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# Gauge Transformation and Reciprocal Link for (2+1)-Dimensional Integrable Field Systems

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**Abstract**

Appropriate restrictions of Lax operators which allows to construction of (2+1)-dimensional integrable field systems, coming from centrally extended algebra of pseudo-differential operators, are reviewed. The gauge transformation and the reciprocal link between three classes of Lax hierarchies are established.

## 1 Introduction

It is well known that approach of the classical $R$-matrix formalism to the specific infinite-dimensional Lie algebras can be used for systematic construction of field and lattice integrable dispersive systems (soliton systems) as well as dispersionless integrable field systems (see [1]-[17] and the references there). The Lie algebra of pseudo-differential operators (PDO) leads to the construction of (1+1)-dimensional integrable soliton systems [15, 8, 4]. Considering the (1+1)-dimensional integrable hierarchies with infinitely many fields one can extract from them closed equations for a single field by the elimination of the remaining fields [14, 7, 11]. This method, the so-called Sato approach [14], leads to construction of (2+1)-dimensional integrable one-field equations: Kadomtsev-Petviashvili (KP), modified Kadomtsev-Petviashvili (mKP) and (2+1) Harry-Dym (HD). An analogous method with matrix coefficients and dressing operators, the so-called matrix Sato theory [8], permits a construction of (2+1)-dimensional integrable evolution equations, with more number of fields, like (2+1) AKNS. There is another more effective and systematic method for construction of (2+1)-dimensional integrable systems, the so-called central extension procedure [16, 17, 13]. The central extension approach to integrable field, lattice-field and dispersionless systems was presented in [6, 5] and [3]. In this paper an appropriate restrictions of Lax operators coming from centrally extended PDO algebra, are systematically considered.

In [9, 11] a wide class of gauge, reciprocal, Bäcklund and auto-Bäcklund transformations for (1+1)-dimensional soliton systems and (2+1)-dimensional systems like KP, mKP and HD, originating from the PDO Lie algebra, is presented. Therefore, the investigation of such transformations for (2+1)-dimensional systems, originating from the centrally extended PDO algebra, seems interesting. In this paper the relations between three classes of Lax hierarchies, coming from the centrally extended PDO algebra, are constructed.

## 2 $R$-matrix and the central extension approach

The crucial point of the formalism is the observation that integrable systems can be obtained from Lax equations. Let $\mathfrak{g}$ be a Lie algebra, equipped with the Lie bracket $[\cdot,\cdot]$. A linear map $R:\mathfrak{g}\to\mathfrak{g}$, such that the bracket $[a,b]_R := [Ra,b]+[a,Rb]$ is a second Lie product on $\mathfrak{g}$, is called the classical $R$-matrix. Let $R$ satisfy an Yang-Baxter equation YB($\alpha$): $[Ra,Rb]-R[a,b]_R+\alpha[a,b]=0$, which is a sufficient condition for $R$ to be an $R$-matrix. Assume now that the Lie algebra $\mathfrak{g}$ depends effectively on an independent parameter $y\in\mathbb{S}^1$, which naturally generates the corresponding current operator algebra $\overline{\mathfrak{g}}=\mathcal{C}^\infty(\mathbb{S}^1,\mathfrak{g})$. Then, invariants $A_n\in$, numbered by $n$, of the Novikov-Lax equation, i.e.

$$[A_n,L]+(A_n)_y=0 \qquad L\in\overline{\mathfrak{g}}, \tag{2.1}$$

generate mutually commuting vector fields

$$L_{t_n}=[RA_n,L]+(RA_n)_y\,. \tag{2.2}$$

For a fixed $n$, the remaining systems are considered as its symmetries. In this sense (2.2) represents a hierarchy of integrable dynamical systems.

To construct the simplest $R$-structure let us assume that the Lie algebra $\mathfrak{g}$ can be split into a direct sum of Lie subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$, i.e. $\mathfrak{g}=\mathfrak{g}_+\oplus\mathfrak{g}_-$, where $[\mathfrak{g}_\pm,\mathfrak{g}_\pm]\subset\mathfrak{g}_\pm$. Denoting the projections onto these subalgebras by $P_\pm$, we define the $R$-matrix as $R=P_+-\frac{1}{2}$. Then, straightforward calculation shows that $R$ solves YB($\frac{1}{4}$). Following the above scheme, we are able to construct in a systematic way integrable systems once we fix a Lie algebra.

## 3 Centrally extended PDO Lie algebra

Let us consider the associative algebra of pseudo-differential operators

$$\overline{\mathfrak{g}}=\left\{L=\sum_{i\in\mathbb{Z}}u_i(x,y)\hat{\partial}_x^i\right\}, \tag{3.1}$$

where the coefficients $u_i(x,y)$ are dynamical fields being smooth functions and $y\in\mathbb{S}^1$ is an additional independent variable introduced by the central extension procedure. We use a dash to distinguish the pseudo-differential operators such $\hat{\partial}_x$ and $\hat{\partial}_x^{-1}$ from derivatives $\partial_x$ and from formal integration symbols $\partial_x^{-1}$. Multiplication of two operators in $\overline{\mathfrak{g}}$ uses the generalized Leibniz rule: $\hat{\partial}_x^i u=\sum_{s\geqslant 0}\binom{i}{s}u_{sx}\hat{\partial}_x^{i-s}$, where $\binom{i}{s}=(-1)^s\binom{-i+s-1}{s}$ for $i<0$. Lie structure on $\overline{\mathfrak{g}}$ is given by the commutator $[A,B]=AB-BA$. The algebra (3.1) can be split into a direct sum of its Lie subalgebras $\overline{\mathfrak{g}}_{\geq k}=\{\sum_{i\geq k}u_i\hat{\partial}_x^i\}$ and $\overline{\mathfrak{g}}_{<k}=\{\sum_{i<k}u_i\hat{\partial}_x^i\}$ for $k=0,1,2$. As a result, $R$-matrix is given by $R=P_{\geq k}-\frac{1}{2}$. Then, the hierarchy (2.2) takes the form

$$L_{t_n}=\left[(A_n)_{\geq k}\,,L-\hat{\partial}_y\right] \qquad k=0,1,2. \tag{3.2}$$

We are interested in extracting closed systems for a finite number of fields. To obtain a consistent Lax equation, the Lax operator $L$ has to span a proper subspace of the full algebra $\bar{\mathfrak{g}}$, i.e. the left and right-hand sides of expression (3.2) have to coincide. To construct any integrable equations for a given $L$ we have to know how to solve (2.1), i.e.

$$\left[A_n, L - \hat{\partial}_y\right] = 0. \tag{3.3}$$

It can be done by putting $A_n = \sum_{i \le n} a_i \hat{\partial}_x^i$ for $n \ge k$. Then the function parameters $a_i$ are obtained from (3.3) successively via the recurrent procedure. Note that solutions are in the form of infinite series. In fact we need only the finite parts $(A_n)_{\ge k}$. We will look for the restrictions of Lax operators $L$ given in the general form:

$$L = u_N \hat{\partial}_x^N + u_{N-1} \hat{\partial}_x^{N-1} + ... + u_0 + \hat{\partial}_x^{-1} u_{-1} + ... + \hat{\partial}_x^{-m} u_{-m}. \tag{3.4}$$

There is another important class of restrictions to the Lax operators, which contain additional dynamical fields, the so-called source terms $\psi_i$ and $\phi_i$ [12], given by

$$L = l + \sum_i \psi_i \hat{\partial}_x^{-1} \phi_i, \tag{3.5}$$

where $l$ has an appropriate form as (3.4). For pseudo-differential operator $A = \sum_i a_i \hat{\partial}_x^i$ let $[A]_0 = a_0$, so $[\cdot]$ means the projection to the 0-order. The transposition operation of $A$ is defined as $A^\dagger = \sum_i (a_i \hat{\partial}_x^i)^\dagger = \sum_i (-1)^i \hat{\partial}_x^i a_i$. Hence, from the Lax equation (3.2) and the equivalent transposed Lax formulation

$$L^\dagger_{t_n} = -\left[(A_n)^\dagger_{\ge k}, L^\dagger + \hat{\partial}_y\right] \qquad k = 0, 1, 2, \tag{3.6}$$

it follows that $\psi_i$ and $\phi_i$ are eigenfunctions and adjoint-eigenfunctions of the Lax hierarchy (3.2), i.e. they satisfy

$$(\psi_i)_{t_n} = \left[(A_n)_{\geqslant k} \psi_i\right]_0 \qquad (\phi_i)_{t_n} = -\left[(A_n)^\dagger_{\geqslant k} \phi_i\right]_0.$$

**The case:** $k = 0$**.**

The simplest appropriate Lax operators are given in the form

$$L = \hat{\partial}_x^N + u_{N-2} \hat{\partial}_x^{N-2} + u_{N-3} \hat{\partial}_x^{N-3} + ... + u_1 \hat{\partial}_x + u_0 \qquad N \geqslant 2. \tag{3.7}$$

There are no further simple reductions.

In all examples, in this article, for a given $L$ we will exhibit only the first non-trivial equation from the Lax hierarchy (3.2) .

**Example.** $N = 2$: The KP.

The Lax operator is $L = \hat{\partial}_x^2 + u$. Solving (3.3) for $n = 3$ one finds $(A_3)_{\geqslant 0} = \hat{\partial}_x^3 + \frac{3}{2} u \hat{\partial}_x + \frac{3}{4} u_x + \frac{3}{4} \partial_x^{-1} u_y$. Hence

$$u_{t_3} = \frac{1}{4} u_{3x} + \frac{3}{2} u u_x + \frac{3}{4} \partial_x^{-1} u_{2y}. \tag{3.8}$$

**Example.** $N=3$: The (2+1)-Boussinesq.
Let $L=\hat{\partial}_x^3+u\hat{\partial}_x+v$. Then, for $(A_2)_{\geqslant 0}=\hat{\partial}_x^2+\frac{2}{3}u$

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2}=\begin{pmatrix} 2v_x-u_{2x} \\ \frac{2}{3}u_y-\frac{2}{3}uu_x+v_{2x}-\frac{2}{3}u_{3x} \end{pmatrix}. \tag{3.9}$$

Considering the restriction of the Lax operators to the form (3.5) we find

$$L=\hat{\partial}_x^N+u_{N-2}\hat{\partial}_x^{N-2}+\ldots+u_0+\sum_i \psi_i\hat{\partial}_x^{-1}\phi_i \qquad N\geqslant 1, \tag{3.10}$$

$$L=\sum_i \psi_i\hat{\partial}_x^{-1}\phi_i \qquad N=0. \tag{3.11}$$

The Lax operator (3.11) is pure (2+1)-dimensional effect and does not have (1+1)-dimensional counterpart.

**Example.** $N=1$: The (2+1) AKNS.
The Lax operator is $L=\hat{\partial}_x+\psi\hat{\partial}_x^{-1}\phi$. Then, $(A_2)_{\geqslant 0}=\hat{\partial}_x^2+a$ and

$$\begin{pmatrix} \psi \\ \phi \end{pmatrix}_{t_2}=\begin{pmatrix} a\psi+\psi_{2x} \\ -a\phi-\phi_{2x} \end{pmatrix} \qquad a_x-a_y-2\left(\psi\phi\right)_x=0. \tag{3.12}$$

**Example.** $N=0$: Two-field system.
Let $L=\psi\hat{\partial}_x^{-1}\phi$ and $(A_2)_{\geqslant 0}=\hat{\partial}_x^2-2\partial_y^{-1}(\psi\phi)_x$. Then

$$\begin{pmatrix} \psi \\ \phi \end{pmatrix}_{t_2}=\begin{pmatrix} \psi_{2x}-2\psi\partial_y^{-1}(\psi\phi)_x \\ -\phi_{2x}+2\phi\partial_y^{-1}(\psi\phi)_x \end{pmatrix}. \tag{3.13}$$

**The case: $k=1$.**

The simplest restrictions of Lax operators are given in the following form

$$L=\hat{\partial}_x^N+u_{N-1}\hat{\partial}_x^{N-1}+u_{N-2}\hat{\partial}_x^{N-2}+\ldots+u_0+\hat{\partial}_x^{-1}u_{-1} \qquad N\geqslant 1, \tag{3.14}$$

$$L=u_0+\hat{\partial}_x^{-1}u_{-1} \qquad N=0. \tag{3.15}$$

There are further admissible reductions given by $\{u_{-1}=0\}$ and $\{u_{-1}=u_0=0\}$. The case (3.15) does not exist in (1+1) dimensions.

**Example.** $N=2$: The mKP.
For $L=\hat{\partial}_x^2+2u\hat{\partial}_x$ and for $(A_3)_{\geqslant 1}=\hat{\partial}_x^3+3u\hat{\partial}_x^2+\frac{3}{2}(u^2+u_x+\partial_x^{-1}u_y)\hat{\partial}_x$ we find

$$u_{t_3}=\frac{1}{4}u_{3x}-\frac{3}{2}u^2u_x+\frac{3}{2}u_x\partial_x^{-1}u_y+\frac{3}{4}\partial_x^{-1}u_{2y}. \tag{3.16}$$

**Example.** $N=3$: The (2+1) modified Boussinesq.
Let $L=\hat{\partial}_x^3+3u\hat{\partial}_x^2+v\hat{\partial}_x$. Then, for $(A_2)_{\geqslant 1}=\hat{\partial}_x^2+2u\hat{\partial}_x$

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2}=\begin{pmatrix} \frac{2}{3}v_x-2uu_x-u_{2x} \\ 2u_y-2u_xv+2uv_x-6uu_{2x}+v_{2x}-2u_{3x} \end{pmatrix}. \tag{3.17}$$

**Example.** $N = 1$: The (2+1) Kaup-Broer.

The Lax operator is $L = \hat{\partial}_x + u + \hat{\partial}_x^{-1} v$. For $(A_2)_{\geqslant 1} = \hat{\partial}_x^2 + a\hat{\partial}_x$ we find

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} 2v_x + au_x + u_{2x} \\ a_x v + av_x - v_{2x} \end{pmatrix} \qquad a_x - a_y - 2u_x = 0. \tag{3.18}$$

The reduction $v = 0$ leads to the (2+1) Burgers equation: $u_{t_2} = u_{2x} + au_x$ for $a$ given above.

**Example.** $N = 0$: Two-field system.

For (3.15): $L = u + \hat{\partial}_x^{-1} v$ we construct $(A_2)_{\geqslant 1} = \hat{\partial}_x^2 - 2\partial_y^{-1} u_x \hat{\partial}_x$. Hence

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} 2v_x + u_{2x} - 2u_x \partial_y^{-1} u_x \\ -v_{2x} - 2\left(v\partial_y^{-1} u_x\right)_x \end{pmatrix}. \tag{3.19}$$

Reducing it by the constraint $v = 0$ one obtains a one-field system $u_{t_2} = u_{2x} - 2u_x \partial_y^{-1} u_x$, which, according to the transformation $u' = -\partial_y^{-1} u_x$, becomes the (1+1)-dimensional Burgers equation $u'_t = u'_{2x} + 2u'u'_x$.

The admissible Lax operators (3.5), containing sources, have the form

$$L = \hat{\partial}_x^N + u_{N-1}\hat{\partial}_x^{N-1} + ... + u_0 + \hat{\partial}_x^{-1} u_{-1} + \sum_i \psi_i \hat{\partial}_x^{-1} \phi_i \qquad N \geqslant 1, \tag{3.20}$$

$$L = u_0 + \hat{\partial}_x^{-1} u_{-1} + \sum_i \psi_i \hat{\partial}_x^{-1} \phi_i \qquad N = 0, \tag{3.21}$$

where (3.21) is pure (2+1)-dimensional phenomenon. There is only one further reduction $\{u_{-1} = 0\}$.

**Example.** $N = 1$: Three field system.

Let $L = \hat{\partial}_x + u + \psi\hat{\partial}_x^{-1}\phi$. Then, for $(A_2)_{\geqslant 1} = \hat{\partial}_x^2 + a\hat{\partial}_x$ we find

$$\begin{pmatrix} u \\ \psi \\ \phi \end{pmatrix}_{t_2} = \begin{pmatrix} au_x + 2(\psi\phi)_x + u_{2x} \\ a\psi_x + \psi_{2x} \\ (a\phi)_x - \phi_{2x} \end{pmatrix} \qquad a_x - a_y - 2u_x = 0.$$

**Example.** $N = 0$: Three field system.

For $L = u + \psi\hat{\partial}_x^{-1}\phi$ and $(A_2)_{\geqslant 1} = \hat{\partial}_x^2 - 2\partial_y^{-1} u_x \hat{\partial}_x$ we have

$$\begin{pmatrix} u \\ \psi \\ \phi \end{pmatrix}_{t_2} = \begin{pmatrix} u_{2x} + 2(\psi\phi)_x - 2u_x \partial_y^{-1} u_x \\ \psi_{2x} - 2\psi_x \partial_y^{-1} u_x \\ -\phi_{2x} - 2(\phi\partial_y^{-1} u_x)_x \end{pmatrix}.$$

We will show now, what will be important later, that fields $u_0$ and $u_{-1}$ from the Lax operators (3.14-3.15) and (3.20-3.21) are expressible by the respective eigenfunction and adjoint-eigenfunction in the following way

$$u_0 = \varphi - \partial_x^{-1} u_{-1} \qquad \text{for (3.14-3.15)}, \tag{3.22}$$

$$u_0 = \varphi - \partial_x^{-1} u_{-1} - \sum_i \psi_i \partial_x^{-1} \phi_i \qquad \text{for (3.20-3.21)}, \tag{3.23}$$

where $\varphi$ is an eigenfunction of the Lax hierarchy (3.2) for $k=1$ and $\psi_i$, $\phi_i$ are the source fields. The following relations, fulfilled for an arbitrary operator $A$ and an arbitrary smooth function $v$, will be useful later:

$$v\,[A]_0 = [vA]_0\,, \tag{3.24}$$

$$\left[A\hat{\partial}_x^{-1}v\right]_0 = \left[A\partial_x^{-1}v\right]_0 - [A]_0\,\partial_x^{-1}v, \tag{3.25}$$

$$\left[\hat{\partial}_x^{-1}A\right]_0 = \partial_x^{-1}[A]_0 - \partial_x^{-1}\left[A^\dagger\right]_0. \tag{3.26}$$

It is enough to consider the case (3.23) as the case (3.22) is a simple consequence of (3.23). Let $L = ... + u_0 + \hat{\partial}_x^{-1}u_{-1} + \sum_i \psi_i\hat{\partial}_x^{-1}\phi_i$, then immediately from (3.6) it follows that $(u_{-1})_{t_n} = -[(A)^\dagger_{\geqslant 1}u_{-1}]_0$. Hence, the field $u_{-1}$ is an adjoint-eigenfunction. The time evolution for $u_0$ is obtained from (3.2) as

$$(u_0)_{t_n} = [Bu_0]_0 + \left[B\hat{\partial}_x^{-1}u_{-1}\right]_0 - \left[\hat{\partial}_x^{-1}u_{-1}B\right]_0 + \sum_i\left[B\psi_i\hat{\partial}_x^{-1}\phi_i\right]_0 - \sum_i\left[\psi_i\hat{\partial}_x^{-1}\phi_iB\right]_0, \tag{3.27}$$

where $B=(A_n)_{\geqslant 1}$. We introduce now a new function $\varphi$ defined as $\varphi = u_0 + \partial_x^{-1}u_{-1} + \sum_i \psi_i\partial_x^{-1}\phi_i$ for which from (3.27) we obtain the following time evolution

$$\begin{aligned}\varphi_{t_n} = {}& [B\varphi]_0 - \left[B\partial_x^{-1}\varphi\right]_0 - \sum_i\left[B\psi_i\partial_x^{-1}\phi_i\right]_0 + \left[B\psi_i\hat{\partial}_x^{-1}u_{-1}\right]_0\\ & - \left[\hat{\partial}_x^{-1}u_{-1}B\right]_0 + \sum_i\left[B\psi_i\hat{\partial}_x^{-1}\phi_i\right]_0 - \sum_i\left[\psi_i\hat{\partial}_x^{-1}\phi_iB\right]_0\\ & - \partial_x^{-1}\left[B^\dagger u_{-1}\right]_0 + \sum_i[B\psi_i]_0\,\partial_x^{-1}\phi_i - \sum_i\psi_i\partial_x^{-1}\left[B^\dagger\phi_i\right]_0.\end{aligned}$$

As the following relations are valid:

$$\begin{aligned}&\left[B\hat{\partial}_x^{-1}u_{-1}\right]_0 = \left[B\partial_x^{-1}v\right]_0 && \text{by (3.25)},\\ &\left[\hat{\partial}_x^{-1}u_{-1}B\right]_0 = -\partial_x^{-1}\left[B^\dagger u_{-1}\right]_0 && \text{by (3.26)},\\ &\left[B\psi_i\hat{\partial}_x^{-1}\phi_i\right]_0 = \left[B\psi_i\partial_x^{-1}\phi_i\right]_0 - [B\psi_i]_0\,\partial_x^{-1}\phi_i && \text{by (3.25)},\\ &\left[\psi_i\hat{\partial}_x^{-1}\phi_iB\right]_0 = -\psi_i\partial_x^{-1}\left[B^\dagger\phi_i\right]_0 && \text{by (3.24) and (3.26)},\end{aligned}$$

time evolution of $\varphi$ is $\varphi_{t_n} = [(A_n)_{\geqslant 1}\varphi]_0$. Hence, the field $\varphi$ is indeed an eigenfunction of (3.2) for $k=1$.

**The case:** $k=2$**.**

Appropriate Lax operators are of the form

$$L = u_N\hat{\partial}_x^N + u_{N-1}\hat{\partial}_x^{N-1} + ... + u_1\hat{\partial}_x + u_0 + \hat{\partial}_x^{-1}u_{-1} + \hat{\partial}_x^{-2}u_{-2} \qquad N\geqslant 1 \tag{3.28}$$

and simplest admissible reductions are given by $\{u_{-2}=0\}$, $\{u_{-2}=u_{-1}=0\}$, $\{u_{-2}=u_{-1}=u_0=0\}$, $\{u_{-2}=u_{-1}=u_0=u_1=0\}$.

**Example.** $N = 2$: The (2+1) HD.

For the Lax operator $L = u^2\hat{\partial}_x^2$ and $(A_3)_{\geqslant 2} = u^3\hat{\partial}_x^3 + \frac{3}{2}u^2(u_x + \partial_x^{-1}\frac{u_y}{u^2})\hat{\partial}_x^2$ we find

$$u_{t_3} = \frac{1}{4}u^3 u_{3x} + \frac{3}{4}\frac{1}{u}\left(u^2\partial_x^{-1}\frac{u_y}{u^2}\right)_y. \tag{3.29}$$

**Example.** $N = 3$: Two-field system.

Let $L = u^3\hat{\partial}_x^3 + v\hat{\partial}_x^2$ and $(A_2)_{\geqslant 2} = u^2\hat{\partial}_x^2$. Then

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} \frac{2}{3}v_x - \frac{4}{3}\frac{u_x}{u}v - u^2u_{2x} \\ 2uu_y - 2u_x^2v - 2uu_{2x}v + u^2v_{2x} - 6u^3u_xu_{2x} - 2u^4u_{3x} \end{pmatrix}. \tag{3.30}$$

**Example.** $N = 1$: Three-field system.

For $L = u\hat{\partial}_x + v + \hat{\partial}_x^{-1}w$ and $(A_2)_{\geqslant 2} = a\hat{\partial}_x^2$ one finds

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_2} = \begin{pmatrix} 2av_x + au_{2x} \\ a_xw + 2aw_x + av_{2x} \\ -(aw)_{2x} \end{pmatrix} \qquad a_y - ua_x + 2au_x = 0. \tag{3.31}$$

Considering the restrictions (3.5) one find that the appropriate Lax operators are given by

$$L = u_N\hat{\partial}_x^N + u_{N-1}\hat{\partial}_x^{N-1} + \dots + u_0 + \hat{\partial}_x^{-1}u_{-1} + \hat{\partial}_x^{-2}u_{-2} + \sum_i \psi_i\hat{\partial}_x^{-1}\phi_i \qquad N \geqslant 1. \tag{3.32}$$

The reductions are $\{u_{-2} = 0\}$, $\{u_{-2} = u_{-1} = 0\}$.

## 4 Gauge transformation and reciprocal link

The three classes of Lax hierarchies (3.2) for $k = 0, 1, 2$ are interrelated as shown in the following two theorems.

**Theorem 1.** *Gauge transformation: $k = 0 \to k = 1$. Let $L \in \bar{\mathfrak{g}}$ satisfy the hierarchy $L_{t_n} = [(A_n)_{\geqslant 0}, L - \hat{\partial}_y]$ and let the function $\psi \neq 0$ be an eigenfunction of this hierarchy: $\psi_{t_n} = [(A_n)_{\geqslant 0}\psi]_0$. Then, $\tilde{L} = \psi^{-1}L\psi - \psi^{-1}\psi_y$ satisfies the hierarchy $\tilde{L}_{t_n} = [(\tilde{A}_n)_{\geqslant 1}, \tilde{L} - \hat{\partial}_y]$, where $\tilde{A}_n = \psi^{-1}A_n\psi$.*

**Proof.** First we have to show that $\tilde{A}_n$ is solution of (3.3) for $\tilde{L}$:

$$\begin{aligned}\left[\tilde{A}_n, \tilde{L} - \hat{\partial}_y\right] &= \left[\psi^{-1}A_n\psi, \psi^{-1}L\psi - \psi^{-1}\psi_y - \hat{\partial}_y\right] \\ &= \psi^{-1}\left[A_n, L - \psi^{-1}\psi_y - \psi\hat{\partial}_y\psi^{-1}\right]\psi = \psi^{-1}\left[A_n, L - \hat{\partial}_y\right]\psi = 0.\end{aligned}$$

Next, one observes that for an arbitrary pseudo-differential operator $A$, the following relation is valid: $(\psi^{-1}A\psi)_{\geq 1} = \psi^{-1}(A)_{\geqslant 0}\psi - \psi^{-1}[(A)_{\geqslant 0}\psi]_0$. Then

$$\begin{aligned}\left[\left(\tilde{A}_n\right)_{\geqslant 1}, \tilde{L} - \hat{\partial}_y\right] &= \left[\psi^{-1}(A_n)_{\geqslant 0}\psi - \psi^{-1}\left[(A_n)_{\geqslant 0}\psi\right]_0, \psi^{-1}L\psi - \psi^{-1}\psi_y - \hat{\partial}_y\right] \\ &= \psi^{-1}\left[(A_n)_{\geqslant 0}, L - \hat{\partial}_y\right]\psi - \psi^{-1}\left[\psi^{-1}\left[(A_y)_{\geqslant 0}\psi\right]_0, L - \hat{\partial}_y\right]\psi.\end{aligned}$$

Now, since

$$\tilde{L}_{t_n} = \left(\psi^{-1}L\psi - \psi^{-1}\psi_y\right)_{t_n} = \psi^{-1}L_{t_n}\psi - \psi^{-1}\left[\psi^{-1}\psi_{t_n}, L - \hat{\partial}_y\right]\psi$$

the proof of the theorem is completed. ■

Consider the Lax operator of the form (3.7). Then the gauge transformed operator, by theorem 1, is given by

$$\tilde{L} = \hat{\partial}_x^N + N\psi^{-1}\psi_x\hat{\partial}_x^{N-1} + ... + (...)\hat{\partial}_x + u_0 + \psi^{-1}\left[L\psi\right]_0 - \psi^{-1}\psi_y \qquad N \geqslant 2. \quad (4.1)$$

We will compare (4.1) with Lax operator (3.14) with $\tilde{u}_i$ components, where $\{\tilde{u}_{-1} = \tilde{u}_0 = 0\}$, as then it is spanned by the same number of dynamical fields as (3.7). Hence

$$\begin{aligned} \tilde{u}_{N-1} &= N\psi^{-1}\psi_x \\ \tilde{u}_{N-2} &= u_{N-1} + \frac{N(N-1)}{2}\psi^{-1}\psi_{2x} \\ &\vdots \\ 0 &= u_0 + \psi^{-1}\left[L\psi\right]_0 - \psi^{-1}\psi_y. \end{aligned}$$

Now, eliminating eigenfunction $\psi$ one obtains the Miura transformation between the fields $u_i$ from the Lax operator (3.7) for $k = 0$ and the fields $\tilde{u}_i$ from the Lax operator (3.14) $\{\tilde{u}_{-1} = \tilde{u}_0 = 0\}$ for $k = 1$.

**Example.** A well known Miura map between systems KP (3.8) and mKP (3.16) is given in the form

$$L = \hat{\partial}_x^2 + u \longrightarrow \tilde{L} = \hat{\partial}_x^2 + 2\tilde{u}\hat{\partial}_x \quad \Longrightarrow \quad (u + \tilde{u}^2 + \tilde{u}_x)_x = \tilde{u}_y.$$

**Example.** The Miura map for (2+1) Boussinesq (3.9) and (2+1) modified Boussinesq (3.17) is

$$L = \hat{\partial}_x^3 + u\hat{\partial}_x + v \longrightarrow \tilde{L} = \hat{\partial}_x^3 + 3\tilde{u}\hat{\partial}_x^2 + \tilde{v}\hat{\partial}_x \quad \Longrightarrow \quad \begin{cases} u = \tilde{v} - 3\tilde{u}^2 - 3\tilde{u}_x \\ (v - 2\tilde{u}^3 + \tilde{u}\tilde{v} + \tilde{u}_{2x})_x = \tilde{u}_y. \end{cases}$$

Let us consider the source Lax operators given by the form (3.10). A natural choice of eigenfunctions is the choice of one from the source eigenfunctions $\psi_i$. Let $\psi = \psi_1$, as then the Lax operator (3.10) naturally transforms to the (3.20) form. That is since the gauge transformed operator has the form

$$\tilde{L} = \hat{\partial}_x^N + N\psi^{-1}\psi_x\hat{\partial}_x^{N-1} + ... + u_0 + \psi^{-1}\left[L\psi\right]_0 - \psi^{-1}\psi_y + \hat{\partial}_x^{-1}\phi_1 + \sum_{i\neq 1}\psi_i\hat{\partial}_x^{-1}\phi_i \qquad N \geqslant 1, \quad (4.2)$$

then (3.10) and (3.20) are spanned by the same number of dynamical fields. Analogously, if $\psi = \psi_1$ the Lax operators of the form (3.11) naturally lead to the Lax operators (3.21). If operators (3.10) and (3.11) contain only one pair of eigenfunction and adjoint-eigenfunction $\psi_1, \phi_1$, then by theorem (1) with $\psi = \psi_1$ we construct a Miura map between the fields from the Lax operators (3.10), (3.11) ($i \in \{1\}$) and the Lax operators (3.14), (3.15), respectively.

**Example.** The Miura map between (2+1) AKNS (3.12) and (2+1) KB (3.18) is given by

$$L = \hat{\partial}_x + \psi\hat{\partial}_x^{-1}\phi \longrightarrow \tilde{L} = \hat{\partial}_x + u + \hat{\partial}_x^{-1}v \quad \Longrightarrow \quad \begin{cases} u = \psi^{-1}\psi_x - \psi^{-1}\psi_y \\ v = \psi\phi. \end{cases}$$

**Example.** The transformation between fields for two-field systems (3.13) and (3.19) is

$$L = \psi\hat{\partial}_x^{-1}\phi \longrightarrow \tilde{L} = u + \hat{\partial}_x^{-1}v \quad \Longrightarrow \quad \begin{cases} u = -\psi^{-1}\psi_y \\ v = \psi\phi. \end{cases}$$

**Theorem 2.** *The reciprocal link: $k = 1 \to k = 2$. Let $L = L(x, y, t)$ satisfy $L_{t_n} = [(A_n)_{\geqslant 1}, L - \hat{\partial}_y]$ and the function $\phi(x, y, t)$, such that $\varphi_x \neq 0$ and $\varphi_y \neq 0$, be an eigenfunction of this hierarchy satisfying $\varphi_{t_n} = [(A)_{\geqslant 1}\varphi]_0$. Consider the following transformation $x' = \varphi(x, y, t)$, $y' = y$, $t'_n = t_n$. Then, $L'(x', y', t') = L(x, y, t) - \varphi_y\hat{\partial}_{x'}$ satisfies the hierarchy $L'_{t'_n} = [(A'_n)_{\geqslant 2}, L' - \hat{\partial}_{y'}]$, where $A'_n(x', y', t') = A_n(x, y, t)$.*

**Proof.** Consider transformation: $x' = \varphi(x, y, t)$, $y' = y$, $t'_n = t_n$, then

$$\partial_x = \varphi_x\partial_{x'} \quad \partial_y = \varphi_y\partial_{x'} + \partial_{y'}, \quad \partial_{t_n} = \varphi_{t_n}\partial_{x'} + \partial_{t'_n}.$$

In consequence $L' - \hat{\partial}_{y'} = L - \hat{\partial}_y$ and $[A'_n, L' - \hat{\partial}_{y'}] = [A_n, L - \hat{\partial}_y] = 0$. Let $A = \sum_i a_i\hat{\partial}^i_{x'}$. Observing that for an arbitrary pseudo-differential operator $(A')_{\geqslant 1}$ the lowest coefficient can be obtained by $a_1 = [(A')_{\geqslant 1}x']_0$ one finds the following relation $(A')_{\geqslant 2} = (A)_{\geqslant 1} - [(A)_{\geqslant 1}\varphi]_0\hat{\partial}_{x'}$. Hence

$$\left[\left(A'_n\right)_{\geqslant 2}, L' - \hat{\partial}_{y'}\right] = \left[(A_n)_{\geqslant 1}, L - \hat{\partial}_y\right] - \left[\left[(A_n)_{\geqslant 1}\varphi\right]_0\hat{\partial}_{x'}, L - \hat{\partial}_y\right].$$

Now as

$$\begin{aligned} L'_{t'_n} &= \left[\hat{\partial}_{t'_n}, L'\right] = \left[\hat{\partial}_{t_n} - \varphi_{t_n}\hat{\partial}_{x'}, L - \varphi_y\hat{\partial}_{x'}\right] \\ &= L_{t_n} - \left[\varphi_{t_n}\hat{\partial}_{x'}, L - \hat{\partial}_y\right] \\ &\quad + \left[\varphi_{t_n}(\varphi_x)^{-1}\hat{\partial}_x, \varphi_y(\varphi_x)^{-1}\hat{\partial}_x\right] - \left[\hat{\partial}_{t_n}, \varphi_y(\varphi_x)^{-1}\hat{\partial}_x\right] - \left[\varphi_{t_n}(\varphi_x)^{-1}\hat{\partial}_x, \hat{\partial}_y\right], \end{aligned}$$

where the last three terms cancel to zero, we obtain the result of the theorem. ■

Consider the Lax operators (3.14) with the reduction $\{u_{-1} = u_0 = 0\}$. Then, the linked operator from theorem 2, where $x' = \varphi$, has the form

$$L' = \varphi_x^N\hat{\partial}_{x'}^N + \left[(N-1)\varphi_x^{N-2}\varphi_{2x} + u_{N-1}\varphi_x^{N-1}\right]\hat{\partial}_{x'}^{N-1} + ... + [... + u_1\varphi_x - \varphi_y]\,\hat{\partial}_{x'} \qquad N \geqslant 2. \tag{4.3}$$

We have to compare (4.3) with the Lax operators (3.28) where $\{u_{-2} = u_{-1} = u_0 = u_{-1} = 0\}$ as it is spanned by the same number of dynamical fields as (3.14) for $\{u_{-1} = u_0 = 0\}$. So, it follows that the coefficient of (4.3) standing by the first order term has to be equal zero. Thus eliminating $\varphi$ we construct the reciprocal transformation.

**Example.** A well known reciprocal link between systems mKP (3.16) and (2+1) HD (3.29) is

$$L = \hat{\partial}_x^2 + 2u\hat{\partial}_x \longrightarrow L' = u'^2\hat{\partial}_{x'}^2 \quad \Longrightarrow \quad (2uu' + u'_x)_x = u'_y \qquad x' = \partial_x^{-1}u'.$$

**Example.** The reciprocal link between (2+1) modified Boussinesq (3.17) and two field system (3.30) has the form

$$L = \hat{\partial}_x^3 + 3u\hat{\partial}_x^2 + v\hat{\partial}_x \longrightarrow L' = u'^3\hat{\partial}_{x'}^3 + v'\hat{\partial}_{x'}^2 \quad \Longrightarrow$$
$$\begin{cases} v' = 3uu'^2 + 3u'u_x \\ (vu' + 3uu'_x + u'_{2x})_x = u'_y \end{cases} \qquad x' = \partial_x^{-1}u'.$$

We will now consider the cases of Lax operators (3.14) and (3.15) without reductions. Then, the linked operators for eigenfunction $\varphi$ are:

$$\begin{aligned} L' = \varphi_x^N\hat{\partial}_{x'}^N + \left[(N-1)\varphi_x^{N-2}\varphi_{2x} + u_{N-1}\varphi_x^{N-1}\right]\hat{\partial}_{x'}^{N-1} + ... \\ + [... + u_1\varphi_x - \varphi_y]\,\hat{\partial}_{x'} + u_0 + \hat{\partial}_{x'}^{-1}\tfrac{u_{-1}}{\varphi_x} \qquad N \geqslant 2, \qquad (4.4) \end{aligned}$$

$$L' = (\varphi_x - \varphi_y)\,\hat{\partial}_{x'} + u_0 + \hat{\partial}_{x'}^{-1}\tfrac{u_{-1}}{\varphi_x} \qquad N = 1, \qquad (4.5)$$

$$L' = -\varphi_y\hat{\partial}_{x'} + u_0 + \hat{\partial}_{x'}^{-1}\tfrac{u_{-1}}{\varphi_x} \qquad N = 0. \qquad (4.6)$$

Here the natural choice of eigenfunction $\varphi$ is given by (3.22): $\varphi = u_0 + \partial_x^{-1}u_{-1}$. We will compare linked operators (4.4-4.6) with (3.28), for fields $u'_i$, and reduction $\{u'_{-2} = 0\}$. Thus, $u'_0 = u_0$ and $u'_{-1} = \frac{u_{-1}}{\varphi_x}$. Now, as

$$\varphi_x = \varphi_x(u'_0)_{x'} + u_{-1} \Longleftrightarrow u_{-1} = \varphi_x\left(1 - (u'_0)_{x'}\right)$$

we find constraint $u'_{-1} = (1 - (u'_0)_{x'})$. Hence, we found new appropriate restriction for $k = 2$:

$$L' = u'_N\hat{\partial}_{x'}^N + u'_{N-1}\hat{\partial}_{x'}^{N-1} + ... + u'_0 + \hat{\partial}_{x'}^{-1}\left(1 - (u'_0)_{x'}\right) \qquad N \geqslant 1. \qquad (4.7)$$

Therefore, by theorem 2, we construct reciprocal links for fields from Lax operators (3.14-3.15) to fields from (4.7). But operators (3.14) for $N = 1$ and (3.15) are linked to the same Lax operator (4.7) with $N = 1$.

**Example.** The reciprocal link between (2+1) Kaup-Broer (3.18) and the system (3.31) with the reduction $w = 1 - v_x$. Let $L' = u'\hat{\partial}_{x'} + v' + \hat{\partial}_{x'}^{-1}(1 - v'_{x'})$, then for $(A_2)_{\geqslant 2} = a\hat{\partial}_{x'}^2$ one finds

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} 2av'_{x'} + au'_{2x'} \\ a_{x'}w' - a_{x'}v'_{x'} - av'_{2x'} \end{pmatrix} \qquad a_{y'} - u'a_{x'} + 2au'_{x'} = 0. \qquad (4.8)$$

The reciprocal link is:

$$L = \hat{\partial}_x + u + \hat{\partial}_x^{-1}v \longrightarrow L' = u'\hat{\partial}_{x'} + v' + \hat{\partial}_{x'}^{-1}(1 - v'_{x'}) \quad \Longrightarrow$$
$$\begin{cases} u'_x = u_{2x} - u_{xy} + v_x - v_y \\ v' = u \end{cases} \qquad x' = u + \partial_x^{-1}v.$$

**Example.** The reciprocal link between the two-field system (3.19) and the system (4.8):

$$L = u + \hat{\partial}_x^{-1} v \longrightarrow L' = u' \hat{\partial}_{x'} + v' + \hat{\partial}_{x'}^{-1}(1 - v'_{x'}) \quad \Longrightarrow$$

$$\begin{cases} u'_x = -u_{xy} - v_y \\ v' = u \end{cases} \qquad x' = u + \partial_x^{-1} v.$$

The linked Lax operators (3.20-3.21) have similar form as (4.4-4.6) with additional source terms, so $L' = ... + \sum_i \psi'_i \hat{\partial}_{x'}^{-1} \frac{\phi_i}{\varphi_x}$. We choose eigenfunction $\varphi$ as $\varphi = u_0 + \partial_x^{-1} u_{-1} + \sum_i \psi_i \partial_x^{-1} \phi_i$ by (3.23). We will equate $L'$ to the Lax operators of the form (3.32), for fields $u'_i$, with reduction $\{u'_{-2} = 0\}$. Thus, $u'_0 = u_0$, $u'_{-1} = \frac{u_{-1}}{\varphi_x}$, $\psi'_i = \psi_i$ and $\phi'_i = \frac{\phi_i}{\varphi_x}$. Now, as

$$\varphi_x = \varphi_x (u'_0)_{x'} + u_{-1} + \sum_i \varphi_x \left(\psi'_i \partial_{x'}^{-1} \phi'_i\right)_{x'} \Longleftrightarrow u_{-1} = \varphi_x \left(1 - (u'_0)_{x'} - \sum_i \left(\psi'_i \partial_{x'}^{-1} \phi'_i\right)_{x'}\right)$$

we find constraint $u'_{-1} = 1 - (u'_0)_{x'} - \sum_i \left(\psi'_i \partial_{x'}^{-1} \phi'_i\right)_{x'}$. Hence, we found again new restriction with sources for $k = 2$ given by

$$L' = u'_N \hat{\partial}_{x'}^N + ... + u'_0 + \hat{\partial}_{x'}^{-1} \left(1 - (u'_0)_{x'} - \sum_i \left(\psi'_i \partial_{x'}^{-1} \phi'_i\right)_{x'}\right) + \sum_i \psi'_i \hat{\partial}_{x'}^{-1} \phi'_i \qquad N \geqslant 1. \quad (4.9)$$

Therefore, by theorem 2, we construct reciprocal links for the fields from Lax operators (3.20-3.21) to those fields from (4.9), respectively.

## 5 Summary

In this paper, we have examined the restrictions of Lax operators allowing systematic construction of integrable (2+1)-dimensional systems from three classes of Lax hierarchies (3.2) in centrally extended PDO algebra. It is important to mention that systems (3.2) are Hamiltonian, i.e. we can construct, beside infinite hierarchy of commuting symmetries, also Poisson tensor and infinitely many conserved quantities (see [5, 3]). Then, invariants solving the Novikov-Lax equation (3.3) are differentials of Casimir functionals of the natural Lie-Poisson bracket.

Besides, we have established some relations between the three classes of Lax hierarchies (3.2). Theorem 1 describes the gauge transformation from $k = 0$ to $k = 1$ allowing a construction of Miura maps between respective evolution equations. Theorem 2 shows how to construct reciprocal links from $k = 1$ to $k = 2$ for respective systems. It may be worth further investigation to extend the presented theory to other gauge, reciprocal and as well as Bäcklund and auto-Bäcklund transformation for Lax hierarchies (3.2), in a similar way as it is done for the (1+1)-dimensional case [9, 11].

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