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Vortex Line Representation for the Hydrodynamic Type Equations

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Abstract

In this paper we give a brief review of the recent results obtained by the author and his co-authors for description of three-dimensional vortical incompressible flows in the hydrodynamic type systems. For such flows we introduce a new mixed Lagrangian-Eulerian description - the so called vortex line representation (VLR), which corresponds to transfer to the curvilinear system of coordinates moving together with vortex lines. Introducing the VLR allows to establish the role of the Cauchy invariants from the point of view of the Hamiltonian description. In particular, these (Lagrangian) invariants, characterizing the property of frozenness of the generalized vorticity into fluids, are shown to represent the infinite (continuous) number of Casimirs for the so-called non-canonical Poisson brackets. The VLR allows to integrate partially the equations of motion, to exclude the infinite degeneracy due to frozenness of the primitive Poisson brackets and to establish in new variables the variational principle. It is shown that the original Euler equations for vortical flows coincides with the equations of motion of a charged *compressible* fluid moving due to a self-consistent electromagnetic field. Transition to the Lagrangian description in a new hydrodynamics is equivalent to the VLR. The VLR, as a mapping, turns out to be compressible that gives a new opportunity for collapse in fluid systems - breaking of vortex lines, resulting in infinite vorticity. It is shown that such process is possible for three-dimensional integrable hydrodynamics with the Hamiltonian $\mathcal{H} = \int |\mathbf{\Omega}| d\mathbf{r}$ where $\mathbf{\Omega}$ is the vorticity. We also discuss some arguments in the favor of existence of such type of collapses for the Euler hydrodynamics, based on the results of some numerics.

1 Introduction

Description of vortical flows is one of the main problems in fluid dynamics, especially this question is important for turbulence theory. Up to now it is a big challenge to construct a theory of developed hydrodynamic turbulence. Probably such a theory has not been constructed because of absence of appropriate description of vortical flows.

As well known, at high Reynolds numbers the turbulent flow can be considered with a good accuracy as a flow of an ideal fluid, namely, when the Euler equations can be applied. As it was pointed out by V.I. Arnold [1], the Euler hydrodynamics in many extents represents itself a geometric theory. The Euler equations for ideal fluids demonstrate some features common with the Euler equations for a free rigid body. But if in the three-dimensional case for the rigid body motion the governing group is $SO(3)$, then for an ideal (incompressible) fluid we have the infinite group - the group of diffeomorphisms remaining constant volume (or area in 2D). In both cases the equations of motion can be written in the Hamiltonian form by means of the Poisson brackets. The Poisson brackets for both systems define the corresponding Lie algebras: in the case of rigid body it is $so(3)$ and for fluids we have algebra of divergence-free vector fields. In both cases, however, the Poisson brackets are degenerate. The degeneracy for the rigid body case is connected with conservation of the angular momentum square (this is a Casimir). For the fluid case the fact of the degeneracy of the (noncanonical) Poisson brackets was first established by Mikhailov and the author [10]: the simplest Casimir found in [10] was the helicity $\int(\mathbf{v}\cdot\mathbf{\Omega})d\mathbf{r}$ (here $\mathbf{v}$ and $\mathbf{\Omega}$ are the fluid velocity and vorticity, respectively). This invariant has a topological meaning [19]: up to a constant factor, the helicity coincides with the Hopf invariant - the winding number for any two vortex lines. The most important point is that the number of Casimirs for the noncanonical Poisson brackets is infinite (continuous). This fact has been established sufficiently recently by Ruban and the author [11, 12]. These Casimirs turn out to coincide with the so-called Cauchy invariants. From another side, as known [24] (see also the review [26]), the Cauchy invariants are sequence of the special Noether symmetry - the relabeling symmetry of Lagrangian markers. The invariants can be considered also a consequence of the property of frozenness of vorticity into fluid. According to this property fluid particles are pasted to their own vortex line and can not leave it. Interesting to notice that the constancy of the Cauchy invariants (as Lagrangian or material invariants) is less known than the famous Kelvin theorem about conservation of velocity circulation. However, both conservation laws represent the same. The difference between them is that the Kelvin theorem says about conservation of the *integral* quantity, i.e. the velocity circulation, but the Cauchy invariant is *local*, expressing the same constancy. The latter means that motion of an ideal fluid is very restricted: in each (Lagrangian) point the Euler equations have the conservation law which can be considered as a first integral of the equations. As known, fixing all Casimirs yields a symplectic leave. According to the general theory (see, for instance, the review [26]) introducing coordinates on this leave allows to establish a fully valid Hamiltonian mechanics, in particular, to write down the variational principle. As it was pointed out in [26] introducing Poisson structure for the dynamical system can be considered as the Hamiltonian description in the weakest sense.

For the fluid case this leave generally is an infinitely-dimension manifold embedded in the space of divergence-free fields that makes this problem to be very difficult and complicated. Thus, to get a fully valid Hamiltonian mechanics for the fluid case one needs first to find all Casimirs, secondly, to introduce appropriate coordinates resolving all these constrains (Casimirs) and only after it is possible to have the variational principle, etc. In this paper we give a solution of this problem by introducing the so-called vortex line representation - the mixed Lagrangian-Eulerian description, when each vortex line is labeled by a two-dimensional Lagrangian marker (the Clebsch variables may be used as such markers) and one Eulerian coordinate, for instance, x, given the vortex line. This representation implies transition to the curvilinear system of coordinates moving together with vortex lines. Resulting equations turn out to be resolved with respect to the Cauchy invariants. In the case of the classical Euler equations applying the VLR means their partial integration.

The VLR solves not only the problem with Casimirs for the hydrodynamical type models but also opens a new possibility to treat another very important problem, i.e the problem of collapse - the singularity formation in finite time for smooth initial conditions.

Because of the vorticity frozenness, the equation of motion for vortex lines defines by the velocity component transverse to the vortex lines, that results in compressibility of the VLR, as a mapping. As recently pointed out in [13], compressibility of the mapping for the Euler equations is amenable of a simple interpretation. The equations can be rewritten as the equations of motion for a charged *compressible* fluid moving under the action of effective self-consistent electric and magnetic fields satisfying Maxwell equations. The VLR for the Euler equations corresponds to transition from the Eulerian description to the Lagrangian one in a new charged hydrodynamics.

It is well known that the appearance of singularities in compressible flows is connected with the emergence of shocks, corresponding to the formation of folds in the classical catastrophe theory [2] when the Jacobian of the corresponding mapping vanishes. In the gas-dynamic case the mapping is defined by the transition from the usual Eulerian to the Lagrangian description. Due to the compressible character of VLR, the phenomenon of breaking becomes also possible for vortex lines. The breaking of vortex lines, as we show in the paper, might be for 3D flows but is forbidden in 2D. It should lead to the gradient catastrophe resulting in infinite vorticity.

Although the problem of collapse for the Euler equations remains controversial, there are some arguments in the favor of its existence. First of all this is some numerical experiments [8]-[14] demonstrated blow-up behavior. In particular, in our numerics [28], [14] for the partially integrated Euler equations, which are resolved relative to the Cauchy invariants, we have observed sharp increase of the vorticity corresponding to vanish of the VLR Jacobian in finite time that can be interpreted in the favor of vortex lines breaking. Secondly, among the systems of hydrodynamic type possessing the same symplectic operator as for the Euler equations, there exists one remarkable model - the three-dimensional integrable hydrodynamics [11] with the Hamiltonian $\mathcal{H} = \int |\mathbf{\Omega}| d\mathbf{r}$ where $\mathbf{\Omega}$ is the generalized vorticity. The given model can be integrated by means of combination of the vortex line representation and the inverse scattering transform. By applying the VLR, the Hamiltonian is decomposed into a sum of Hamiltonians of non-interacting vortex lines. Dynamics of each vortex line is described by the integrable one-dimensional Landau-Lifshitz equation for a Heisenberg ferromagnet or by its gauge-equivalent - the nonlinear Schroedinger equation. Thus, the integrable hydrodynamics represents a hydrodynamics of free vortex lines. For continuous distribution of vortex lines this fact is the main reason of their breaking [15]. First time breaking happens when one vortex line touches another. In this sense it is analogous completely to the breaking in the hydrodynamics of dust with a null pressure (see, e.g. [25]).

The plan of the paper is as follows. In the next section we present the main equations, and discuss their properties, in particular, the Cauchy invariants and their generated relabeling symmetry, Poisson brackets and its degeneracy. The third section deals with the Clebsch variables for the generalized Euler equations and their connection with the mixed Lagrangian-Eulerian description of vortex lines. The section 5 is devoted to the general formulation of the vortex line representation. Here we get the variational principle in terms of the VLR. In the sections 5 and 6 we deal with the 3D integrable hydrodynamics and construct solutions of collapsing type for the model. Numerical experiments on the observation of collapse in the Euler equations are discussed briefly in the section 7.

2 General remarks

As well known (see, for instance, [24], [26]) the Euler equations for the velocity $\mathbf{v}$ and the pressure p of an ideal incompressible fluid,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\nabla p, \qquad \mathrm{div}\ \mathbf{v} = 0, \tag{2.1}$$

in both two-dimensional and three-dimensional cases possess the infinite (continuous) number of integrals of motion. These are the so-called Cauchy invariants. The most simple way to derive the Cauchy invariants is one to use the Kelvin theorem about conservation of the velocity circulation,

$$\Gamma = \oint (\mathbf{v} \cdot d\mathbf{l}), \tag{2.2}$$

where the integration contour $C[\mathbf{r}(t)]$ moves together with a fluid. If in this expression one makes a transform from the Eulerian coordinate $\mathbf{r}$ to the Lagrangian ones $\mathbf{a}$ then Eq. (2.2) can be rewritten as follows:

$$\Gamma = \oint \dot{x}_i \cdot \frac{\partial x_i}{\partial a_k}\, da_k \ ,$$

where a new contour $C[\mathbf{a}]$ is already immovable. Hence, due to arbitrariness of the contour $C[\mathbf{a}]$ and using the Stokes formula one can conclude that the quantity

$$\mathbf{I} = \mathrm{curl}_a \left(\dot{x}_i \frac{\partial x_i}{\partial \mathbf{a}} \right) \tag{2.3}$$

conserves in time at each point $\mathbf{a}$. This is just the Cauchy invariant. If the Lagrangian coordinates $\mathbf{a}$ in (2.3) coincide with the initial positions of fluid particles the invariant $\mathbf{I}$ is equal to the initial vorticity $\mathbf{\Omega}_0(\mathbf{a})$. In the two-dimensional case the vorticity, being the Lagrangian (one component) quantity, coincides with the Cauchy invariant. In the three-dimensional situation the Cauchy invariant is the whole conservative vector given at each (Lagrangian) point.

Conservation of these invariants, as it was shown first by Salmon [24], is consequence of the special (infinite) symmetry - the so-called relabeling symmetry. The Cauchy invariants characterize the frozenness of the vorticity into fluid. This is a very important property according to which fluid (Lagrangian) particles can not leave its own vortex line where they were initially. Thus, the Lagrangian particles have one independent degree of freedom – motion along vortex line. From another side, such a motion as it follows from the equation for the vorticity

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \mathrm{curl}\ [\mathbf{v} \times \mathbf{\Omega}], \tag{2.4}$$

does not change its value. From this point of view a vortex line represents the invariant object and therefore it is natural to seek for such a transformation when this invariance is seen from the very beginning. Such type of description - the vortex line representation - was introduced in the papers [11, 12] by Ruban and the author of this paper.

The vortex line representation can be introduced also to the whole family of equations (sometimes called as the Arnold equations):

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \text{curl}\left[\text{curl}\frac{\delta \mathcal{H}}{\delta \mathbf{\Omega}} \times .\mathbf{\Omega}\right] = \{\mathbf{\Omega}, \mathcal{H}\}, \tag{2.5}$$

where the noncanonical Poisson brackets are given by the expression [10]:

$$\{F, G\} = \int \left(\mathbf{\Omega}\left[\text{curl}\,\frac{\delta F}{\delta \mathbf{\Omega}} \times \text{curl}\,\frac{\delta G}{\delta \mathbf{\Omega}}\right]\right) d\mathbf{r}. \tag{2.6}$$

Here $\mathbf{\Omega}$ is the (generalized) vorticity,

$$\mathbf{v} = \text{curl}\,\frac{\delta \mathcal{H}}{\delta \mathbf{\Omega}} \tag{2.7}$$

has the meaning of the fluid velocity, $\mathcal{H} = \mathcal{H}[\mathbf{\Omega}]$ is the fluid Hamiltonian. In particular, if $\mathcal{H}$ coincides with kinetic energy, $1/2 \int \mathbf{v}^2 d\mathbf{r}$, vorticity is expressed through velocity by the standard formula: $\mathbf{\Omega} = \text{curl}\,\mathbf{v}$, and, respectively, the equation (2.5) becomes the Euler equations (2.4). For the equations (2.5) the Cauchy invariants have the same form (2.3) and the property of the vorticity frozenness can be established by the same way as for the original Euler equations.

The brackets (2.6) allow to describe flows with arbitrary topology, but its main lack is a degeneracy. The simplest Casimir, annulating (2.6), is the helicity $I_h = \int (\mathbf{\Omega} \cdot \text{curl}^{-1}\mathbf{\Omega}) d\mathbf{r}$: $\{I_h, .\} = 0$. This integral has topological meaning: up to the constant it coincides with the winding number (Hopf invariant) of any two vortex lines. However, as it was shown in [12], a more deep cause of the degeneracy, i.e, presence of Casimirs, is connected with existence of the special symmetry formed the whole group - the relabeling group of Lagrangian markers [24]. By this reason it is impossible to formulate the variational principle on the whole space of divergence-free vector fields.

3 Clebsch variables and mixed Lagrangian-Eulerian description

Consider the vortical flow ($\mathbf{\Omega} \neq 0$) of an ideal fluid given by the Clebsch variables λ and μ:

$$\mathbf{\Omega} = [\nabla\lambda \times \nabla\mu]. \tag{3.1}$$

The geometrical meaning of these variables is well known: intersection of any two surfaces $\lambda = \text{const}$ and $\mu = \text{const}$ yields a vortex line. If λ and μ are one-valued functions of space coordinates then vortex lines will be closed.

As it follows from (3.1) the Clebsch variables are defined up to the point change of variables for which the Jacobian $\partial(\lambda, \mu)/\partial(\lambda', \mu') = 1$. This change of variables is nothing more that the canonical transformation, if one considers λ and μ canonically conjugated quantities. And they are indeed.

It is known that the Clebsch variables are Lagrangian invariants, being unchanged along trajectories of fluid particles:

$$\frac{\partial\lambda}{\partial t} + (\mathbf{v}\nabla)\lambda = 0; \qquad \frac{\partial\mu}{\partial t} + (\mathbf{v}\nabla)\mu = 0. \tag{3.2}$$

This fact is valid not only for the Euler equation but also for the whole family (2.5), with the velocity given by (2.7).

Due to the velocity definition (2.7), two following identities are fulfilled:

$$(\mathbf{v}\nabla)\lambda = -\frac{\delta\mathcal{H}}{\delta\mu}, \qquad (\mathbf{v}\nabla)\mu = \frac{\delta\mathcal{H}}{\delta\lambda}.$$

The latter means that λ and μ are canonically conjugated quantities.

Being Lagrangian, these variables can be taken as markers for vortex lines. It is easily to establish that transition in (3.1) to new variables

$$\lambda = \lambda(x,y,z), \qquad \mu = \mu(x,y,z), \qquad s = s(x,y,z), \tag{3.3}$$

where s is the parameter given the vortex line, leads to the expression [13]:

$$\mathbf{\Omega}(\mathbf{r},t) = \frac{1}{J}\cdot\frac{\partial\mathbf{R}}{\partial s} \tag{3.4}$$

where

$$J = \frac{\partial(x,y,z)}{\partial(\lambda,\mu,s)} \tag{3.5}$$

is the Jacobian of the mapping

$$\mathbf{r} = \mathbf{R}(\lambda,\mu,s). \tag{3.6}$$

The transform (3.6) inverse to (3.3) defines the corresponding transition to the curvilinear, connected with vortex lines, system of coordinates.

Note, that the expression for the vorticity (3.4) is invariant under reparameterization: $\tilde{s} = \tilde{s}(\lambda,\mu,s,t)$. By another words, we have a freedom in choice of the parameter s. For instance, we can take instead of s any Cartesian coordinate, say, x. Therefore, unlike λ and μ, the parameter s represents the Eulerian variable.

The equations of motion for vortex lines - the equations for $\mathbf{R}(\lambda,\mu,s,t)$ – can be obtained directly from the equation of motion for the vorticity (2.4) (see [12]). However, the most simple way to derive them is to use the combination of the equations (3.2):

$$\nabla\mu\left[\frac{\partial\lambda}{\partial t} + (\mathbf{v}\nabla)\lambda\right] - \nabla\lambda\left[\frac{\partial\mu}{\partial t} + (\mathbf{v}\nabla)\mu\right] = 0, \tag{3.7}$$

which is identical to (3.2) due to a linear independence of the vectors $\nabla\lambda$ and $\nabla\mu$.

Performing in (3.7) the transformations (3.3), we arrive at the equation of motion for vortex lines [11]:

$$\left[\frac{\partial\mathbf{R}}{\partial s} \times \left(\frac{\partial\mathbf{R}}{\partial t} - \mathbf{v}(\mathbf{R},t)\right)\right] = 0. \tag{3.8}$$

This equation has one important property: any motion along a vortex line does not change the line itself.

Because of a freedom in choice of the parameter s, when any change in it leads to addition $s_t\partial_s\mathbf{R}$ into $\partial_t\mathbf{R}$, we can require, without any loss of generality, the vector $\partial_t\mathbf{R}$ to be orthogonal to $\partial_s\mathbf{R}$. Then the (3.8) becomes equivalent to the equation

$$\frac{\partial\mathbf{R}}{\partial t} = \mathbf{v}_n(\mathbf{R}, t), \tag{3.9}$$

where $\mathbf{v}_n$ is the velocity component normal to the vorticity vector.

The obtained equations (3.4), (3.8) or (3.9) can be considered as the result of applying the vortex line representation to the Euler equations, in its local version [11]. The vortex line representation, as the mapping (3.6), has the Jacobian which values are not fixed as it was for transition from the Eulerian to Lagrangian variables. The latter demonstrates the VLR as a compressible mapping.

Locality of the VLR (3.6) follows from the definition of the Clebsch variables: in accordance with the Darboux theorem, they can be introduced locally always but not globally. It is well known also (see, i.g., [26]) that the flows parameterized by the Clebsch variables has a zero helicity integral, characterizing the flow topology. Therefore to introduce the vortex line representation for flows with nontrivial topology it is necessary to come back to the original equations of motion (2.1) and (2.5) for velocity and vorticity.

4 Vortex line representation: general formulation

According to the equation (2.4) the tangent to the vector $\boldsymbol{\Omega}$ velocity component $\mathbf{v}_\tau$ does not effect (directly) on the vorticity dynamics, i.e., in (2.4) we can replace $\mathbf{v}$ by its transverse component $\mathbf{v}_n$.

The equation of motion for the transverse velocity $\mathbf{v}_n$ follows directly from the equation (2.1). It has the form of the equation of motion of charged particle moving in an electromagnetic field [13]:

$$\frac{\partial\mathbf{v}_n}{\partial t} + (\mathbf{v}_n\nabla)\mathbf{v}_n = \mathbf{E} + [\mathbf{v}_n \times \mathbf{H}], \tag{4.1}$$

where the effective electric and magnetic fields are given by the standard formulas accepted in electrodynamics

$$\mathbf{E} = -\nabla\varphi - \frac{\partial\mathbf{A}}{\partial t}, \tag{4.2}$$

$$\mathbf{H} = \text{curl } \mathbf{A} \tag{4.3}$$

with the scalar φ and vector $\mathbf{A}$ potentials given by the expressions:

$$\varphi = p + \frac{\mathbf{v}_\tau^2}{2}, \qquad \mathbf{A} = \mathbf{v}_\tau, \tag{4.4}$$

so that two Maxwell equations

$$\text{div } \mathbf{H} = 0, \qquad \frac{\partial\mathbf{H}}{\partial t} = -\text{curl } \mathbf{E}$$

satisfy automatically. In this case the vector potential $\mathbf{A}$ has the gauge div $\mathbf{A} = -\text{div } \mathbf{v}_n$, which is equivalent to the condition div $\mathbf{v} = 0$.

Two other Maxwell equations can be written also but they can be considered as definition of the charge density ρ and the current $\mathbf{j}$ which follow from the relations (4.2) and (4.3). The basic equation in the new hydrodynamics is the equation of motion (4.1) for the normal component of the velocity which represents the equation of motion for nonrelativistic particle with a charge and a mass equal to unity, the light velocity in this units is equal to 1.

New terms in the right hand side of Eq. (4.1) have also mechanical interpretation. Lorenz force $[\mathbf{v_n} \times \mathbf{H}]$ is nothing more than Coriolis force. Addition in ϕ to pressure p, equal to ${\mathbf{v}_\tau}^2/2$, has direct connection with the Bernoulli formula. The term $\partial_t \mathbf{v}_\tau$ appears due to transition to movable non-inertial system of coordinates.

The equation of motion (4.1) is written in the Eulerian representation. To transfer to its Lagrangian formulation one needs to consider the equations for "trajectories" given by the velocity $\mathbf{v}_n$:

$$\frac{d\mathbf{R}}{dt} = \mathbf{v}_n(\mathbf{R}, t) \tag{4.5}$$

with initial conditions $\mathbf{R}|_{t=0} = \mathbf{a}$. Solution of the equation (4.5) yields the mapping

$$\mathbf{r} = \mathbf{R}(\mathbf{a}, t), \tag{4.6}$$

which defines transition from the Eulerian description to a new Lagrangian one.

The equations of motion in new variables are the Hamilton equations:

$$\dot{\mathbf{P}} = -\frac{\partial h}{\partial \mathbf{R}}, \qquad \dot{\mathbf{R}} = \frac{\partial h}{\partial \mathbf{P}}, \tag{4.7}$$

where dot means differentiation with respect to time for fixed $\mathbf{a}$, $\mathbf{P} = \mathbf{v}_n + \mathbf{A} \equiv \mathbf{v}$ is the generalized momentum, and the Hamiltonian of a particle h being a function of momentum $\mathbf{P}$ and coordinate $\mathbf{R}$ is given by the standard expression:

$$h = \frac{1}{2}(\mathbf{P} - \mathbf{A})^2 + \varphi \equiv p + \frac{\mathbf{v}^2}{2},$$

i.e., coincides with the Bernoulli "invariant".

The first equation of the system (4.7) is the equation of motion (4.1), written in terms of $\mathbf{a}$ and t, and the second equation coincides with (4.5).

For new hydrodynamics (4.1) or for its Hamilton version (4.7) it is possible to formulate a "new" Kelvin theorem (it is also the Liouville theorem):

$$\Gamma = \oint (\mathbf{P} \cdot d\mathbf{R}), \tag{4.8}$$

where integration is taken along a loop moving together with the "fluid". Hence, analogously as it was made before while derivation of (2.3) we get the expression for a new Cauchy invariant:

$$\mathbf{I} = \text{curl}_a \left(P_i \frac{\partial x_i}{\partial \mathbf{a}} \right). \tag{4.9}$$

Its difference from the original Cauchy invariant (2.3) consists in that in the equation of motion (4.5) instead of the velocity $\mathbf{v}$ stands its normal component $\mathbf{v}_n$. As consequence, the "new" hydrodynamics becomes compressible: div $\mathbf{v}_n \neq 0$. Therefore on the Jacobian J of the mapping (4.6) there are imposed no restrictions. The Jacobian J can take arbitrary values.

¿From the formula (4.9) it is easily to get the expression for the vorticity $\mathbf{\Omega}$ in the given point $\mathbf{r}$ at the instant t (compare with [11, 12]):

$$\mathbf{\Omega}(\mathbf{r},t) = \frac{(\mathbf{\Omega}_0(\mathbf{a})\cdot\nabla_a)\mathbf{R}(\mathbf{a},t)}{J(\mathbf{a},t)}, \tag{4.10}$$

where J is the Jacobian of the mapping (4.6) equal to

$$J(\mathbf{a},t) = \frac{\partial(x_1,x_2,x_3)}{\partial(a_1,a_2,a_3)}.$$

Here we took into account that the generalized momentum $\mathbf{P}$ coincides with the velocity $\mathbf{v}$, including the moment of time $t=0$: $\mathbf{P}_0(\mathbf{a}) \equiv \mathbf{v}_0(\mathbf{a})$. $\mathbf{\Omega}_0(\mathbf{a})$ in this relation is the "new" Cauchy invariant with zero divergence: $\mathrm{div}_a\mathbf{\Omega}_0(a) = 0$.

The representation (4.10) generalizes the relation (3.1) to arbitrary topology of vortex lines. The variables $\mathbf{a}$ in this expression can be considered locally as a set of ν and s.

The equations (4.10), (4.5) together with the definition (2.7) form the complete set of equations written in the vortex line representation. Because of frozenness of vorticity the equations (4.5) is nothing more than the equation of motion of vortex lines. It can be written also in the form analogous to (3.8):

$$\left[\mathbf{b}\times\frac{\partial\mathbf{R}}{\partial t}\right] = [\mathbf{b}\times\mathbf{v}(\mathbf{R},t)], \tag{4.11}$$

where $\mathbf{b} = (\mathbf{\Omega}_0(\mathbf{a})\cdot\nabla_a)\mathbf{R}(a,t)$ is the tangent vector to the vortex line.

The equations of motion (4.5) (or (4.11)) together with the relation (4.10) can be considered as the result of partial integration of the Euler equation (2.1). These new equations are resolved with respect to the Cauchy invariants – an infinite number of integrals of motion, that is a very important issue for numerical integration (see [28, 14]). For the partially integrated system the Cauchy invariants conserve automatically that, however, for direct numerical integration of the Euler equations one needs to test in which extent these invariants remain constant. Probably, this is one of the main restrictions defining accuracy of discrete algorithms for direct integration of the Euler equations.

Another very important property of the vortex line representation is absence of any restrictions on the value of the Jacobian J which do exist, for instance, for transition from the Eulerian description to the Lagrangian one in the original Euler equation (2.1) (when Jacobian in the simplest situation is equal to unity). The value $1/J$ for the system (4.5) (or (4.11)), (4.10) has a meaning of a density n of vortex lines. This quantity as a function of $\mathbf{r}$ and t , according to (4.5), obeys the continuity equation:

$$\frac{\partial n}{\partial t} + \mathrm{div}_r(n\mathbf{v}_n) = 0.$$

In this equation $\mathrm{div}_r\mathbf{v}_n \neq 0$ because only the total velocity has zero divergence.

The vortex line representation as a local change of variables $\mathbf{r} = \mathbf{r}(\mathbf{a}, t)$ does not work in singular points, where the vorticity is equal to zero and, respectively, the normal velocity occurs uncertain. Due to the frozenness of vorticity such points remain in time, advected by the fluid. Really, let us consider the point $\mathbf{r} = \mathbf{r}(t)$ which defines from the equation $\mathbf{\Omega}(\mathbf{r}(t), t) = 0$. Differentiate this equation with respect to time we arrive at the equation,

$$\frac{\partial \mathbf{\Omega}}{\partial t} + (\dot{\mathbf{r}}(t) \cdot \nabla)\mathbf{\Omega} = 0, \tag{4.12}$$

coinciding with the Euler equation for vorticity in this partial case, $\mathbf{\Omega}(\mathbf{r}(t), t) = 0$. Here $\dot{\mathbf{r}}(t) = \mathbf{v}(\mathbf{r}(t), t)$. This proves that these points are advected by flows and can not dissipate or, for instance, transform into cuts.

The velocity $\mathbf{v}$ in these points is defined by inverting the curl operator: $\mathbf{v} = \mathrm{curl}^{-1}\mathbf{\Omega}$. However, the normal component of the velocity $\mathbf{v}_n$ is not defined in these points. By this reason for the vector field $\tau(\mathbf{r}) \equiv \mathbf{\Omega}/|\Omega|$, i.e., for the unit tangent vector to vortex lines, the null points represent topological singularities which can be classified by means of topological methods. This classification can be determined by the topological charge as a degree of mapping $\mathcal{S}^2 \to \mathcal{S}^2$, given by the integral,

$$\int_{\partial V} \epsilon_{\alpha\beta\gamma} \, (\tau \cdot [\partial_\beta \tau \times \partial_\gamma \tau]) \, dS_\gamma = 4\pi m, \tag{4.13}$$

where integration is performed over the boundary ∂V of the region V containing the points and the topological charge m takes integer numbers.

Thus, the equations (4.10), (4.5) together with the condition (4.13) constitute a complete system of equations that provides a vortex line representation for the Euler equations in the general case. One should note that the generalized Euler equations (2.5) in the VLR have the same form as for the original Euler equations, i.e., (4.10), (4.5) (or (4.11)).

It turns out that the equations of motion for vortex lines (4.11) follow from the variational principle for the action with the Lagrangian [12]:

$$\mathcal{L} = \frac{1}{3}\int ([\mathbf{R}_t(\mathbf{a}) \times \mathbf{R}(\mathbf{a})] \cdot (\mathbf{\Omega}_0(\mathbf{a})\nabla_{\mathbf{a}})\mathbf{R}(\mathbf{a})) - \mathcal{H}(\{\mathbf{\Omega}\{\mathbf{R}\}\})d\mathbf{a}. \tag{4.14}$$

This fact can be verified by direct calculations with the help of the following equality, holding for functionals depending only on $\mathbf{\Omega}$:

$$\left[\mathbf{b} \times \mathrm{curl}\left(\frac{\delta F}{\delta \mathbf{\Omega}(\mathbf{R})}\right)\right] = \frac{\delta F}{\delta \mathbf{R}(\mathbf{a})}\Big|_{\mathbf{\Omega}_0}. \tag{4.15}$$

Note also, due to this equality, the right-hand-side of (4.11) equals to the variational derivative $\delta\mathcal{H}/\delta\mathbf{R}$:

$$[(\mathbf{\Omega}_0(\mathbf{a})\nabla_{\mathbf{a}})\mathbf{R}(\mathbf{a}) \times \mathbf{R}_t(\mathbf{a})] = \frac{\delta \mathcal{H}\{\mathbf{\Omega}\{\mathbf{R}\}\}}{\delta \mathbf{R}(\mathbf{a})}\Big|_{\mathbf{\Omega}_0}. \tag{4.16}$$

¿From another side, the formulation of the variational principle (4.14) for the equations of motion for vortex lines imply that by introducing the vortex line representation we simultaneously have solved the problem with Casimirs. Moreover, the vortex line representation for the equation (4.10), as a formal change of variables from from the divergence-free field

$\mathbf{\Omega r}$ to new fields $\mathbf{R}(\mathbf{a})$ and $\mathbf{\Omega_0}(\mathbf{a})$, should give us desired coordinates on the symplectic leaf when we fix all Casimirs, i.e., we are very close to a final answer.

Let us show now that desired Casimirs are Cauchy invariants.

¿From (4.15) follows the property that the vector $\mathbf{b}$ and $\delta F/\delta\mathbf{R}(\mathbf{a})$ are orthogonal. In other words the variational derivative of the gauge-invariant functionals should be understood (specifically, in (4.15)) as $\hat{P}(\delta F/\delta\mathbf{R}(\mathbf{a}))$, where $\hat{P}_{ij} = \delta_{ij} - \tau_i\tau_j$ is a projector transverse to vortex line. Using this property as well as the transformation formula (4.15) it is possible, by a direct calculation of the bracket (2.6), to obtain the Poisson bracket (between two gauge-invariant functionals) expressed in terms of vortex lines:

$$\{F, G\} = \int \frac{d\mathbf{a}}{|\mathbf{b}|^2}\left(\mathbf{b}\cdot\left[\hat{P}\frac{\delta F}{\delta\mathbf{R}(\mathbf{a})}\times\hat{P}\frac{\delta G}{\delta\mathbf{R}(\mathbf{a})}\right]\right). \tag{4.17}$$

The new bracket (4.17) does not contain variational derivatives with respect to $\mathbf{\Omega}_0(\mathbf{a})$. Therefore, with respect to the initial bracket (2.6) the Cauchy invariant $\mathbf{\Omega}_0(\mathbf{a})$ is a Casimir fixing a symplectic leaf on which it is possible to introduce the variational principle (4.14). Respectively, the variables $\mathbf{R}(\mathbf{a})$ serve coordinates on these leaves.

5 Three-dimensional integrable hydrodynamics

In this and next sections, we will show how and why collapse is possible in 3D integrable hydrodynamics. This model was introduced in [11, 12]. The Hamiltonian of this model is expressed through the absolute value of $\mathbf{\Omega}(\mathbf{r}, t)$

$$\mathcal{H} = \int |\mathbf{\Omega}(\mathbf{r})| d\mathbf{r}, \tag{5.1}$$

and the equation of motion coincides with the frozenness equation (2.5) with the velocity $\mathbf{v} = \mathrm{curl}\,\vec{\tau}$ where $\vec{\tau} = (\mathbf{\Omega}/\Omega)$ is the unit tangent vector along the vortex line. Assuming all the lines closed and substituting the representation (3.4) into (5.1), it is easy to see that the Hamiltonian is decomposed as a sum of Hamiltonians for set of vortex lines [1]:

$$\mathcal{H}\{\mathbf{R}\} = \int d2\nu \int \left|\frac{\partial\mathbf{R}}{\partial s}\right| ds. \tag{5.2}$$

Here, the integral over s is the length of the vortex line with the marker ν. The equation of motion for the vector $\mathbf{R}(\nu, s)$, in accordance with (4.16), is local in these variables – it doesn't contain an interaction with other vortices:

$$[\mathbf{R}_s \times \mathbf{R}_t] = [\vec{\tau} \times [\vec{\tau} \times \vec{\tau}_s]]. \tag{5.3}$$

By this reason, the energy and momentum for each vortex loop are conservative quantities, corresponding to its geometrical characteristics: its length

$$L \equiv \mathcal{H}(\nu) = \int |\mathbf{R}_s(\nu)| ds,$$

[1] It is worth to notice that this property is common for all systems with the Hamiltonians of the type $\mathcal{H} = \int F(\tau, (\tau\nabla)\tau, (\tau\nabla)2\tau, \dots)|\mathbf{\Omega}|d\mathbf{r}$. To explain the idea of collapse of vortex lines, we have chosen the simplest example (5.1), which has a physical meaning.

the oriented area **S** spanned on the vortex loop coincides with its momentum:

$$\mathbf{P}(\nu) = \frac{1}{2}\int [\mathbf{R}(\nu) \times \mathbf{R}_s(\nu)]ds.$$

By introducing new variables, the filament length l ($dl = |\mathbf{R}_s|ds$) and time $t' = t$, the equation of motion for one vortex line (5.3) can be reduced to the integrable one-dimensional (1D) Landau-Lifshitz equation for a Heisenberg ferromagnet [11]:

$$\frac{\partial \vec{\tau}}{\partial t} = \left[\vec{\tau} \times \frac{\partial^2 \vec{\tau}}{\partial l^2}\right]. \tag{5.4}$$

As known [27], this equation is gauge equivalent to the 1D nonlinear Shrödinger equation [27] and, for instance, can be reduced to the NLSE by means of the Hasimoto transformation [7].

The system under consideration has direct relation to hydrodynamics. As it is known [5] (see also [21]), the local induction approximation for a thin vortex filament, under assumption of smallness of the filament width to the characteristic longitudinal scale, leads to the Hamiltonian (4.7), but only for a single separate line. The essence of this approximation is in replacing the logarithmic interaction law by a delta-functional one. When the widths of the filaments are small comparable with distances between them, in the same approximation, the Hamiltonian of vortex lines transforms into the sum of the Hamiltonians of independent vortex loops, yielding in a "continuous" limit the Hamiltonian (5.1).

By such a way, we have the model of 3D integrable hydrodynamics of free vortex filaments that is a main reason of collapse - a singularity formation in a finite time. In the given case this process is analogous to the phenomenon of wave breaking in gas-dynamics.

Consider the simplest solution of the equation (5.3) - a stationary propagation of a closed vortex line: $\mathbf{R}_t = \mathbf{V} \equiv \text{const}$. In this case the velocity $\mathbf{V}$ is determined from solution of the equation

$$[\mathbf{R}_s \times \mathbf{V}] = [\vec{\tau} \times [\vec{\tau} \times \vec{\tau}_s]]. \tag{5.5}$$

It is easily to check that this equation follows from the variational principle

$$\delta(\mathcal{H}(\nu) - \mathbf{V} \cdot \mathbf{P}(\nu)) = 0, \tag{5.6}$$

i.e., any solution of (5.5) represents a stationary point of the Hamiltonian for a fixed momentum $\mathbf{P}(\nu)$. The equation (5.5) can be simply integrated, being rewritten in terms of the binormal $\mathbf{b}$ and the curvature κ of the line as follows

$$[\tau \times \mathbf{V}] = \kappa[\tau \times b], \tag{5.7}$$

that gives

$$\mathbf{V} = \kappa \mathbf{b}. \tag{5.8}$$

A constant value of the velocity $\mathbf{V}$ in this expression implies constancy of the curvature κ, i.e. the vortex line must be a ring of radius $r = 1/\kappa$ and

$$V = 1/r. \tag{5.9}$$

The direction of the ring motion is perpendicular to its plane. It is interesting to note that the exact answer to the velocity of a thin (with width $d \ll r$) vortex ring in ideal hydrodynamics ([18]) coincides with Eq.(5.8) up to the logarithmic accuracy that just differs the considering model from the Euler equation.

The found solution in the form of moving ring turns out to be stable. Their stability (in the Lyapunov sense) follows from the fact that the oriented surface S spanned on the vortex loop, coinciding with its momentum $\mathbf{P}$, for fixed loop length ($\equiv \mathcal{H}(\nu)$) attains its maximal value at the perfect circle.

6 Collapse in integrable hydrodynamics

The solution (5.8), (5.9) enables us to construct the simplest mappings $\mathbf{R} = \mathbf{R}(\nu, s, t)$.

Let all vortex lines be circle-shaped and oriented in the same direction, for instance, along z-axis. Because collapse in our model is a purely local phenomenon, it is sufficient to consider some vortex tube (which can be imagined as a torus) to find a mapping. Let vortex rings be distributed continuously inside the tube. We label each vortex line by the two-dimensional parameter $\nu = (\lambda, \mu)$, which values coincide with coordinate of some cross section of the tube at $t = 0$. We will use the ring arc-length as longitudinal parameter s ($ds = r d\phi$, where ϕ is the polar angle around z-axis). Then, with the help of (5.8), the desired mapping can be written as follows

$$\mathbf{R} = \mathbf{R_0}(\nu) + r(\nu)\cos\phi \cdot \mathbf{e}_x + r(\nu)\sin\phi \cdot \mathbf{e}_y + V(\nu)t \cdot \mathbf{e}_z \tag{6.1}$$

where $\mathbf{e}_{x,y,z}$ are unit vectors along the corresponding axes.

It is easy to see that the Jacobian of this mapping is a linear function of time

$$J = \frac{\partial(X, Y, Z)}{\partial(\lambda, \mu, s)} = J_0(\nu, s) + A(\nu, s)t. \tag{6.2}$$

Here $A(\nu, s)$ is a coefficient linearly dependent on the velocity derivatives with respect to ν and J_0 the initial value of Jacobian.

Dependence J (6.2) on time means that for every fixed $\mathbf{a} = (\nu, s)$ there exists such a moment of time $t = \tilde{t}(\mathbf{a})$, when the Jacobian is equal to zero: $J(\mathbf{a}, t) = 0$. Denote as t_0 the minimal value of $t = \tilde{t}(\nu, s)$ at $t > 0$. And let this minimum be attained at some point $\mathbf{a} = \mathbf{a_0}$ (here we denote a point (ν_1, ν_2, s) as $\mathbf{a}$). It is evident that at $t \to t_0$ at the small vicinity of the minimal point $\mathbf{a_0}$ the expansion of J has the form:

$$J(\mathbf{a}, t) = \alpha(t_0 - t) + \gamma_{ij}\Delta a_i \Delta a_j + ..., \tag{6.3}$$

where

$$2\gamma_{ij} = \left. \frac{\partial^2 J}{\partial a_i \partial a_j} \right|_{a_0}$$

is a positive definite tensor, $\alpha > 0$ and $\Delta\mathbf{a} = \mathbf{a} - \mathbf{a_0}$.

Geometrically, the above expansion corresponds to the following. The (hyper-) surface $J = J(\mathbf{a}, t)$ is deforming with time in such a manner, that its minimum reaches the (hyper-) plane $J = 0$ at $t = t_0$, when two surfaces touch each other. Obviously, for smooth mappings in a typical case this touching takes place in one separate point. For instance, in a degenerated situation, touching is possible in a few points simultaneously, or even at a curve. A case, when two eigenvalues of the Jacoby matrix $\hat{J}$ tend to zero simultaneously at the collapse point, should be regarded also as degenerated.

In accordance with (3.4), the equality $J = 0$ at the singular point means the formation of a singularity for the vorticity at the moment $t = t_0$:

$$\mathbf{\Omega}(\mathbf{r}, t) = \frac{\Omega_0(\nu)\mathbf{R}_s}{\alpha(t_0 - t) + \gamma_{ij}\Delta a_i \Delta a_j}. \tag{6.4}$$

Therefore, the vorticity at the singular point blows up as $(t_0 - t)^{-1}$, and the characteristic size of the collapsing distribution in a-coordinates decreases as $\sqrt{t_0 - t}$.

The above type of collapse arises as a result of vortex line breaking when one vortex overtakes another [15]. For flows of the general type (without symmetries) a singularity must arise at the first time always in one separate point. In this case the vorticity near singular point will be described by means of (6.4).

The above formula (6.4) describes self-similar compression in the auxiliary **a**-space. However, the latter does not assume, that compression in the physical, **r**-space will be the same as in **a**-space.

In order to understand such a difference, consider first the one-dimensional gasdynamics of dust (with zero pressure):

$$\partial_t \rho + (\rho v)_x = 0, \quad \partial_t v + v v_x = 0,$$

where ρ is the dust density and v its velocity.

These equations are easily integrated in Lagrangian variables:

$$v = v_0(a), \qquad \rho = \frac{\rho_0(a)}{J}$$

where the mapping is the linear function of time: $x = a + v_0(a)t$. Near the breaking point the Jacobian will have the expansion (6.3):

$$J \equiv \frac{\partial x}{\partial a} = \alpha\tau + \gamma a2, \tag{6.5}$$

where we put the coordinate center at the point a_0. Hence, after integration,

$$x = \alpha\tau a + \frac{\gamma a3}{3},$$

we get another, than in a-space, spatial similarity: $x \sim \tau^{3/2}$, that gives respectively $J = \tau f(x \cdot \tau^{-3/2})$.

¿From this consideration it becomes evident that for the 3D case in **r**-space we should expect a strong anisotropy connected with different behavior of eigen-values of the Jacoby matrix. Vanishing J implies that an eigenvalue of the Jacoby matrix (say, λ_1) vanishes, and generically the other two eigenvalues (λ_2 and λ_3) are non-zero. Therefore, there exist one "soft" direction associated with λ_1, and two "hard" directions associated with λ_2 and λ_3. Along the soft direction contraction has to be familiar to that in 1D case, i.e. $X_1 \sim \tau^{3/2}$. Compression along two hard directions is the same as in the auxiliary **a**-space:

$X_{2,3} \sim \tau^{1/2}$. The different contraction along soft and hard directions leads to formation of pancake structure for the vorticity maximum:

$$\mathbf{\Omega} = \tau^{-1}\mathbf{G}(\zeta_1, \zeta_2, \zeta_3) \tag{6.6}$$

where $\zeta_1 = X_1/\tau^{3/2}$, $\zeta_{2,3} = X_{2,3}/\tau^{1/2}$ are self-similar variables. As $t \to t_0$ the vorticity ω becomes to lie in the plane of the pancake.

7 Numerical experiment

Considering the hydrodynamic model with the Hamiltonian $\mathcal{H} = \int |\mathbf{\Omega}| d\mathbf{r}$, we have arrived at the conclusion that each vortex line in the given system moves independently of other lines. Just this property makes possible to form singularity in a finite time for the generalized vorticity $\mathbf{\Omega}(\mathbf{r}, t)$ starting from smooth initial data. A typical singularity of this kind looks like an infinite condensation of the vortex lines near some point. Thus, the collapse in the integrable hydrodynamics has purely inertial origin. If one will assume that this type of collapse is possible also in the Euler hydrodynamics, then the asymptotics of the vorticity near the singular point will be given by (6.4) in a non-degenerated situation and the curl of the velocity will blow up as $(t_0 - t)^{-1}$. Exactly this dependence for vorticity near a singular point has been observed practically in all numerical simulations of the Euler equation, including the above cited. Concerning the spatial structure of the collapsing domain, only a qualitative agreement takes place. The numerical results [4] for short-time dynamics showed for the initial conditions in the form of the Taylor-Green vortex and for random initial conditions formation of thin vortex layers with high vorticity that support our predictions. The results of Kerr [8], as well as his next paper [9] seem to support our theory also. In particular, the analysis of numerical data [9] gave two distinguished scales, one scale being contracted as the square root: $l_1 \sim (t_0 - t)^{1/2}$, and another as the first power of time: $l_2 \sim t_0 - t$. In the work of Grauer, Marliani and Germaschewsky [6], the successful attempt was undertaken to observe collapse for the initial condition not possessing a low symmetry. The initial vorticity was concentrated in the vicinity of a cylinder, and was modulated over the angle in such a way so that the simplest symmetries were absent. In the present time this experiment has the best spatio-temporal resolution. In this simulation appearance of a separate collapsing region was observed with the vorticity growth at the maximum as $(t_0 - t)^{-1}$.

However, from all these simulations based on direct integration of the Euler equations it is impossible to make a conclusion whether this blowup process can be regarded to vortex line breaking. This flaw has been supplied in Refs [28, 14] where numerical solving have been performed within the partially integrated Euler equations resolved with respect to the Cauchy invariants. In spite of a lack of spatial resolution (1283 grids) it was shown that collapse took place due to vanishing of the Jacobian at one separate point. The Jacobian at the collapse point with a good accuracy vanished *linearly* in time as t tended to t_0. While approaching the collapse time, the coefficients $\gamma_{\alpha\beta}$ at the minimum of J practically did not vary in time. Thus, these results can be interpreted as the evidence of the vortex line breaking which was not related to any symmetry of the initial vorticity distribution and occurred at a single point.

8 Concluding remarks

In this brief review we have presented a new sight to the collapse problem in hydrodynamics based on the vortex line representation. Using this approach we have demonstrated on the example of three-dimensional integrable hydrodynamics formation of singularity in a finite time due to breaking of vortex lines. Although this mechanism has inertial origin in this model - all vortex lines turn out free objects, it happens in spite of incompressibility of both velocity and vorticity fields. The main reason of appearance of singularity is connected with compressibility of the VLR, as the mapping from the Eulerian to the mixed Lagrangian-Eulerian variables. Now analogs of the VLR have been found for many incompressible hydrodynamic systems, including 2D ideal hydrodynamics [16, 17], MHD [22, 12, 16] and EMHD [23], and for viscous fluids also [13]. For all these systems there exist one or two frozen-in fields, so that their equations of motion has the form (2.5). The corresponding field changes due to the *normal* component velocity only, the velocity component parallel to the field plays a passive role providing incompressibility of the flow. Thus, divergence of the normal velocity is not equal zero and just this is the origin of compressibility of mappings and possibility of the frozen-in field contraction which may lead to collapse in such systems.

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