermi particles. However, the collection of morphisms of supervarieties (locally, of its superalgebra of functions F) — supersymmetries — is not the largest possible group of automorphisms of the algebra F , with superstructure ignored. Besides, not every subalgebra or a quotient of a supercommutative superalgebra is supercommutative, whereas they are metaabelian and the notion of superscheme was first given ([7]) in terms of such, not necessarily homogeneous, subalgebras and quotients of supercommutative superalgebras. Recall that a ring M is said to be *metaabelian* if $[a, [b, c]] = 0$ for all $a, b, c \in M$, where $[a, b] = ab - ba$. Volichenko showed (see [8]) that every metaabelian algebra can be realized as a nonhomogeneous subalgebra of a universal supercommutative superalgebra, called its *supercommutative envelope*.

The purpose of this note is to construct an appropriate analog of differential operators on metaabelian algebras, more precisely, viewing a metaabelian algebra M as an analog of the algebra of functions, construct the corresponding algebra of vector fields.

Lunts and Rosenberg ([9]) constructed algebras of differential operators on (graded) noncommutative algebras. In particular, one can study differential operators on superalgebras. Superderivations of a superalgebra, which are first order differential operators, form a Lie superalgebra.

In a work aborted by his death, Volichenko gave a conjectural intrinsic description of nongraded subalgebras of Lie superalgebras. In his memory then, Leites and Serganova ([8]) called such subalgebras *Volichenko algebras* and (under a technical assumption) listed *simple* Volichenko algebras (finite dimensional and of vector fields). Like the list of simple

Lie superalgebras about doubles the list of simple Lie algebras of the same type [6], the list of simple Volichenko algebras about doubles the list of simple Lie superalgebras.

We construct differential operators on metaabelian algebras so that certain first order differential operators form a Volichenko algebra.

Preliminaries on differential operators defined by Lunts and Rosenberg are covered in section 2 followed by a section introducing superalgebras. In section 4, we study different kinds of differential operators on metaabelian algebras. Here, (4.1.2) we also construct a special tensor product of two metaabelian algebras. Note that the usual tensor product of metaabelian algebras is not necessarily metaabelian.

Given two metaabelian algebras M and N , let S_M and S_N be their respective supercommutative envelopes. Then, their *metaabelian tensor product*, $M \overset{m}{\otimes} N$ is defined as a natural nonhomogeneous subalgebra of $S_M \overset{s}{\otimes} S_N$. Molotkov has already defined a special tensor product of two metaabelian algebras (see appendix in [8]). Our constructions are different but we arrive at isomorphic metaabelian algebras because of the universal property (Proposition 4.1.3.)

Using the machinery of super differential operators on superalgebras, we define the Volichenko differential operators VD in section 4.2. That is, let S_M denote the supercommutative envelope of a metaabelian algebra M , along with a natural odd derivation d on S_M given by Volichenko. We let $D_\beta(S_M)$ denote the algebra of superdifferential (this is a specialization for the super case of the β -differential introduced in [9]) operators on S_M . Then,

$$VD(M) = \{\varphi_0 + [d, \varphi_0] \mid \varphi_0 \in D_\beta(S) \text{ is even}\}.$$

It then follows (Proposition 4.2.1) that $VD(M) = \cup_{n \geq 0} VD^n(M)$ is a filtered algebra with $VD^0(M)$ being the subalgebra of left multiplication homomorphisms in M and $VD^1(M)$ the direct sum as vector spaces of $VD^0(M)$ and a Volichenko algebra. We also show that there is a natural homomorphism $VD(M) \overset{m}{\otimes} VD(N) \longrightarrow VD(M \overset{m}{\otimes} N)$ for M, N metaabelian algebras.

Finally, in section 4.2.3, we describe the Volichenko differential operators on the metaabelian algebra $\mathbb{k}[Z]$, the subalgebra of $\mathbb{k}[X; n] \otimes \Lambda[Y; n]$ generated by Z , where $z_i = x_i + y_i$.

2 Preliminaries from [9]

2.1 $D_\beta(R)$, the space of β -differential operators.

Let Γ be an abelian group. Fix a bicharacter $\beta : \Gamma \times \Gamma \longrightarrow \mathbb{k}^*$. Let R be a Γ -graded \mathbb{k} -algebra and M a Γ -graded R -bimodule. Let $Z_\beta(M)$ denote the β -center of M defined as the \mathbb{k} -span of homogeneous elements $m \in M$ such that

$$mr = \beta(d_m, d_r)rm \text{ for any homogeneous } r \in R,$$

where d_x denotes the degree of x . The β -differential part of M is defined as follows: Let M_0 be the R -bimodule generated by $Z_\beta(M)$. Having defined M_i , define M_{i+1} as the R -bimodule generated by the homogeneous $m \in M$ such that

$$mr - \beta(d_m, d_r)rm \in M_i \text{ for any homogeneous } r \in R.$$

Each M_i is called the i th β -differential part of M . The β -differential part of M is the filtered, Γ -graded R -bimodule

$$M_\beta = \cup_{i \geq 0} M_{\beta i}.$$

The β -differential operators on R are the β -differential parts of the elements of the R -bimodule $M = \text{grHom}_{\mathbb{k}}(R, R)$. The R -bimodule structure on M is as follows:

$$(r\varphi s)(t) = r\varphi(st) \quad \text{for any } r, s, t \in R, \varphi \in \text{grHom}_{\mathbb{k}}(R, R).$$

The R -bimodule of β -differential operators is denoted by $D_\beta(R)$; the i th β -differential part of $\text{grHom}_{\mathbb{k}}(R, R)$ is denoted by $D_\beta^i(R)$ and the elements in $D_\beta^i(R)$ are called β -differential operators of order i . One can see that

$$D_\beta^i(R)D_\beta^j(R) \subset D_\beta^{i+j}(R).$$

Thus, $D_\beta(R)$ is a \mathbb{k} -algebra.

Remark 2.1.1. Let Γ_1 and Γ_2 be two abelian groups, equipped with their respective bicharacters $\beta_i : \Gamma_i \times \Gamma_i \longrightarrow \mathbb{k}$ for $i = 1, 2$. Let R and S be two finitely generated \mathbb{k} -algebras which are Γ_1 - and Γ_2 -graded, respectively, such that elements of \mathbb{k} are of degree 0. Then the usual tensor product $R \otimes S$ is a $\Gamma_1 \times \Gamma_2$ -graded \mathbb{k} -algebra. Similar to the proof of Theorem 3.1.1 of [4], we can show that $D_{\beta_1}(R) \otimes D_{\beta_2}(S) \cong D_{\beta_1\beta_2}(R \otimes S)$ as $R \otimes S$ -bimodules and as \mathbb{k} -algebras with the bicharacter given by

$$\begin{aligned} \beta_1\beta_2 : (\Gamma_1 \times \Gamma_2) \times (\Gamma_1 \times \Gamma_2) &\longrightarrow \mathbb{k}^*, \\ \beta_1\beta_2((a, b), (c, d)) &= \beta_1(a, c)\beta_2(b, d) \quad \text{for any } a, c \in \Gamma_1; b, d \in \Gamma_2. \end{aligned}$$

2.2 $D_q(R)$, the space of quantum differential operators.

These operators are defined for the same set-up as before, i.e., $\beta : \Gamma \times \Gamma \longrightarrow \mathbb{k}^*$ is a fixed bicharacter.

Let R be a Γ -graded \mathbb{k} -algebra and M be a Γ -graded R -bimodule. Denoted by $Z_q(M)$ the q -center of M defined as the \mathbb{k} -span of homogeneous elements $m \in M$ for which there exists a $d \in \Gamma$ such that

$$mr = \beta(d, d_r)rm \quad \text{for any homogeneous } r \in R.$$

Note that d depends on m , but is not necessarily the degree of m , and neither does it have to be unique. The rest of the construction follows as before for β -differential operators. We denote by $M_q = \cup_{i \geq 0} M_{q_i}$ the q -differential part of M , where M_{q_i} denotes the i th q -differential part. Likewise, $D_q(R) = \cup_{i \geq 0} D_q^i(R)$ is the ring of q -differential operators, where $D_q^i(R)$ is the R -bimodule of q -differential operators of order i on R . For each $a \in \Gamma$, we define the grading map $\sigma_a \in D_q^0(R)$ by setting

$$\sigma_a(r) = \beta(a, d_r)r.$$

Remark 2.2.1. (1) In the section covering preliminaries in [5], it is shown that $D_q^0(R)$ is a Γ -graded \mathbb{k} -algebra generated by homomorphisms given by left multiplications, right multiplications, and grading (Lemma 1.2.1 of [5]). The R -bimodule $D_q^{n+1}(R)$ is generated over $D_q^0(R)$ by the \mathbb{k} -span of homogeneous φ such that $(\varphi r - r\varphi) \in D_q^n(R)$ (Corollary 1.2.1 of [5]).

(2) For the same set-up as that of Remark 2.1.1, we have $D_q(R \otimes S) \cong D_q(R) \otimes D_q(S)$ as $R \otimes S$ -bimodules and as \mathbb{k} -algebras.

2.3 $D(R)$, the space of the usual differential operators.

For a non-graded set-up, in the above definitions, we have to merely consider R as trivially graded. That is, we consider $R = R_0$, $\beta \equiv 1$, M is an R -bimodule, and the ring of *differential operators* on R , denoted by $D(R)$, is the *differential part* of $\text{Hom}_{\mathbb{k}}(R)$.

Remark 2.3.1. (1) By Theorem 3.1.1 of [4], if R and S are finitely generated \mathbb{k} -algebras, then $D(R \otimes S) \cong D(R) \otimes D(S)$ as $R \otimes S$ -bimodules and as \mathbb{k} -algebras.

(2) If R is a Γ -graded \mathbb{k} -algebra and finitely generated as an algebra by homogeneous elements, then $D(R) \subset \text{grHom}_{\mathbb{k}}(R, R)$ by Theorem 3.2.1 of [4].

For the rest of this article, we fix a few notations: For any $r \in R$, let $\lambda_r, \rho_r \in \text{Hom}(R, R)$ be defined as

$$\lambda_r(s) = rs \quad \text{and} \quad \rho_r(s) = sr \quad \text{for any } s \in R.$$

For any homomorphism $\varphi \in \text{Hom}(R, R)$ and $x \in R$, we denote $[\varphi, x] := [\varphi, \lambda_x]$.

3 Superalgebras

Let $\mathbb{Z}_2 := \mathbb{Z}/2$. A \mathbb{Z}_2 -graded \mathbb{k} -algebra $R = R_0 \oplus R_1$ is called a *superalgebra*. Elements of R_0 are called *even* and those of R_1 are called *odd*. For a homogeneous element $a \in R$, we let $p(a)$ denote its *parity* (which is the same as *degree* in this case). The ring R is called *supercommutative* if, for any homogeneous elements $a, b \in R$, we have $ab = (-1)^{p(a)p(b)}ba$. Throughout our paper, the underlying field \mathbb{k} is trivially \mathbb{Z}_2 -graded, i.e., all the scalars are even.

We define a bicharacter $\beta : \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{k}^*$ by setting $\beta(x, y) = (-1)^{xy}$ and define the β -differential operators on supercommutative superalgebras as the elements of \mathbb{k} -algebra $D_\beta(R)$.

Remark 3.0.2. (1) The algebra of 0th order β -differential operators on a superalgebra R is generated by λ_r, ρ_s^β for homogeneous $r, s \in R$, where

$$\rho_s^\beta(t) = (-1)^{p(t)p(s)}ts.$$

When R is supercommutative, we have $\rho_s^\beta = \lambda_s$ for any homogeneous $s \in R$.

(2) Let φ be a *left superderivation* on a superalgebra R . That is, φ is a homogeneous homomorphism such that

$$\varphi(rs) = \varphi(r)s + (-1)^{p(\varphi)p(r)}r\varphi(s) \quad \text{for } r \text{ homogeneous.}$$

That is,

$$\varphi r - \beta(p(\varphi), p(r))r\varphi = \lambda_{\varphi(r)} \in D_{\beta}^0(R).$$

Hence $\varphi \in D_{\beta}^1(R)$, so first order β -differential operators on a supercommutative superalgebra form a Lie superalgebra. For preliminaries on (Lie) superalgebras, see [6], [2]. Let

$$[m, n]_{\beta} = mn - (-1)^{p(m)p(n)}nm.$$

Proposition 3.0.1. *Let $R = R_0 \oplus R_1$ be a supercommutative superalgebra. A homogeneous homomorphism φ belongs to $D_{\beta}^n(R)$ if and only if*

$$[\cdots [[\varphi, a_1]_{\beta}, a_2]_{\beta}, \cdots, a_{n+1}]_{\beta} = 0 \text{ for all homogeneous } a_i \in R.$$

Proof. Let a homogeneous $\varphi \in D_{\beta}^0(R)$ be a generator. That is, $[\varphi, a]_{\beta} = 0$ for any homogeneous $a \in R$ or $\varphi = \rho_r^{\beta}$. But $\rho_r^{\beta} = \lambda_r$. Thus, the proposition is true for $n = 0$. The rest follows by induction. ■

Corollary 3.0.1. *Let $R = R_0 \oplus R_1$ be a supercommutative superalgebra. Then $D_{\beta}^1(R) = D_{\beta}^0(R) \oplus \text{Der}_{\beta}(R)$ as vector spaces where $\text{Der}_{\beta}(R)$ denotes the left R -module of left superderivations on R .*

Proof. The sum is direct because superderivations vanish at 1. Given homogeneous $\varphi \in D_{\beta}^1(R)$, we let $\psi = \varphi - \rho_{\varphi(1)}^{\beta}$. Clearly, ψ is homogeneous with $\psi(1) = 0$, and

$$[\psi, r]_{\beta} = [\varphi, r]_{\beta} \text{ for all } r \in R.$$

Hence, $\psi \in D_{\beta}^1(R)$. Since $[[\psi, r_1]_{\beta}, r_2]_{\beta} = 0$, we see that $[\psi, r]_{\beta} = \lambda_{f_r}$. Since $\psi(1) = 0$, we have $\psi(r) = f_r$. In particular, $[\psi, r] = \lambda_{\psi(r)}$, hence ψ is a left superderivation. ■

Lemma 3.0.1. *Let $\{R_{\alpha}\}$ be a family of supercommutative superalgebras. Let $\prod_{\alpha} R_{\alpha}$ denote the unital supercommutative superalgebra, the direct product of the R_{α} . Then, $D_{\beta}(\prod_{\alpha} R_{\alpha}) \cong \prod_{\alpha} D_{\beta}(R_{\alpha})$ as filtered algebras.*

Proof. Note, $D_{\beta}^0(\prod_{\alpha} R_{\alpha}) \cong \prod_{\alpha} D_{\beta}^0(R_{\alpha})$. By induction we show that $D^n(\prod R_{\alpha}) \subset \prod_{\alpha} \text{End}_{\mathbb{k}}(R_{\alpha})$ which inturn will imply the result.

Consider a homogeneous

$$\Phi = \prod_{\alpha_1, \alpha_2} \varphi_{\alpha_1, \alpha_2} \in \text{End}_{\mathbb{k}}(\prod R_{\alpha}),$$

where $\varphi_{\alpha_1, \alpha_2} \in \text{Hom}(R_{\alpha_1}, R_{\alpha_2})$. Fix an $a_{\alpha_0} \in R_{\alpha_0}$. We have

$$[\Phi, a_{\alpha_0}]_{\beta} = [\varphi_{\alpha_0, \alpha_0}, a_{\alpha_0}]_{\beta} + \prod_{\alpha_2 \neq \alpha_0} \varphi_{\alpha_0, \alpha_2} a_{\alpha_0} - (-1)^c \prod_{\alpha_1 \neq \alpha_0} a_{\alpha_0} \varphi_{\alpha_1, \alpha_0} \in \prod \text{End}_{\mathbb{k}}(R_{\alpha})$$

for some $c \in \{\pm 1\}$. Hence, $\varphi_{\alpha_1, \alpha_2} a_{\alpha_0} = a_{\alpha_0} \varphi_{\alpha_1, \alpha_2} = 0$ for all $a_{\alpha_0} \in R_{\alpha_0}$. In particular, we have $\varphi_{\alpha_0, \alpha_2} = 0 = \varphi_{\alpha_1, \alpha_0}$ for $\alpha_1 \neq \alpha_0$ and $\alpha_2 \neq \alpha_0$. Hence the result. ■

4 Metaabelian algebras

Let R be a metaabelian \mathbb{k} -algebra. A suitable algebra of differential operators on R is $D(R)$ computed for R viewed as a non-graded algebra.

Proposition 4.0.2. (1) *Let R be a metaabelian \mathbb{k} -algebra. If $\varphi \in D^n(R)$, then*

$$[\cdots [[\varphi, x_1], x_2], \cdots, x_{2(n+1)}] = 0$$

for all $x_i \in R$.

(2) *In particular, a homomorphism φ belongs to $D(R)$ if and only if*

$$[\cdots [[\varphi, x_1], x_2], \cdots, x_n] = 0 \text{ for some } n \geq 0 \text{ and all } x_i \in R.$$

Proof. By definition of $D(R)$, if $[\cdots [[\varphi, x_1], x_2], \cdots, x_{n+1}] = 0$ for all $x_i \in R$, then $\varphi \in D^n(R)$. Hence the second statement follows from the first one. We prove the first one by induction on n . Since $D^0(R)$ is generated, as algebra, by $\{\lambda_r, \rho_s \mid r, s \in R\}$, we see that the result holds for $n = 0$. Assuming that the result holds true for $n - 1$, let $\lambda_r \varphi \in D^n(R)$ be such that $[\varphi, x] \in D^{n-1}(R)$ for all $x \in R$. For $x_1, x_2 \in R$, we have

$$[[\lambda_r \varphi, x_1], x_2] = \lambda_{[r, x_1]}[\varphi, x_2] + \lambda_{[r, x_2]}[\varphi, x_1] + \lambda_r [[\varphi, x_1], x_2]$$

which is in $D^{n-1}(R)$. Hence the proposition. ■

Corollary 4.0.2. *Let R be a supercommutative superalgebra. Then*

$$D_\beta(R) \subset D(R) \subset D_q(R).$$

Proof. The proof of Proposition 3.0.1 shows that $D_\beta^0(R) \subset D(R)$. Assume that $D_\beta^n(R) \subset D(R)$. Let $\varphi \in D_\beta^{n+1}(R)$ be homogeneous. Then, by Proposition 3.0.1, $[\varphi, a]_\beta \in D_\beta^n(R) \subset D(R)$. If φ is an even morphism (that is, $p(\varphi) = 0$), then $[\varphi, a] = [\varphi, a]_\beta \in D^n(R)$ for all $a \in R$, hence $\varphi \in D(R)$. If φ is an odd homomorphism and a homogeneous, then

$$[\varphi, a] = ((-1)^{p(a)} - 1)a\varphi + [\varphi, a]_\beta$$

which, in turn, is in $D(R)$ since $a\varphi$ is even when $p(a) = 1$. Hence the first inclusion. The second inclusion follows from Remark 2.3.1 and the definition of $D_q(R)$. ■

Remark 4.0.3. The inclusions in the corollary need not be filtration preserving. In general, these inclusions can be strict inclusions. For instance, for odd $r \in R$, the operator ρ_r lies in $D(R)$ but need not lie in $D_\beta(R)$, and σ_1 lies in $D_q(R)$ but need not be in $D(R)$ if R is not finitely generated.

4.1 A special case

Let $R = \mathbb{k}[\mathbf{X}; n] \otimes \Lambda[\mathbf{Y}; m]$ be a supercommutative superalgebra defined by $p(x_i) = 0$ and $p(y_i) = 1$.

Let us compute $D_\beta(R)$, $D_q(R)$ and $D(R)$. Since $\mathbb{k}[\mathbf{X}; n]$ is trivially graded,

$$D_\beta(\mathbb{k}[\mathbf{X}; n]) = D_q(\mathbb{k}[\mathbf{X}; n]) = D(\mathbb{k}[\mathbf{X}; n]).$$

Note, $D(\mathbb{k}[\mathbf{X}; n])$ is the Weyl algebra for $\text{char } \mathbb{k} = 0$. By Remarks 2.1.1, 2.2.1 and 2.3.1, it suffices to compute $D_\beta(\Lambda[\mathbf{Y}; m])$, $D_q(\Lambda[\mathbf{Y}; m])$ and $D(\Lambda[\mathbf{Y}; m])$. We already know that

$$D_\beta(\Lambda[\mathbf{Y}; m]) \subset D(\Lambda[\mathbf{Y}; m]) \subset D_q(\Lambda[\mathbf{Y}; m]).$$

For each $i \in \{1, 2, \dots, m\}$, we have $\lambda_{y_i}, \partial_i \in D_\beta(\Lambda[\mathbf{Y}; m])$, where $\partial_i(y_j) = \delta_{ij}$; extend ∂_i as a superderivation. It is easy to show that the \mathbb{k} -algebra generated by λ_{y_i} and ∂_i generates $\text{End}_{\mathbb{k}}(\Lambda[\mathbf{Y}; m])$ (the proof requires m to be finite). Hence,

$$D_\beta(R) = D(R) = D_q(R) = D(\mathbb{k}[\mathbf{X}; n]) \otimes \text{End}_{\mathbb{k}}(\Lambda[\mathbf{Y}; m]).$$

More is true.

Proposition 4.1.1. *Let R be a supercommutative superalgebra such that the number of odd generators is finite. Then*

$$D_\beta(R) = D(R) = D_q(R).$$

Proof. Let $\{a_1, a_2, \dots, a_m\}$ be the set of odd generators of R . By induction on m we show that $D_q^m(R) \subset D_\beta(R)$.

Note that $\rho_r = \lambda_r$ for r even and $\rho_r = \lambda_r \sigma_1$ for r odd. Since $D_\beta^0(R) = \{\lambda_r | r \in R\}$, it suffices to show that $\sigma_1 \in D_\beta(R)$. Since σ_1 is an even homomorphism, $[\sigma_1, r]_\beta = [\sigma_1, r]$. For r even, $[\sigma_1, r]_\beta = 0$. For r odd, say $r = a_1$, we have $[\sigma_1, a_1]_\beta = -2a_1\sigma_1$. Further, for r even, we see that $[a_1\sigma_1, r]_\beta = 0$ and for r odd, say $r = a_2$, we have $[a_1\sigma_1, a_2]_\beta = -2a_1a_2\sigma_1$. Proceeding similarly, we see that

$$[\dots [[\sigma_1, a_1]_\beta, a_2]_\beta, \dots a_n]_\beta = (-2)^n a_1 a_2 \dots a_n \sigma_1.$$

From this it follows that $\sigma_1 \in D_\beta^{n+1}(R)$.

Assume that $D_q^m(R) \subset D_\beta(R)$. By Remark 2.2.1, we need to consider $\varphi \in D_q^{m+1}(R)$ such that

$$[\varphi, r] \in D_q^m(R) \text{ for all } r \in R.$$

If φ is an even homomorphism, then $[\varphi, r]_\beta = [\varphi, r] \in D_q^m(R)$, and hence $\varphi \in D_\beta(R)$. If φ is an odd homomorphism, then consider

$$[\varphi, a_1]_\beta = \varphi a_1 + a_1 \varphi = [\varphi, a_1] + 2a_1 \varphi$$

which is an even homomorphism with $[\varphi, a_1] \in D_\beta(R)$. We next see that

$$[[\varphi, a_1]_\beta, a_2]_\beta = 4a_1 a_2 \varphi + \text{something in } D_\beta(R).$$

Proceeding thus, we have

$$[\dots [[\varphi, a_1]_\beta, a_2]_\beta, \dots a_n]_\beta = 2^n a_1 a_2 \dots a_n \varphi + \text{something in } D_\beta(R).$$

One further β -commutator places φ in $D_\beta(R)$. Hence the proposition. ■

Let R, S be two superalgebras. Then $R \overset{s}{\otimes} S$ is a superalgebra whose superspace is $R \otimes S$ with the usual addition, and the multiplication given by

$$(a \otimes b)(c \otimes d) = (-1)^{p(b)p(c)}(ac \otimes bd). \quad (4.1.1)$$

If R and S are supercommutative, then $R \overset{s}{\otimes} S$ is also supercommutative.

The natural maps

$$\mathrm{Hom}(R, R) \longrightarrow \mathrm{Hom}(R \overset{s}{\otimes} S, R \overset{s}{\otimes} S), \quad \varphi \mapsto (r \otimes s \mapsto \varphi(r) \otimes s) \quad (4.1.2)$$

$$\mathrm{Hom}(S, S) \longrightarrow \mathrm{Hom}(R \overset{s}{\otimes} S, R \overset{s}{\otimes} S), \quad \psi \mapsto (r \otimes s \mapsto r \otimes \psi_0(s) + \sigma_1(r) \otimes \psi_1(s)), \quad (4.1.3)$$

where ψ_0 is the even part and ψ_1 is the odd part of ψ , give rise to a map

$$\mathrm{Hom}(R, R) \overset{s}{\otimes} \mathrm{Hom}(S, S) \longrightarrow \mathrm{Hom}(R \overset{s}{\otimes} S, R \overset{s}{\otimes} S).$$

Note that the tensor product on the left is the tensor product of superalgebras, which is ensured because of the altered action (4.1.3) of $\mathrm{Hom}(S, S)$ on $R \overset{s}{\otimes} S$ above.

Remark 4.1.1. If d_R, d_S are superderivations of superalgebras R, S respectively, then the morphism $d_{R \otimes S} \in \mathrm{Hom}(R \overset{s}{\otimes} S, R \overset{s}{\otimes} S)$ defined by

$$d_{R \otimes S}(r \otimes s) = d_R(r) \otimes s + r \otimes (d_S)_0(s) + \sigma_1(r) \otimes (d_S)_1(s)$$

gives a superderivation of $R \overset{s}{\otimes} S$.

The following proposition is similar to Theorem 3.1.1 of [4]. We present it here for completeness of discussion.

Proposition 4.1.2. *If R, S are finitely generated by homogeneous elements, then there is an isomorphism of filtered superalgebras*

$$D_\beta(R) \overset{s}{\otimes} D_\beta(S) \cong D_\beta(R \overset{s}{\otimes} S).$$

Proof. For any $\varphi \in D_\beta(R)$ and $\psi \in D_\beta(S)$, we have

$$[\varphi \otimes \psi, r \otimes s]_\beta = (-1)^{p(\psi)p(r)} \left([\varphi, r]_\beta \otimes \psi s + (-1)^{p(\varphi)p(r)} r \varphi \otimes [\psi, s]_\beta \right).$$

Hence, $D_\beta^a(R) \overset{s}{\otimes} D_\beta^b(S) \subset D_\beta^{a+b}(R \overset{s}{\otimes} S)$.

It remains to show that if a homogeneous $\varphi \in \mathrm{Hom}(R \overset{s}{\otimes} S)$ is such that $[\varphi, r \otimes s]_\beta \in D_\beta(R) \overset{s}{\otimes} D_\beta(S)$ for all $r \otimes s \in R \overset{s}{\otimes} S$, then $\varphi \in D_\beta(R) \overset{s}{\otimes} D_\beta(S)$.

By changing φ to $\varphi - \lambda_{\varphi(1 \otimes 1)}$, we assume that $\varphi(1 \otimes 1) = 0$.

First consider the case where $[\varphi, r \otimes 1]_\beta = 0$ for all $r \in R$. Let $\{s_1, \dots, s_n\}$ be a finite set of homogeneous generators of S over \mathbb{k} . Let η_i denote $[\varphi, 1 \otimes s_i]_\beta$. For each i , let $\eta_i = \sum \eta_{i(1)} \otimes \eta_{i(2)} \in D_\beta(R) \overset{s}{\otimes} D_\beta(S)$, where $\eta_{i(1)}$ and $\eta_{i(2)}$ are homogeneous.

Note that for homogeneous r ,

$$[[\varphi, 1 \otimes s_i]_\beta, r \otimes 1]_\beta = (-1)^{p(s_i)p(r)} [[\varphi, r \otimes 1]_\beta, 1 \otimes s_i]_\beta = 0.$$

So, we may assume that $[\eta_{i(1)}, r]_\beta = 0$ for all $r \in R$.

Let $V_1 \subset D_\beta(R)$ be the finite dimensional super vector space spanned by the $\eta_{i(1)}$. Every element of V_1 commutes with $r \in R$. For $s, s' \in S$, we have

$$[\varphi, 1 \otimes ss']_\beta = [\varphi, 1 \otimes s]_\beta(1 \otimes s') + (-1)^{p(\varphi)p(s)}(1 \otimes s)[\varphi, 1 \otimes s']_\beta.$$

Hence, every $[\varphi, 1 \otimes s]_\beta$ can be expressed as $\sum_{i=1}^l f_i \otimes g_i^s$, where $\{f_i\}_{i=1}^l \subset D_\beta(R)$ is a homogeneous basis of V_1 , and $g_i^s \in D_\beta(S)$, where $p(g_i^s) = p(\varphi) + p(s) + p(f_i)$. For each $i = 1, \dots, l$, define a function $h_i : S \rightarrow S$ setting $h_i(s) = g_i^s(1)$. Since $[\varphi, 1 \otimes s]_\beta$ is linear in s , it follows that $h_i \in \text{Hom}(S, S)$. Also, $p(h_i) = p(f_i) + p(\varphi)$, and hence is independent of s .

For $s, t \in S$, consider

$$[h_i, s]_\beta(t) = h_i(st) - (-1)^{p(h_i)p(s)} s h_i(t) = g_i^{st}(1) - (-1)^{p(h_i)p(s)} s g_i^t(1).$$

Since

$$[\varphi, 1 \otimes st]_\beta = [\varphi, 1 \otimes s]_\beta(1 \otimes t) + (-1)^{p(\varphi)p(s)}(1 \otimes s)[\varphi, 1 \otimes t]_\beta,$$

we have

$$\begin{aligned} \sum_i f_i \otimes g_i^{st} &= \left(\sum_i f_i \otimes g_i^s \right) (1 \otimes t) + (-1)^{p(\varphi)p(s)}(1 \otimes s) \sum_i f_i \otimes g_i^t \\ &= \sum_i f_i \otimes g_i^s t + (-1)^{p(\varphi)p(s)+p(f_i)p(s)} \sum_i f_i \otimes s g_i^t \\ &= \sum_i f_i \otimes g_i^s t + (-1)^{(p(\varphi)+p(f_i))p(s)} \sum_i f_i \otimes s g_i^t. \end{aligned}$$

That is, $g_i^{st} = g_i^s t + (-1)^{(p(\varphi)+p(f_i))p(s)} s g_i^t$. Thus, $[h_i, s]_\beta(t) = g_i^s \in D_\beta(S)$ which further implies that $h_i \in D_\beta(S)$. Hence, $\psi = \sum f_i \otimes h_i \in D_\beta(R) \overset{s}{\otimes} D_\beta(S)$. One can see that $[\psi, r \otimes 1]_\beta = [\varphi, r \otimes 1]_\beta = 0$ and $[\psi, 1 \otimes s]_\beta = [\varphi, 1 \otimes s]_\beta$. Since $\psi(1 \otimes 1) = \varphi(1 \otimes 1) = 0$, we have $\varphi = \psi \in D_\beta(R) \overset{s}{\otimes} D_\beta(S)$.

Now consider any $\varphi \in D_\beta(R \overset{s}{\otimes} S)$ such that $[\varphi, r \otimes s]_\beta \in D_\beta(R) \overset{s}{\otimes} D_\beta(S)$ with $\varphi(1 \otimes 1) = 0$. Let $\{r_1, r_2, \dots, r_m\}$ be a set of homogeneous generators of R . Let

$$[\varphi, r_i \otimes 1] = \mu_i = \sum \mu_{i(1)} \otimes \mu_{i(2)} \in D_\beta(R) \overset{s}{\otimes} D_\beta(S).$$

Let $V_2 \subset D_\beta(S)$ denote the finite dimensional super vector space spanned by the $\mu_{i(2)}$. Let $\{g_1, g_2, \dots, g_p\}$ be a basis of homogeneous elements in V_2 . Let $[\varphi, r \otimes 1] = \sum_i f_i^r \otimes g_i$ for $f_i^r \in D_\beta(R)$. For each i , let $e_i \in \text{Hom}(R, R)$ denote the homogeneous homomorphism given by $e_i(r) = f_i^r(1)$. As before, we can see that $[e_i, r]_\beta = f_i^r \in D_\beta(R)$. Hence, $\psi = \sum e_i \otimes g_i \in D_\beta(R) \overset{s}{\otimes} D_\beta(S)$. We have,

$$(\varphi - \psi)(1 \otimes 1) = 0, \quad [(\varphi - \psi), r \otimes 1]_\beta = 0, \quad \text{and} \quad [(\varphi - \psi), r \otimes s]_\beta \in D_\beta(R) \overset{s}{\otimes} D_\beta(S).$$

As before, $\varphi - \psi \in D_\beta(R) \overset{s}{\otimes} D_\beta(S)$ which proves the result. \blacksquare

4.1.1 The universal supercommutative envelope

Every metaabelian algebra is a nonhomogeneous subalgebra of a supercommutative superalgebra. Its universal supercommutative envelope was constructed by Volichenko as follows [8].

For a metaabelian M , let $M^{(+)}$ denote the related *Jordan algebra*. That is, as a set $M^{(+)} = \{m^{(+)} \mid m \in M\}$. The vector space structure on $M^{(+)}$ is given by $(m^{(+)} + n^{(+)} = (m + n)^{+})$ and $\alpha m^{(+)} = (\alpha m)^{+}$ for any $\alpha \in \mathbb{k}$ and $m, n \in M$. The multiplication on $M^{(+)}$ is defined as

$$m^{(+)} \circ n^{(+)} = \frac{1}{2}(mn + nm)^{+}.$$

Let C denote the subalgebra of M generated by the elements $[m, n]$ for any $m, n \in M$. Let $\Omega_{M^{(+)}/C}^1$ denote the $M^{(+)}$ -module of differentials, i.e., $\Omega_{M^{(+)}/C}^1$ is the quotient of the free $M^{(+)}$ -module with basis dm where $m \in M$, modulo the submodule generated by

$$d(m + n) - dm - dn, \quad d(mn) - m^{(+)}dn - n^{(+)}dm, \quad \text{for any } m, n \in M.$$

Note that $dc = 0$ for any $c \in C$ in $\Omega_{M^{(+)}/C}^1$. The universal supercommutative envelope of M is $S_M = M^{(+)} \oplus \Omega_{M^{(+)}/C}^1$, whose even component is $M^{(+)}$ and the odd component is $\Omega_{M^{(+)}/C}^1$; the product of odd elements being $dm \cdot dn := \frac{1}{2}[m, n]^{+}$. The metaabelian algebra M can be seen as a nonhomogeneous subspace $\{m^{(+)} + dm \mid m \in M\}$ of S_M .

4.1.2 Tensor products of metaabelian algebras

A special tensor product of metaabelian algebras was constructed by V Molotkov in an Appendix to [8]. Here we present another, simpler, construction which also satisfies the universal property of Proposition 4.1.3.

We first describe metaabelian tensor product of two (possibly nonhomogeneous) subalgebras of superalgebras.

Let S and T be two \mathbb{k} -superalgebras. Let $A \subset S$ and $B \subset T$ be subalgebras, possibly nonhomogeneous. We have an inclusion of vector spaces $A \otimes B \subset S \overset{s}{\otimes} T$. The multiplication (4.1.1) on $S \overset{s}{\otimes} T$ with a natural parity makes $S \overset{s}{\otimes} T$ into a superalgebra. The vector space $A \otimes B$ is not necessarily closed under this multiplication. For instance

$$(1 \otimes b)(a \otimes 1) = (a \otimes b - 2a_1 \otimes b_1) \notin A \otimes B,$$

where a_1, b_1 are the *odd* parts of a, b , respectively, for $a \in A$ and $b \in B$. Therefore, define the metaabelian tensor product of algebras A and B to be the subalgebra of $S \overset{s}{\otimes} T$ generated by $A \otimes B$ and denote it by $A \overset{m}{\otimes} B$.

There are natural maps of algebras $i_A : A \longrightarrow A \overset{m}{\otimes} B$ and $i_B : B \longrightarrow A \overset{m}{\otimes} B$ given by $i_A(a) = a \otimes 1$ and $i_B(b) = 1 \otimes b$.

Remark 4.1.2. The metaabelian tensor product is dependent on its ambient space. For instance, consider the following two examples:

- (1) Let $\mathbb{k}[x]$ and $\mathbb{k}[y]$ be purely even. Then, $\mathbb{k}[x] \otimes \mathbb{k}[y] = \mathbb{k}[x, y]$.

(2) Let $\mathbb{k}[z_1] = \mathbb{k}[x_0 + x_1]$ and $\mathbb{k}[z_2] = \mathbb{k}[y_0 + y_1]$ be metaabelian algebras in their supercommutative envelopes $\mathbb{k}[x_0] \otimes \Lambda(x_1)$ and $\mathbb{k}[y_0] \otimes \Lambda(y_1)$. Then,

$$\mathbb{k}[z_1, z_2] := \mathbb{k}[z_1] \overset{m}{\otimes} \mathbb{k}[z_2] = \mathbb{k}[x_0 + x_1] \overset{m}{\otimes} \mathbb{k}[y_0 + y_1] = \mathbb{k}[x_0 + x_1, y_0 + y_1]$$

is a metaabelian subalgebra in $\mathbb{k}[x_0, y_0] \otimes \Lambda(x_0, y_0)$. Observe that $\mathbb{k}[z_1, z_2]$ is no longer commutative. Hence, it is not algebraically isomorphic to $\mathbb{k}[x, y]$ even though $\mathbb{k}[x] \cong \mathbb{k}[z_1]$ and $\mathbb{k}[y] \cong \mathbb{k}[z_2]$ as algebras.

Let M and N be metaabelian algebras. Let S_M and S_N be their supercommutative envelopes.

The supercommutative envelopes are universal constructions. So, we have natural inclusions $M \subset S_M$ and $N \subset S_N$. By construction $M \overset{m}{\otimes} N$ is metaabelian. This product possesses universal properties:

Proposition 4.1.3. *Let A, B, C be three metaabelian algebras. Let $f : A \rightarrow C$ and $g : B \rightarrow C$ be algebra homomorphisms. Then there exists a unique algebra homomorphism $h : A \overset{m}{\otimes} B \rightarrow C$ such that $f = h \circ i_A$ and $g = h \circ i_B$.*

Proof. An algebra homomorphism $f : A \rightarrow C$ naturally gives an *even* algebra homomorphism of supercommutative superalgebras $S_f : S_A \rightarrow S_C$ such that the restriction of S_f on A is f . Likewise, we have an *even* algebra homomorphism of supercommutative superalgebras $S_g : S_B \rightarrow S_C$ such that the restriction of S_g on B is g .

Since $S_A \overset{s}{\otimes} S_B$ is a tensor product with universal properties, we have the unique map $S_h : S_A \overset{s}{\otimes} S_B \rightarrow S_C$. By construction, $S_h(a \otimes 1) = S_f(a) = f(a) \in C$ and $S_h(1 \otimes b) = S_g(b) = g(b) \in C$ for any $a \in A$ and $b \in B$. Denote by h the restriction of S_h on $A \overset{m}{\otimes} B$. Then we have $f = i_A \circ h$ and $g = i_B \circ h$.

Let a map $l : A \overset{m}{\otimes} B \rightarrow C$ be given such that $f = l \circ i_A$ and $g = l \circ i_B$. Then, we have $h(a \otimes 1) = S_f(a \otimes 1) = l(a \otimes 1)$ and $h(1 \otimes b) = S_g(1 \otimes b) = l(1 \otimes b)$. Hence, $h = l$. \blacksquare

Proposition 4.1.4. *Let M, N be two metaabelian algebras, S_M, S_N their corresponding supercommutative envelopes. Then $S_M \overset{s}{\otimes} S_N$ is the supercommutative envelope of $M \overset{m}{\otimes} N$.*

Proof. We write $S_M = M^{(+)} \oplus \Omega_{M^{(+)} / C_M}^1$ and $S_N = N^{(+)} \oplus \Omega_{N^{(+)} / C_N}^1$, where C_M and C_N are the commutator subalgebras of M and N respectively. We need to show that

- $(M \overset{m}{\otimes} N)^{(+)} = (S_M \overset{s}{\otimes} S_N)_0$;
- $(S_M \overset{s}{\otimes} S_N)_1 = \Omega_{(M \overset{m}{\otimes} N)^{(+)} / C_{M \overset{m}{\otimes} N}}^1$ is the module of differentials over $(M \overset{m}{\otimes} N)^{(+)}$ generated by $d_{S_M \otimes S_N} r$ for $r \in (M \overset{m}{\otimes} N)$;
- $C_{M \overset{m}{\otimes} N} \subset \text{Ker}(d_{S_M \otimes S_N})$,

where $d_{S_M \otimes S_N}$ is the derivation of $S_M \overset{s}{\otimes} S_N$ defined in Remark (4.1.1). Note that $d_{S_M \otimes S_N}$ is odd and $d_{S_M \otimes S_N}^2 = 0$. We will use the letter d to mean any of the derivations $d_M, d_N, d_{S_M \otimes S_N}$ as will be clear from the context. We have an isomorphism

$$\begin{aligned} (M \overset{m}{\otimes} N)^{(+)} &\longrightarrow (S_M \overset{s}{\otimes} S_N)_0 \\ (m \otimes n)^{(+)} &\mapsto (m^{(+)} \otimes n^{(+)} + (dm \otimes dn)) \end{aligned}$$

for any $m \in M, n \in N$. Similarly, note that

$$d(m \otimes n) = dm \otimes n^{(+)} + m^{(+)} \otimes dn.$$

Hence $(S_M \overset{s}{\otimes} S_N)_1$ is generated over $(M \overset{m}{\otimes} N)^{(+)}$ by dr for $r \in (M \overset{m}{\otimes} N)$. Lastly, we see that $C_{M \overset{m}{\otimes} N}$ is generated by elements of the kind $e[r, s]$ for $r, s \in (M \overset{m}{\otimes} N)$ and e an even element, multiple of terms of the form $dm \otimes dn$ for $m \in M$ and $n \in N$. This implies that $d(e) = 0$. So, it remains to show that $d([r, s]) = 0$ for and $r = (m \otimes n)$ and $s = (a \otimes b)$ where $m, a \in M$ and $n, b \in N$; this follows from a straightforward calculation. \blacksquare

Remark 4.1.3. Let $S = \mathbb{k}[\mathbf{X}; n] \otimes \Lambda[\mathbf{Y}; n]$ and $T = \mathbb{k}[\mathbf{A}; m] \otimes \Lambda[\mathbf{B}; m]$. Let $\mathbb{k}[Z]$ be the non-graded subalgebra in S generated by all the elements $z_i = x_i + y_i$ and $\mathbb{k}[C]$ be the non-graded subalgebra in T generated by all the elements $c_i = a_i + b_i$. Then, the tensor product of the two metaabelian algebras $\mathbb{k}[Z]$ and $\mathbb{k}[C]$ is the metaabelian algebra $\mathbb{k}[Z, C]$ contained in $\mathbb{k}[\mathbf{X}, \mathbf{A}; n + m] \otimes \Lambda[\mathbf{Y}, \mathbf{B}; n + m]$.

4.2 Volichenko algebras as algebras of differential operators

Volichenko algebras are to Lie superalgebras what metaabelian algebras are to supercommutative superalgebras ([8]): a *Volichenko algebra* is defined to be a non-homogeneous subspace of a Lie superalgebra closed under the superbracket. No intrinsic definition of Volichenko algebras (in terms of defining identities) is available yet, see [1], [8].

Recall that the algebra of usual differential operators on M is

$$D(M) := \{\varphi \in \text{End}_{\mathbb{k}}(M) \mid \cdots [[\varphi, m_1], m_2], \cdots m_n] = 0 \text{ for some } n \geq 0 \text{ and any } m_i \in M\}.$$

We also introduce other algebras of differential operators on M , $D_\beta(M)$ and $VD(M)$.

4.2.1 β -differential operators on M

We define the algebra of β -differential operators on M as

$$D_\beta(M) = D_\beta(S) \cap \text{End}_{\mathbb{k}}(M) = \{\varphi \in D_\beta(S) \mid \varphi(M) \subset M\}.$$

Remark 4.2.1. The following are immediate:

- (1) $D_\beta(M) \subset D(M)$;
- (2) $D_\beta(M)$ is a filtered algebra, $D_\beta(M) = \cup_n D_\beta^n(M)$, where

$$D_\beta^n(M) = \{\varphi \in D_\beta^n(S) \mid \varphi(M) \subset M\}.$$

For any $\varphi = \varphi_0 + \varphi_1 \in \text{End}_{\mathbb{k}}(S)$ and $a = a_0 + a_1 \in S$, we let $\bar{a} = a_0 - a_1$ and $\bar{\varphi} = \varphi_0 - \varphi_1$. Consider the β -commutator,

$$[\varphi, a]_{\beta} = \varphi a - (a_0 \varphi + a_1 \bar{\varphi}) = \varphi a - (a \varphi_0 + \bar{a} \varphi_1).$$

It is not clear whether $[\varphi, a]_{\beta} \in D_{\beta}(M)$ if $\varphi \in D_{\beta}(M)$ and $a \in M$. The situation is remedied by introducing the Volichenko differential operators.

4.2.2 Volichenko differential operators on M

The construction of a supercommutative envelope of M gives us a natural superderivation $d : S \rightarrow S$ of parity 1 given on the generators by

$$d(m^{(+)}) = dm \quad \text{and} \quad d(dm) = 0.$$

Clearly, $d^2 = 0$. For an even $\varphi_0 \in D_{\beta}^n(S)$, one can show by induction on n that $[d, \varphi_0] \in D_{\beta}^n(S)$. Define the algebra of *Volichenko differential operators* as

$$VD(M) = \{\varphi_0 + [d, \varphi_0] \mid \varphi_0 \in D_{\beta}(S) \text{ is even}\}.$$

The following proposition justifies the terminology used above.

Proposition 4.2.1. (1) For any even $\varphi_0 \in D_{\beta}(S)$, we have $(\varphi_0 + [d, \varphi_0])(m) \in M$ for all $m \in M$. In particular, $VD(M) \subset D_{\beta}(M)$.

(2) The vector space $VD(M)$ is a filtered algebra, $VD(M) = \cup_{n \geq 0} VD^n(M)$, where

$$VD^n(M) = \{\varphi_0 + [d, \varphi_0] \mid \varphi_0 \in D_{\beta}^n(S) \text{ is even}\}.$$

In particular, $VD(M) \subset D_{\beta}(M)$ as filtered algebras.

(3) $VD^0(M) = \{\lambda_m \mid m \in M\}$ and $VD^1(M) = VD^0(M) \oplus V$ as vector spaces where V is a Volichenko algebra.

(4) For any $\varphi = \varphi_0 + [d, \varphi_0] \in VD^n(M)$, we have $[\varphi, m]_{\beta} \in VD^{n-1}(M)$ for any $m \in M$. In particular, if $\varphi \in VD^n(M)$, then $[\cdots [\varphi, m^0]_{\beta}, m^1]_{\beta}, \cdots, m^n]_{\beta} = 0$, where $m^i \in M$.

Proof. (1) We write $m = m^{(+)} + dm$ and

$$(\varphi_0 + [d, \varphi_0])(m) = (\varphi_0(m^{(+)}) + [d, \varphi_0](dm)) + (\varphi_0(dm) + [d, \varphi_0](m^{(+)})).$$

From direct checking it follows that

$$d(\varphi_0(m^{(+)}) + [d, \varphi_0](dm)) = (\varphi_0(dm) + [d, \varphi_0](m^{(+)})).$$

(2) For $\varphi_0, \psi_0 \in D_{\beta}(S)$ even, we see that $[d, \varphi_0][d, \psi_0]$ is an even homomorphism and $[d, [d, \varphi_0][d, \psi_0]] = 0$. Thus, $[d, \varphi_0][d, \psi_0] \in VD(M)$. Further

$$(\varphi_0 + [d, \varphi_0])(\psi_0 + [d, \psi_0]) = (\varphi_0 \psi_0 + [d, \varphi_0 \psi_0]) + [d, \varphi_0][d, \psi_0] \in VD(M).$$

Hence, $VD(M)$ is an algebra.

(3) The first part follows from $[d, \lambda_{m^{(+)}}] = \lambda_{dm}$. As spaces, $D_{\beta}^1(S) = D_{\beta}^0(S) \oplus \text{Der}_{\beta}(S)$, where $\text{Der}_{\beta}(S)$ is a Lie superalgebra (see Corollary 3.0.1). The odd derivation $d \in D_{\beta}^1(S)$

is *homologic* (i.e., $[d, d]_\beta = 0$). Hence, by Example 2.5 of [8], $VD^1(M)$ is a Volichenko algebra.

(4) For $m = m_0 + dm \in M$ and $\varphi = \varphi_0 + [d, \varphi_0]$, we have

$$[\varphi, m]_\beta = [\varphi_0, m_0] + [d, \varphi_0]dm + dm[d, \varphi_0] + [\varphi_0, dm] + [[d, \varphi_0], m_0].$$

It is easy to check that

$$[d, [\varphi_0, m_0] + [d, \varphi_0]dm + dm[d, \varphi_0]] = [\varphi_0, dm] + [[d, \varphi_0], m_0].$$

■

Remark 4.2.2. The result (4) of the above proposition, is not an if and only if statement. That is, if $[\varphi, m] \in VD^{n-1}(M)$, then φ need not be in $VD^n(M)$. For example, consider $S = \mathbb{k}[x] \otimes \Lambda(y)$ a supercommutative superalgebra with $p(x) = 0$ and $p(y) = 1$, and let $M = \mathbb{k}[x + y]$, be the polynomial ring sitting diagonally in S . Let $\psi_1 = x\partial_y + y\partial_x \in Der_\beta(S)$. We see that $[\psi_1, m]_\beta = \lambda_{\psi_1(m)} \in VD^0(M)$ and ψ keeps M invariant, but ψ_1 is a nonzero purely odd operator. Hence $\psi_1 \notin VD^1(M)$.

Theorem 4.2.1. (1) $VD(\prod_\alpha M_\alpha) \cong \prod_\alpha VD(M_\alpha)$ as filtered algebras, where M_α is a metaabelian algebra for each α .

(2) There is an inclusion $VD(M) \otimes^m VD(N) \longrightarrow VD(M \otimes^m N)$ of filtered algebras, where M, N are finitely generated metaabelian algebras.

Proof. (1) For each α , let S_α denote the supercommutative envelope of M_α with d_α the associated superderivation. The direct product $\prod_\alpha S_\alpha$ along with the superderivation $\prod_\alpha d_\alpha$ is the supercommutative envelope of $\prod_\alpha M_\alpha$. The result follows from Lemma 3.0.1.

(2) Let S_M, S_N be the supercommutative envelopes of M, N respectively. Since M and N are finitely generated, S_M, S_N are finitely generated. Let the corresponding natural derivations be denoted by d_M and d_N respectively. Consider their tensor derivation $d_{M \otimes N} : S_M \overset{s}{\otimes} S_N \longrightarrow S_M \overset{s}{\otimes} S_N$ given by

$$d_{M \otimes N}(a \otimes b) = d_M(a) \otimes b + (-1)^{p(a)} a \otimes d_N(b).$$

As in extensions (4.1.2) and (4.1.3), we extend operators in $VD(M)$ and $VD(N)$ to $\text{Hom}(M \overset{m}{\otimes} N, M \overset{m}{\otimes} N)$. By the nature of the extension, one can see that these extended operators are in $VD(M \overset{m}{\otimes} N)$. Proposition 4.1.2 completes the proof. ■

4.2.3 An example

Let $\mathbb{k}[Z]$ be the nonhomogeneous subalgebra of $S = \mathbb{k}[\mathbf{X}; n] \otimes \Lambda[\mathbf{Y}; n]$ generated by variables $Z = (z_1, z_2, \dots, z_n)$, with $z_i = x_i + y_i$.

Note that $D_\beta(S) = D_\beta(\mathbb{k}[\mathbf{X}; n]) \otimes D_\beta(\Lambda[\mathbf{Y}; n])$. The algebra of differential operators on $\mathbb{k}[\mathbf{X}; n]$ is the Weyl algebra. That is,

$$D_\beta(\mathbb{k}[\mathbf{X}; n]) = \mathbb{k} \langle x_1, x_2, \dots, x_n, \partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n} \rangle / Rel,$$

where Rel is the set of relations generated by $[\partial_{x_i}, x_j] = \delta_{i,j}$, $[x_i, x_j] = 0 = [\partial_{x_i}, \partial_{x_j}]$. Likewise,

$$D_\beta(\Lambda[\mathbf{Y}; n]) = \mathbb{k} \langle y_1, y_2, \dots, y_n, \partial_{y_1}, \partial_{y_2}, \dots, \partial_{y_n} \rangle / Rel',$$

where Rel' is the set of relations generated by $[\partial_{y_i}, y_j]_\beta = \delta_{i,j}, [y_i, y_j]_\beta = 0 = [\partial_{y_i}, \partial_{y_j}]_\beta$. Here, the operators ∂_{y_i} and $y_i = \lambda_{y_i}$ are odd operators. The operators ∂_{x_i} (and ∂_{y_i}) is an even (respectively, odd) superderivation on S satisfying $\partial_{y_i}(y_j) = \delta_{i,j}, \partial_{y_i}(x_j) = 0$ (respectively, $\partial_{y_i}(y_j) = \delta_{i,j}, \partial_{y_i}(x_j) = 0$).

Note that S is the supercommutative envelope of $\mathbb{k}[Z]$, with the natural odd derivation $d = \sum_i y_i \partial_{x_i}$. Hence,

$$VD^n(\mathbb{k}[Z]) = \{\varphi_0 + [d, \varphi_0] \mid \varphi_0 \in D_\beta^n(S) \text{ is even}\}.$$

The subalgebra $(D_\beta(S))_0$ of even elements in $D_\beta(S)$ is generated by

$$\{x_i, \partial_{x_i}\}_i \cup \{y_i y_j, \partial_{y_i} \partial_{y_j}\}_{i < j} \cup \{y_i \partial_{y_j}\}_{i,j}.$$

The following formulae are easy to check:

$$\begin{aligned} [d, x_i] &= y_i, & [d, \partial_{x_i}] &= 0, & [d, y_i y_j] &= 0; \\ [d, \partial_{y_i} \partial_{y_j}] &= \partial_{x_i} \partial_{y_j} - \partial_{y_i} \partial_{x_j}, & [d, y_i \partial_{y_j}] &= -y_i \partial_{x_j}. \end{aligned}$$

Hence, the Volichenko differential operators on $\mathbb{k}[Z]$ are generated by the operators, for $1 \leq i, j \leq n$,

- $x_i + y_i$ (this operator acts as left-multiplication by z_i on $\mathbb{k}[Z]$),
- ∂_{x_i} (this operator acts as the derivation ∂_{z_i} on $\mathbb{k}[Z]$),
- $\partial_{y_i} \partial_{y_j} + \partial_{x_i} \partial_{y_j} - \partial_{y_i} \partial_{x_j}$ (this is a second order Volichenko differential operator acting nontrivially),
- $y_i \partial_{y_j} - y_i \partial_{x_j}$ (this is called a *ghost operator*, since it is a non-zero operator whose action on $\mathbb{k}[Z]$ is 0).

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References

- [1] Baranov A A, Volichenko Algebras and nonhomogeneous subalgebras of Lie superalgebras, *Siberian mathematical Journal* **36** (1995), 859–868, (Eng. trans.), *Sibirskii Mat. Z.* **36** (1995), 998–1009.
- [2] Deligne P, Etingof P, Freed D, Jeffrey L, Kazhdan D, Morgan J, David R, Morrison D, and Witten E, Quantum fields and strings: a course for mathematicians, Vol. 1, 2, American Mathematical Society, Providence, 1999.
- [3] Duplij S, Bagger J, Siegel W (eds.) Concise Encyclopedia of Supersymmetry and Noncommutative Structures in Mathematics and Physics, Kluwer, Dordrecht, 2003.

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- [4] Iyer U N, Differential Operators on Hopf algebras and some functorial properties, *Manuscripta math.* **109** (2002), 121–129.
 - [5] Iyer U N, McCune T C, Quantum differential operators on $\mathbb{k}[x]$, *International Journal of Mathematics* **13** (2002), 395–413.
 - [6] Kac V G, Lie Superalgebras, *Advances in Math.* **26** (1997), 8–96.
 - [7] Leites D, Spectra of graded-commutative rings, *Uspehi Matem. Nauk* **29** (1974), 157–158 (in Russian).
 - [8] Leites D, Serganova V, Symmetries wider than supersymmetry, in Noncommutative structures in Mathematics and Physics, Editors: Duplij S and Wess J, 2001, 13–30.
 - [9] Lunts V A, Rosenberg A L, Differential operators on noncommutative rings, *Selecta Math.(N.S)* **3** (1997), 335–359.
 - [10] Manin Yu I, Gauge field theory and complex geometry, Second edition, Springer-Verlag, Berlin, 1997.