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A Note on Surface Profiles for Symmetric Gravity Waves with Vorticity

Mats EHRNSTRÖM

Department of Mathematics, Lund University, PO Box 118, 221 00 Lund, Sweden
E-mail: mats.ehrnstrom@math.lu.se

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Abstract

We consider a nontrivial symmetric periodic gravity wave on a current with nondecreasing vorticity. It is shown that if the surface profile is monotone between trough and crest, it is in fact strictly monotone. The result is valid for both finite and infinite depth.

1 Introduction

The mathematical study of water waves is to a large extent concerned with studying steady wave trains travelling at the surface of the open sea (see the review [11]). Two-dimensional periodic waves propagating on currents with vorticity have been investigated in [2, 3, 4, 5, 6, 8, 12, 17]. Recent results indicate conditions under which such waves are symmetric [3, 4]: monotonicity of the surface profile between trough and crest guarantees symmetry around the crest for steady periodic gravity waves on a current with vorticity decreasing with greater depth. It will be shown here that in such cases the surface profile is strictly increasing from trough to crest, unless the wave has a trivial flat surface. The proof uses sharp maximum principles for elliptic PDE's and relies on the behaviour of the vertical velocity component for symmetric gravity waves.

2 Mathematical formulation of the problem

2.1 Preliminaries

We consider a steady surface wave travelling at speed $c > 0$ across the sea in one fixed direction. The wave is assumed to be periodic with period $L > 0$. Since the wave is supposed to be identical in the direction perpendicular to the propagation direction it is enough to study a cross-section of the fluid domain. We thus fix a Cartesian coordinate system where x denotes the direction of propagation and y is the vertical direction pointing from the bottom to the surface. We define the surface to be $y = \eta(t, x)$ and for a fixed $t = t_0$ we understand the fluid domain Ω_η to be

$$\Omega_\eta = \{(x, y)\,;\, x \in \mathbb{R},\ -d < y < \eta(t_0, x)\}.$$

Here $y = -d$ is the bottom and we accept also $d = \infty$. $y = 0$ is the mean water level, i.e. $\int_0^L \eta(t,x)\,dx = 0$ for all $t \geq 0$.

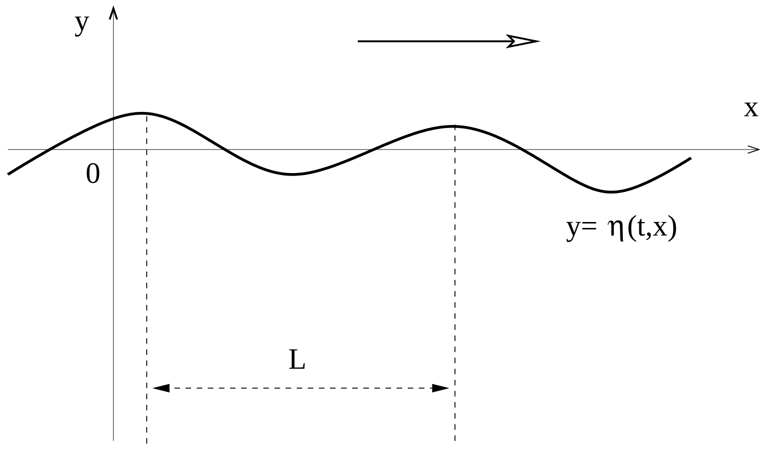


Figure 1.

Under the assumption of a non-viscid fluid, appropriate for water, the equation of motion is the Euler equation

$$\begin{cases} u_t + uu_x + vu_y &= -P_x \\ v_t + uv_x + vv_y &= -P_y - g \end{cases} \tag{2.1}$$

valid within the fluid domain Ω_η for any $t \geq 0$. Here, $(P,u,v) \in C^1(\overline{\Omega_\eta}) \times C^2(\overline{\Omega_\eta}) \times C^2(\overline{\Omega_\eta})$ is required.

For water, density changes very little with depth, and homogeneity is a good approximation [7, 13] implying the equation of mass conservation

$$u_x + v_y = 0 \tag{2.2}$$

(i.e. water is incompressible). We neglect the contribution of surface tension - as is suitable for gravity waves - and postulate that i) the water's free surface is impermeable in the sense that the same particles constitute the free surface $y = \eta(t,x)$ for $t \geq 0$, and ii) the pressure equals the constant atmospheric pressure on the free surface. This gives the boundary conditions

$$\left.\begin{array}{l} P = P_0 \\ v = \eta_t + \eta_x u \end{array}\right\} \quad \text{on the free surface } y = \eta(x). \tag{2.3}$$

In the case of infinite depth (deep water) field data show that the motion vanishes with great depths [10, 14]. In the case of a flat bed $y = -d \in \mathbb{R}$, the impermeability of the bed translates into $v = 0$ on the bottom $y = -d > -\infty$. In both cases we have that

$$v \to 0 \quad \text{as } y \to -d \quad \text{uniformly for } x \in \mathbb{R}. \tag{2.4}$$

Two auxilliary assumptions are suitable in the present context. First, measurements show that for a wave not near breaking or spilling, the velocity of a single particle is considerably smaller than the velocity of the wave itself [13]. We therefore assume

$$u < c \quad \text{in } \overline{\Omega}_\eta. \tag{2.5}$$

Second, the open sea is dominated by deep water waves, and the primary source of ocean currents are long duration winds [11]. Such a current's vorticity distribution is mostly confined to a near-surface layer, and it takes time for the current to reach great depths [11]. It is therefore reasonable to assume that

$$\partial_y \omega \geq 0 \quad \text{in } \overline{\Omega}_\eta \tag{2.6}$$

where $\omega(t,x,y)$ is the vorticity $\omega = v_x - u_y$. (2.1)-(2.6) then constitute our mathematical setting for the water-wave problem.

2.2 Reformulation

In order to handle the problem, a reformulation introduced in [6] is convenvient. Define a stream function $\psi(x,y)$ by

$$\psi_x = -v, \quad \psi_y = u - c.$$

Then $\psi_y < 0$ by (2.5) and the stream function can be explicitly calculated up to a constant $\psi_0 \in \mathbb{R}$:

$$\psi(x,y) = \psi_0 + \int_{-d}^{y} [u(x,\xi) - c]\, d\xi - \int_0^x v(\xi,-d)\, d\xi,$$

where $y \leq \eta(x)$ and d is chosen so that the line $y = -d$ lies totally within the fluid. Note that $\psi \in C^3(\overline{\Omega_\eta})$ and that $\frac{d}{dx}[\psi(x+L,y) - \psi(x,y)] = 0$. As can be seen from the explicit formula this expression is independent also of y so that the asymptotic behaviour of v yields that ψ is periodic in the x-variable.

The wave moving with constant speed c means that for u, v and ψ a change in the x-variable corresponds to one in the t-variable, according to the equation $ct = x$. The map $(x - ct, y) \mapsto (x,y)$ therefore eliminates time from our governing equations. Indeed, the problem reduces to

$$\begin{cases} \psi_y\psi_{xy} - \psi_x\psi_{yy} &= -P_x & \text{in } \Omega_\eta, \\ -\psi_y\psi_{xx} + \psi_x\psi_{xy} &= -P_y - g & \text{in } \Omega_\eta, \\ P &= P_0 & \text{on } y = \eta(x), \\ \psi_x &= -\eta_x\psi_y & \text{on } y = \eta(x), \\ \nabla\psi &\to (0,-c) & \text{as } y \to -d \text{ uniformly for } x \in \mathbb{R}. \end{cases}$$

where the L-periodic function (η, P, ψ) is in $C^3(\mathbb{R}) \times C^1(\overline{\Omega_\eta}) \times C^3(\overline{\Omega_\eta})$. By the boundary condition $\psi_x = -\eta_x\psi_y$ on $y = \eta(x)$, we have $\frac{d}{dx}\psi(x,\eta(x)) = 0$. For convenience we can therefore choose ψ_0 so that $\psi \equiv 0$ on the surface $y = \eta(x)$.

We now introduce a vorticity function. The vorticity of the flow is given by

$$\omega(x,y) = v_x(x,y) - u_y(x,y) \in C^1(\overline{\Omega_\eta})$$

and it is immediate that $\Delta\psi = -\omega$ for $y < \eta(x)$. For a fixed x_0, $\psi_y = u - c < 0$ implies that $y_{x_0} \leftrightarrow \psi_{x_0}$ is a bijection. Yet another transformation $(x, y) \mapsto (q, p)$ defined by $q = x, p = -\psi$ then maps Ω_η into the half-plane $\{(q, p); q \in \mathbb{R}, p \leq 0\}$. Then

$$\begin{cases} \frac{\partial\psi}{\partial y} & = & -\frac{\partial p}{\partial y} \\ \frac{\partial}{\partial x} & = & \frac{\partial}{\partial q} - \frac{\partial}{\partial p}\frac{\partial\psi}{\partial x} \end{cases}$$

Using this, we find that $\frac{\partial\omega}{\partial q} = (\frac{\partial}{\partial x} - v\frac{\partial}{\partial p})\omega = (\frac{\partial}{\partial x} - \frac{v}{c-u}\frac{\partial}{\partial y})\omega$. Applying $(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x})$ to the Euler equation we find that $\omega_q \equiv 0$ (see [6] for details). Hence $\omega = \omega(p) = \gamma(\psi)$ is independent of q. It follows that $\gamma : \mathbb{R}^+ \to \mathbb{R}$ is continuously differentiable. The function γ was termed vorticity function in [6].

The transformed Euler equation (2.1) also implies that $E = \frac{\psi_x^2+\psi_y^2}{2} + gy + P + \int_0^\psi \gamma(\xi)\, d\xi$ is constant throughout the fluid domain Ω_η (this is Bernoulli's law). Since $P = P_0$ and $\psi = 0$ are constant on the free surface $y = \eta(x)$ it is clear that $|\nabla\psi|^2 + 2gy = 2(E - P_0)$ is constant on the surface. All together, we have the free boundary problem

$$\begin{cases} \Delta\psi & = & -\gamma(\psi) & \text{in} & -\infty < y < \eta(x), \\ |\nabla\psi|^2 + 2gy & = & C & \text{on} & y = \eta(x), \\ \psi & = & 0 & \text{on} & y = \eta(x), \\ \nabla\psi & \to & (0, -c) & \text{as} & y \to -d \quad \text{uniformly for } x \in \mathbb{R}, \end{cases} \tag{2.7}$$

to be satisfied for $\eta \in C^3(\mathbb{R})$ and $\psi \in C^3(\overline{D}_\eta)$, both L-periodic in the x-variable.

3 Main Result

Recently Constantin and Escher proved the following result, valid for both finite and infinite depth [3, 4]:

Theorem 1. *A steady periodic water-wave with a monotone profile between crests and troughs, propagating against a current with a vorticity that is non-decreasing with depth and has bounded first-order partial derivatives, must be symmetric.*

In this section it is proved that unless such a wave is trivial, its surface profile is in fact strictly monotone between crest and trough. We will need the following two lemmas:

Lemma 1 (The strong maximum principle). *Let Ω be a connected open set in $\mathbb{R}^2$, and let $\mathcal{L} = \Delta + c(x, y)$, where $c : \overline{\Omega} \to \mathbb{R}$ is continuous and $c \leq 0$. Also, let $u \in C^2(\overline{\Omega})$. Suppose $\mathcal{L}u \geq 0$ in Ω and $\sup_\Omega u \geq 0$. Then, if u attains its maximum at an inner point of Ω, u is necessarily constant in Ω.*

Lemma 2 (The Serrin corner point lemma). *Let $\Omega, \mathcal{L}, c$ and u be as above. Suppose that Ω is bounded and let T be the line containing the outward normal n at a point $p \in \partial\Omega$. Moreover, assume that the boundary $\partial\Omega$ is C^1 near the point p. Then, if $u(x) < u(p)$ for all x lying on one side of T in Ω with $u(p) \geq 0$, either $\frac{\partial u}{\partial m}(p) > 0$ or $\frac{\partial^2 u}{\partial m^2}(p) < 0$ for any vector m pointing outwards from the part of Ω lying on the same side of T.*

For a detailed discussion of these principles, see e.g. [9, 15]. We will apply them for $c(x,y) = \gamma'(\psi(x,y))$. Our main result is the following.

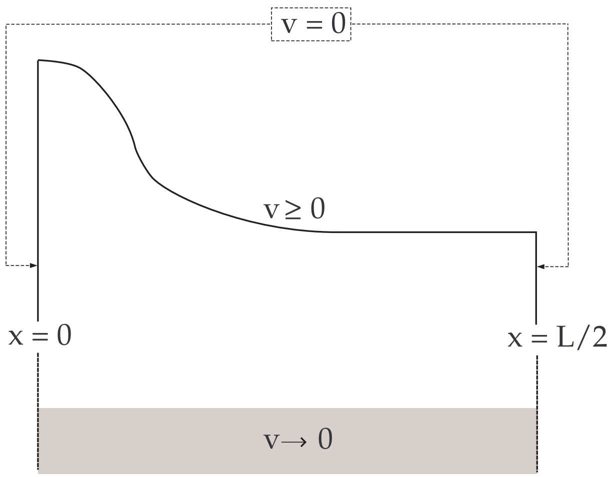
Theorem 2. *Let (η, u, v) be a steady periodic water wave, symmetric around the crest, with a nondecreasing profile from trough to crest on a current with nondecreasing vorticity $\omega_y \geq 0$. Then either the wave is trivial or the surface profile is in fact strictly monotone between trough and crest.*

Proof. Define the fluid domain between the crest and the trough to be

$$\Omega_\eta^{0<x<L/2} = \{(x,y);\ 0 < x < L/2, -d < y < \eta(x)\}.$$

The symmetry result of Theorem 1 means that ψ is symmetric around the crest. This is equivalent to u and η being symmetric and v being anti-symmetric. The crest is here assumed to be located at $x = 0$ and the trough is then at $x = L/2$. Using also the periodicity of v this forces $v(0,\cdot) = 0 = v(L/2,\cdot)$. Differentiating the boundary condition $\psi(x,\eta(x)) = 0$ in (2.7) yields $\psi_x + \psi_y\eta' = 0$. By assumption $\psi_y = u - c < 0$ in Ω_η and for $0 < x < L/2$ the surface profile is decreasing so that $\eta' \leq 0$. Then $v = -\psi_x$ implies $v(x,\eta(x)) \geq 0$ for $0 < x < L/2$. Also, by (2.4), $v(x,y) \to 0$ as $y \to -d$ uniformly for $x \in [0, L/2]$ holds for both finite and infinite depth so that in fact

$$v \geq 0 \quad \text{on the boundary of} \quad \Omega_\eta^{0<x<L/2}. \tag{3.1}$$



We will now use maximum principles to show that either $v > 0$ or $v \equiv 0$ in $\Omega_\eta^{0<x<L/2}$. To see this, we consider the x-derivative of the first line in (2.7). This is $\psi_{xxx} + \psi_{yyx} = -\gamma'(\psi)\,\psi_x$, or

$$(\Delta + \gamma')(-v) = 0. \tag{3.2}$$

Since $\gamma(\psi) = \omega$, by assumptions (2.5-2.6) we have $\gamma'(\psi) = \frac{\omega_y}{\psi_y} \leq 0$. Considering the regularity assumptions and the smoothness of the free surface $y = \eta(x)$ we see that the situation of Lemmas 1 and 2 is at hand, with p being the top right point $(L/2, \eta(L/2))$. Note that since in this case $\mathcal{L}v = 0$ also the minimal counterparts of the maximal principles apply.

Now, suppose $v = -\alpha < 0$ at a point ξ in $\Omega_\eta^{0<x<L/2}$. Let $n \in \mathbb{R}$. For any cut-off at $y = -n$ of $\overline{\Omega}_\eta^{0<x<L/2}$ including ξ, v has a minimal value and by Lemma 1 this cannot be at an interior point. If so, v would equivalently equal a negative constant in $\overline{\Omega}_\eta^{0<x<L/2}$ cut off at $y = -n$, contradicting (3.1). But by the asymptotic behaviour of v near the bottom - or at great depths - we can choose the n such that $v > -\alpha$ on the boundary of this cut-off. The contradiction attained shows that $v \geq 0$ in $\Omega_\eta^{0<x<L/2}$.

Applying Lemma 1 once again shows that that either $v > 0$ or $v \equiv 0$ in $\Omega_\eta^{0<x<L/2}$. This follows since if $v = 0$ at an interior point then v attains its minimum there and thus $v \equiv 0$ in $\Omega_\eta^{0<x<L/2}$. The latter case reduces the wave to a flat surface wave and thus we have

$$v > 0 \quad \text{in} \quad \Omega_\eta^{0<x<L/2} \quad \text{for the nontrivial case.} \tag{3.3}$$

We will now show that this discards the possibility of the surface $\eta(x)$ being constant on an open interval. Note that, by monotonicity, $\eta(x)$ being constant on an open interval is equivalent to $\eta(x)$ not being strictly monotone. Without loss of generality, suppose $\eta(x) \equiv 0$ near the trough, $L/2 - \epsilon \leq x \leq L/2$, $\epsilon > 0$.[1] Let $x_0 = L/2 - \epsilon/2$ be the horizontal coordinate in the middle of this flat region, and let $p = (x_0, \eta(x_0))$ be the corresponding surface point. Define

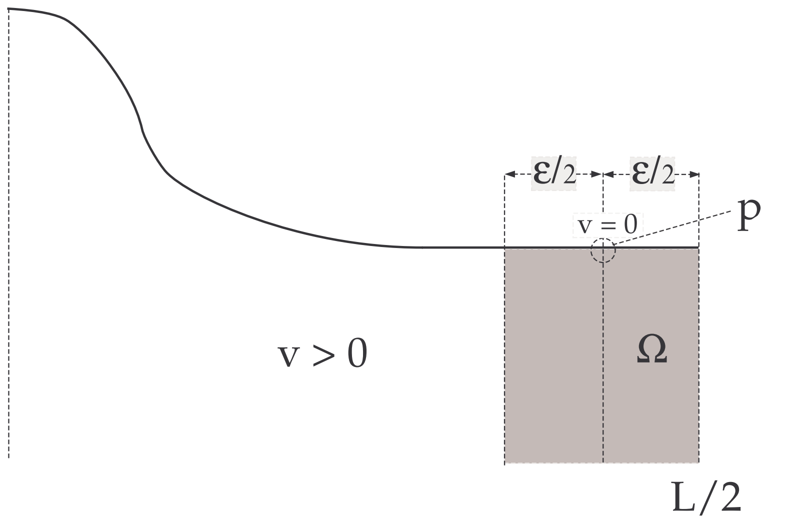
$$w(x,y) = \psi(x,y) - \psi(2x_0 - x, y), \quad x_0 < x < L/2, \quad -d < y < \eta(x) = 0. \tag{3.4}$$

Denote this region simply by $\Omega = \Omega_\eta^{x_0<x<L/2}$. Once again we will use maximum principles to show that the present situation is impossible. Applying the mean value theorem to $\gamma(\psi)$ gives $\gamma(\psi(x,y)) - \gamma(\psi(2x_0 - x, y)) = \gamma'(\xi)(\psi(x,y) - \psi(2x_0 - x, y))$ for some $\xi \geq 0$. Thus $w = \psi(x,y) - \psi(2x_0 - x, y)$ satisfies the conditions for Lemma 2 with $c = \gamma'(\xi) \leq 0$ and p as chosen:

$$(\Delta + \gamma')w = 0. \tag{3.5}$$

We now show that all partial derivatives of the first and second order vanish at p.

[1] The positioning of the flat part to be near the trough is never used, nor the choice of the constant 0 for the surface.



By (3.3) the strictness of $v = -\psi_x > 0$ in Ω forces $w < 0$ in Ω. At p, $w = w_y = w_{yy} = w_{xx} = 0$ by definition, and differentiating the boundary condition of (2.7) $\psi(x, \eta(x)) = 0$ yields $\psi_x + \psi_y \eta' = 0$. The surface being flat at p implies that $\eta'(x_0) = 0$, so that we have $v(p) = -\psi_x(p) = 0$, forcing $w_x = 0$ at p. Differentiating also the Bernoulli boundary condition in (2.7) with respect to x we get

$$\psi_x(\psi_{xx} + \psi_{xy}\eta') + \psi_y(\psi_{xy} + \psi_{yy}\eta') + g\eta' = 0 \quad \text{on} \quad y = \eta(x). \tag{3.6}$$

It has just been shown that at p, $\psi_x = 0$, while $\eta'(x_0) = 0$. Hence (3.6) reduces to $\psi_y(p)\psi_{xy}(p) = 0$, and by the assumption $\psi_y = u - c < 0$, we infer that all first and second partial derivatives of w vanish at p.

Now, $w > 0$ in Ω, $w(p) = 0$ and $\frac{\partial u}{\partial m}(p) = 0 = \frac{\partial^2 u}{\partial m^2}(p)$ for any vector m. Applying Lemma 2 as is suggested by (3.5) we must have $w \equiv 0$ in Ω. The obtained contradiction with our earlier deduction $w < 0$ in Ω completes the proof. ■

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