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The Generalized Version of Dressing Method with Applications to AKNS Equations

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Abstract

The generalized dressing method is extended to variable-coefficient AKNS equations, including a variable-coefficient coupled nonlinear Schrödinger equation and a variable-coefficient coupled mKdV equation. A general variable-coefficient KP equation is proposed and decomposed into the two 1+1 dimensional variable-coefficient soliton equations. As applications, we obtain exact solutions of these variable-coefficient soliton equations in 1+1- and 2+1- dimensions.

1 Introduction

The dressing method [1-5] is a powerful tool in the investigation of soliton equations, which can not only derive the Lax pairs of soliton systems, but also present their explicit solutions. Since its discovery this method has been applied to study a great variety of nonlinear evolution equations which arise in various fields of physics [6-11]. In refs. [12,13], a generalized version of this method has been developed to solve a more general case such as the variable coefficient KdV equation.

The AKNS hierarchy [14-16] is the isospectral class of the Zhakharov-Shabat (ZS) eigenvalue problem. A number of research on the hierarchy has been conducted. For example, its inverse scattering transformation, Bäcklund transformation, Darboux transformation, conserved quantities, and others have been discussed in refs [17-24]. In this paper, we shall extended the generalized dressing method to variable-coefficient AKNS equations, which include a variable-coefficient coupled nonlinear Schrödinger (NSL) equation

$$\begin{aligned} u_{t_2} &= \alpha(u_{xx} - 2u^2 v) + \rho_1 (xu)_x + 2\rho_0 xu, \\ v_{t_2} &= \alpha(-v_{xx} + 2uv^2) + \rho_1 (xv)_x - 2\rho_0 xv, \end{aligned} \tag{1.1}$$

and a variable-coefficient coupled mKdV equation

$$\begin{aligned} u_{t_3} &= \beta(u_{xxx} - 6uu_xv) + \rho_1(xu)_x + 2\rho_0 xu, \\ v_{t_3} &= \beta(v_{xxx} - 6uvv_x) + \rho_1(xv)_x - 2\rho_0 xv. \end{aligned} \tag{1.2}$$

These two coupled equations are the second and third members of the nonisospectral AKNS hierarchy, respectively. We propose a general variable-coefficient KP equation

$$\begin{aligned} w_{tx} &= \frac{\beta}{4}(w_{xxx} + 6ww_x)_x + \frac{3}{4}\gamma w_{yy} + (\frac{3}{4}\gamma\rho_1^2 x^2 + \rho_1 x)w_{xx} - \frac{3}{2}\gamma\rho_1 x w_{xy} \\ &\quad + (\frac{21}{4}\gamma\rho_1^2 x + 3\rho_1 - 3\alpha\gamma\rho_0)w_x - \frac{9}{2}\gamma\rho_1 w_y + 6\gamma\rho_1^2 w, \end{aligned} \tag{1.3}$$

which is a generalization of the KP equation [25], where $\rho_0 = \rho_0(t), \rho_1 = \rho_1(t), \gamma = \beta/\alpha^2$ and α, β are arbitrary constants. Based on the idea in [26, 27], the variable-coefficient KP equation (1.3) is decomposed into the variable-coefficient coupled NSL equation (1.1) and the variable-coefficient coupled mKdV equation (1.2). With the aid of the above results and the decomposition, we obtain exact solutions of these variable-coefficient solition equations.

The present paper is organized as follows. In section 2, we briefly describe the generalized version of dressing method. In section 3, the generalized dressing method is utilized to discuss the variable-coefficient coupled NLS equation and the variable-coefficient coupled mKdV equation. In section 4, the variable-coefficient KP equation is decomposed into the variable-coefficient coupled NSL equation and the variable-coefficient coupled mKdV equation. As applications, we obtain exact solutions of these variable-coefficient soliton equations in 1+1- and 2+1- dimensions

2 A generalized version of the dressing method

For the sake of convenience, we introduce the upper and lower Volterra operators $\mathbf{K}_\pm$ and the integral operator $\mathbf{F}$ by

$$\begin{aligned} \mathbf{K}_+\psi(x) &\equiv \int_x^\infty K_+(x,z)\psi(z)dz, \\ \mathbf{K}_-\psi(x) &\equiv \int_{-\infty}^x K_-(x,z)\psi(z)dz, \\ \mathbf{F}\psi(x) &\equiv \int_{-\infty}^\infty F(x,z)\psi(z)dz, \end{aligned} \tag{2.1}$$

where $K_\pm(x,z)$ and $F(x,z)$ are $n \times n$ matrices and depend on the variables t_m, $\psi(x)$ is any $n \times 1$ matrix. For the kernels $K_\pm(x,z)$, the property is required that $K_+(x,z) = 0$ for $z < x$ and $K_-(x,z) = 0$ for $z > x$. Assume that $(\mathbf{I} + \mathbf{K}_+)^{-1}$ exits and $\mathbf{F}$ admits the triangular factorization

$$\mathbf{I} + \mathbf{F} = (\mathbf{I} + \mathbf{K}_+)^{-1}(\mathbf{I} + \mathbf{K}_-),$$

where $\mathbf{I}$ is the identity operator. We say $\frac{\partial^{m+n}}{\partial x^m \partial z^n}K(x,z) \to 0$ and $\frac{\partial^{m+n}}{\partial x^m \partial z^n}F(x,z) \to 0$, as $z \to \pm\infty (m, n = 0, 1, 2, \cdots)$. In addition, for all $x_0 > -\infty$, we assume that

$$\sup \int_{x_0}^{\infty} |K_{\pm}(x,z)|\psi(z)dz < \infty, \quad \sup \int_{x_0}^{\infty} |F(x,z)|\psi(z)dz < \infty.$$

From the above triangular factorization, we can get the Gel'fand-Levitan equation

$$K_+(x,z) + F(x,z) + \int_x^{\infty} K_+(x,s)F(s,z)ds = 0, \qquad z > x, \tag{2.2}$$

and

$$K_-(x,z) = F(x,z) + \int_x^{\infty} K_+(x,s)F(s,z)ds, \qquad z < x. \tag{2.3}$$

from which $K_-(x,z)$ is defined in terms of $K_+(x,z)$ and $F(x,z)$. Therefore, in the later of paper, we will only consider the case of $K_+(x,z)$ and omit the subscript "+".

We now consider a pair of "bare" operators $\mathbf{M}_1$ and $\mathbf{M}_2$, which have the relation

$$[\mathbf{M}_1, \mathbf{M}_2] = c_1\mathbf{M}_1 + c_2\mathbf{M}_2, \tag{2.4}$$

where $c_1 \equiv c_1(y,t), c_2 \equiv c_2(y,t)$ are functions. According to Refs.[12,13], we introduce the transformation from $\mathbf{M}_j$ to $\widetilde{\mathbf{M}}_j$ by

$$\widetilde{\mathbf{M}}_j(\mathbf{I} + \mathbf{K}) - (\mathbf{I} + \mathbf{K})\mathbf{M}_j = 0, \quad j = 1, 2. \tag{2.5}$$

This implies

$$[\widetilde{\mathbf{M}}_\mathbf{1}, \widetilde{\mathbf{M}}_\mathbf{2}] = c_1\widetilde{\mathbf{M}}_\mathbf{1} + c_2\widetilde{\mathbf{M}}_\mathbf{2}, \tag{2.6}$$

which provides the nonlinear evolution equations with respect to the variables K_{jl}. Notice that the commuting relation

$$[\mathbf{M}_j, \mathbf{F}] = 0, \quad j = 1, 2, \tag{2.7}$$

which determines the matrix $F(x,z)$. Then explicit solutions for the variables K_{jl} can be obtained by solving the Gel'fand-Levitan equation (2.2).

Suppose that (2.7) has solutions in the form of separation of variables [6,8,28]

$$F(x,z) = \sum_{j=1}^{N} f_j(x)g_j(z), \tag{2.8}$$

where f_j, g_j are some $n \times n$ matrices. On the other hand, we assume

$$K(x,z) = \sum_{j=1}^{N} k_j(x)g_j(z). \tag{2.9}$$

Inserting (2.8) and (2.9) into (2.2) yields

$$K(x,x) = \sum_{j=1}^{N} k_j(x)g_j(x) = -(f_1, \cdots, f_N)L^{-1}\begin{pmatrix} g_1 \\ \vdots \\ g_N \end{pmatrix}, \tag{2.10}$$

where the block of L is given by

$$L_{jl} \equiv \delta_{jl} + \int_x^\infty g_j(s) f_l(s) ds, \quad 1 \le j,\ l \le N,$$

and δ_{jl} is Kronecker's delta. In the case of $N = 1$ and let

$$F(x, z) = f(x)g(z) = \begin{pmatrix} 0 & f_{12}(x) \\ f_{21}(x) & 0 \end{pmatrix} \begin{pmatrix} g_{11}(z) & 0 \\ 0 & g_{22}(z) \end{pmatrix}. \tag{2.11}$$

Then we have from (2.10) that

$$\begin{aligned} K(x, x) = {} & [1 - \int_x^\infty f_{12}(s) g_{11}(s) ds \int_x^\infty f_{21}(s) g_{22}(s) ds]^{-1} \\ & \times \begin{pmatrix} f_{12}(x) g_{11}(x) \int_x^\infty f_{21}(s) g_{22}(s) ds & -f_{12}(x) g_{22}(x) \\ -f_{21}(x) g_{11}(x) & f_{21}(x) g_{22}(x) \int_x^\infty f_{12}(s) g_{11}(s) ds \end{pmatrix}. \end{aligned} \tag{2.12}$$

3 The variable-coefficient AKNS equations

In this section we shall discuss the variable-coefficient AKNS equations, including the variable-coefficient coupled NLS equation and variable-coefficient coupled mKdV equation, with the help of generalized dressing method. We consider the nonisospectral Zakharov-Shabat eigenvalue problem [29]

$$\psi_x = \begin{pmatrix} \lambda & -u \\ -v & -\lambda \end{pmatrix} \psi, \tag{3.1}$$

which is written as

$$\{\sigma_3 \partial_x + \begin{pmatrix} 0 & u \\ -v & 0 \end{pmatrix}\} \psi = \lambda \psi, \tag{3.2}$$

where $\partial_z^n = \frac{\partial^n}{\partial z^n}, n \in \mathbf{N}$. Utilizing (3.2), we introduce two differential operators $\mathbf{M}_1$ and $\mathbf{M}_m$ by

$$\mathbf{M}_1 \equiv \sigma_3 \partial_x + a, \quad \mathbf{M}_m \equiv \partial_{t_m} + B \partial_x^m - \rho_1 x \partial_x - \rho_0 x \sigma_3, \tag{3.3}$$

where $a = a(t_m), \rho_i = \rho_i(t), i = 0, 1$, are functions, and B is a constant diagonal matrix. A direct calculation shows that the relation $[\mathbf{M}_1, \mathbf{M}_m] = -\rho_1 \mathbf{M}_1$ holds if $a_{t_m} = \rho_1 a - \rho_0$. In the following, we give the first two non-trivial evolution equations of the variable-coefficient AKNS hierarchy by using the above scheme. Firstly, we consider the case $m = 2, B = -2\alpha\sigma_3$ in (3.3). Using (2.5), we have

$$\begin{aligned} \widetilde{\mathbf{M}}_1 &= \mathbf{M}_1 + \begin{pmatrix} 0 & u \\ -v & 0 \end{pmatrix}, \\ \widetilde{\mathbf{M}}_2 &= \mathbf{M}_2 - 2\alpha \begin{pmatrix} 0 & u \\ -v & 0 \end{pmatrix} \partial_x - \alpha \begin{pmatrix} -uv & u_x \\ -v_x & uv \end{pmatrix}, \\ \widehat{K} &\equiv [K(x, z)]_{z=x} = (\hat{k}_{ij}), \end{aligned} \tag{3.4}$$

where $\hat{k}_{12} = \frac{1}{2}u$, $\quad \hat{k}_{21} = \frac{1}{2}v$, $\quad \hat{k}_{11x} = \hat{k}_{22x} = -\frac{1}{2}uv$, with the relation

$$[k_{12x} - k_{12z}]_{z=x} = -u\hat{k}_{22}, \quad [k_{21x} - k_{21z}]_{z=x} = -v\hat{k}_{11}.$$

We will find in the later that $\hat{k}_{11x}$ has a special meaning. The relation $[\widetilde{\mathbf{M}}_1, \widetilde{\mathbf{M}}_2] = -\rho_1\widetilde{\mathbf{M}}_1$ yields the variable-coefficient coupled NLS equation

$$\begin{aligned} u_{t_2} &= \alpha(u_{xx} - 2u^2v) + \rho_1(xu)_x + 2\rho_0 xu, \\ v_{t_2} &= \alpha(-v_{xx} + 2uv^2) + \rho_1(xv)_x - 2\rho_0 xv. \end{aligned} \tag{3.5}$$

In the following, we consider another case $m = 3$ and $B = -4\beta$ in eq.(3.3). With the same procedure, we can get

$$\begin{aligned} \widetilde{\mathbf{M}}_1 &= \mathbf{M}_1 + \begin{pmatrix} 0 & u \\ -v & 0 \end{pmatrix}, \\ \widetilde{\mathbf{M}}_3 &= \mathbf{M}_3 - 6\beta \begin{pmatrix} -uv & u_x \\ v_x & -uv \end{pmatrix} \partial_x \\ &\quad -\beta \begin{pmatrix} u_xv - 2uv_x - 6uv\hat{k}_{11} + 4a_1 & 3u_{xx} \\ 3v_{xx} & uv_x - 2u_xv - 6uv\hat{k}_{22} + 4a_2 \end{pmatrix}. \end{aligned} \tag{3.6}$$

where $a_i = \frac{d}{dx}([2k_{iix} - k_{iiz}]_{z=x})$, and the evolution equations

$$\begin{aligned} & u_{t_3} - \beta(u_{xxx} - 3uu_xv - 3u^2v_x + 6(\hat{k}_{22} - \hat{k}_{11})u^2v - 4(a_2 - a_1)u) - (\rho_1 xu)_x - 2\rho_0 xu = 0, \\ & v_{t_3} - \beta(v_{xxx} - 3uvv_x - 3u_xv^2 - 6(\hat{k}_{22} - \hat{k}_{11})uv^2 + 4(a_2 - a_1)v) - (\rho_1 xv)_x + 2\rho_0 xv = 0, \\ & 4u_{xx}v + 5u_xv_x + uv_{xx} - 6(uv)_x\hat{k}_{11} + 3(uv)^2 + 4a_{1x} = 0, \\ & -u_{xx}v - 5u_xv_x - 4uv_{xx} + 6(uv)_x\hat{k}_{22} - 3(uv)^2 - 4a_{2x} = 0. \end{aligned} \tag{3.7}$$

The equation (3.7) imply the variable-coefficient coupled mKdV equation

$$\begin{aligned} u_{t_3} &= \beta(u_{xxx} - 6uu_xv) + \rho_1(xu)_x + 2\rho_0 xu, \\ v_{t_3} &= \beta(v_{xxx} - 6uvv_x) + \rho_1(xv)_x - 2\rho_0 xv, \end{aligned} \tag{3.8}$$

If we set $u = v, \rho_0 = 0$, then (3.8) reduces to the variable-coefficient mKdV equation

$$u_{t_3} = \beta(u_{xxx} - 6u^2u_x) + \rho_1(xu)_x, \tag{3.9}$$

which is the generalization of mKdV equation [30].

4 Applications

In this section, we shall construct exact solutions of variable-coefficient coupled NLS equation (3.5), variable-coefficient coupled mKdV equation (3.8) and the variable-coefficient KP equation (1.1) by means of the generalized dressing method. To this end, we first prove the following proposition.

Proposition 1. *Let $u(x,y,t), v(u,y,t)$ be a compatible solution of the variable-coefficient coupled NLS equation (3.5) and the variable-coefficient coupled mKdV equation (3.8), $(t_2 = y, t_3 = t)$. Then*

$$w(x,y,t) = -2u(x,y,t)v(x,y,t) \tag{4.1}$$

solves the variable-coefficient KP equation

$$\begin{aligned} w_{tx} = {} & \tfrac{\beta}{4}(w_{xxx} + 6ww_x)_x + \tfrac{3}{4}\gamma w_{yy} + (\tfrac{3}{4}\gamma\rho_1^2 x^2 + \rho_1 x)w_{xx} - \tfrac{3}{2}\gamma\rho_1 x w_{xy} \\ & +(\tfrac{21}{4}\gamma\rho_1^2 x + 3\rho_1 - 3\alpha\gamma\rho_0)w_x - \tfrac{9}{2}\gamma\rho_1 w_y + 6\gamma\rho_1^2 w, \end{aligned} \tag{4.2}$$

where $\gamma = \beta/\alpha^2$.

Proof. A direct calculation gives

$$w_{xt} = \beta w_{xxxx} + \frac{3}{2}\beta(w^2)_{xx} + \rho_1 x w_{xx} + 3\rho_1 w_x + 6\beta(u_{xxx}v_x + 2u_{xx}v_{xx} + u_x v_{xxx}), \tag{4.3}$$

$$\begin{aligned} \frac{3}{4}w_{yy} = {} & \frac{3}{4}\alpha^2 w_{xxxx} + \frac{3}{4}\alpha^2(w^2)_{xx} - \frac{3}{4}\rho_1^2 x^2 w_{xx} + \frac{3}{2}\rho_1 x w_{xy} + (3\alpha\rho_0 - \frac{21}{4}\rho_1^2 x)w_x + \frac{9}{2}\rho_1 w_y \\ & - 6\rho_1^2 w + 6\alpha^2(u_{xxx}v_x + 2u_{xx}v_{xx} + u_x v_{xxx}), \end{aligned} \tag{4.4}$$

which imply the variable-coefficient KP equation (4.2) by combining the above expressions. ■

According to (2.7), we have the evolution equations of integral kernel F

$$\begin{aligned} & \sigma_3 F_x + F_z\sigma_3 = 0, \\ & F_y - 2\alpha(\sigma_3 F_{xx} - F_{zz}\sigma_3) - \rho_1(xF_x + zF_z) - \rho_1 F + \rho_0(x\sigma_3 F - zF\sigma_3) = 0 \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} & \sigma_3 F_x + F_z\sigma_3 = 0, \\ & F_t - 4\beta(F_{xxx} + F_{zzz}) - \rho_1(xF_x + zF_z) - \rho_1 F + \rho_0(x\sigma_3 F - zF\sigma_3) = 0. \end{aligned} \tag{4.6}$$

If $F(x,z)$ has the form of separation of variables

$$F(x,z) = \sum_{k=1}^{N} f_k(x)g_k(z) \tag{4.7}$$

$$= \sum_{k=1}^{N}\begin{pmatrix} 0 & \sqrt{m_k^1}e^{-m_k^2 x} \\ \sqrt{n_k^1}e^{-n_k^2 x} & 0 \end{pmatrix}\begin{pmatrix} \sqrt{n_k^1}e^{-n_k^2 z} & 0 \\ 0 & \sqrt{m_k^1}e^{-m_k^2 z} \end{pmatrix}$$

$$= \sum_{k=1}^{N}\begin{pmatrix} 0 & m_k^1 e^{-m_k^2(x+z)} \\ n_k^1 e^{-n_k^2(x+z)} & 0 \end{pmatrix}, \tag{4.8}$$

where $m_k^i = m_k^i(y,t), n_k^i = n_k^i(y,t), (i = 1,2)$, substituting (4.7) into (4.5) yields

$$m_k^2(y) = \exp(\int \rho_1 dy)(\eta_k + \int \rho_0 \exp(-\int \rho_1 dy)dy), \quad m_k^1(y) = \exp(\int 4\alpha(m_k^2)^2 + \rho_1 dy),$$

(4.9)

and

$$n_k^2(y)=\exp(\int\rho_1dy)(\xi_k-\int\rho_0\exp(-\int\rho_1dy)dy),\quad n_k^1(y)=\exp(-\int 4\alpha(n_k^2)^2+\rho_1dy). \tag{4.10}$$

From (4.6), it is easy to verify that m_k^i and n_k^i have the form

$$m_k^2(t)=\exp(\int\rho_1dt)(\eta_k+\int\rho_0\exp(-\int\rho_1dt)dt),\quad m_k^1(t)=\exp(\int 8\beta(m_k^2)^3+\rho_1dt), \tag{4.11}$$

and

$$n_k^2(t)=\exp(\int\rho_1dt)(\xi_k-\int\rho_0\exp(-\int\rho_1dt)dt),\quad n_k^1(t)=\exp(\int 8\beta(n_k^2)^3+\rho_1dt), \tag{4.12}$$

where ξ_k and η_k are arbitrary positive constants.

From (2.8) to (2.10), we know that $u(x,y,t)$ and $v(x,y,t)$ are the compatible solutions of (3.5) and (3.8), which implies that (4.5) and (4.6) have the compatible solutions. Set ρ_1 and ρ_2 are constants. Then we can obtain a compatible solution $F(x,z,y,t)$, where m_k^i, n_k^i are determined by

$$\begin{aligned} &m_k^2(y,t)=\eta_k\exp(\rho_1(y+t))-2\rho_0/\rho_1,\\ &m_k^1(y,t)=\exp(\int 4\alpha(m_k^2)^2dy+\int 8\beta(m_k^2)^3dt+\rho_1(y+t)), \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} &n_k^2(y,t)=\xi_k\exp(\rho_1(y+t))+2\rho_0/\rho_1,\\ &n_k^1(y,t)=\exp(-\int 4\alpha(n_k^2)^2dy+\int 8\beta(n_k^2)^3dt+\rho_1(y+t)). \end{aligned} \tag{4.14}$$

Furthermore, the solution $w(x,y,t)$ of KP equation (4.2) is exactly $4\hat{k}_{11x}$, where explicit form of $\widehat{K}$ is given by (2.10). If $N=1$ and omit the subscripts of m_1^i, n_1^i, by virtue of (2.12), we have

$$K(x,x)=[1-\frac{m^1n^1}{(m^2+n^2)^2}e^{-2(m^2+n^2)x}]^{-1}\begin{pmatrix}\frac{m^1n^1}{m^2+n^2}e^{-2(m^2+n^2)x} & -m^1e^{-2m^2x}\\ -n^1e^{-2n^2x} & \frac{m^1n^1}{m^2+n^2}e^{-2(m^2+n^2)x}\end{pmatrix}. \tag{4.15}$$

Then compatible solutions of the variable-coefficient coupled NLS equation (3.5) and coupled mKdV equation (3.8) are as follows

$$\begin{aligned} &u=-2m^1e^{-2m^2x}[1-\frac{m^1n^1}{(m^2+n^2)^2}e^{-2(m^2+n^2)x}]^{-1},\\ &v=-2n^1e^{-2n^2x}[1-\frac{m^1n^1}{(m^2+n^2)^2}e^{-2(m^2+n^2)x}]^{-1}. \end{aligned} \tag{4.16}$$

Noticing the proposition 1, we arrive at a line one-soliton solution of the variable-coefficient KP equation (4.2)

$$w = 2\partial_x^2 \ln[1 - \frac{m^1 n^1}{(m^2+n^2)^2} e^{-2(m^2+n^2)x}]. \tag{4.17}$$

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