

Boundary conditions and Conserved densities for potential Zabolotskaya-Khokhlov equation

V ROSENHAUS

Department of Mathematics and Statistics, California State University, Chico, CA 95929, USA

E-mail: vrosenhaus@csuchico.edu

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Abstract

We study local conservation laws and corresponding boundary conditions for the potential Zabolotskaya-Khokhlov equation in (3+1)-dimensional case. We analyze an infinite Lie point symmetry group of the equation, and generate a finite number of conserved quantities corresponding to infinite symmetries through appropriate boundary conditions.

1 Introduction

The Zabolotskaya-Khokhlov equation [Z-K]

$$v_{xt} - v_x^2 - vv_{xx} - v_{yy} - v_{zz} = 0$$

describes the propagation of a confined three-dimensional sound beam in a slightly non-linear medium without absorption or dispersion. Classical symmetries of the Zabolotskaya-Khokhlov (Z-K) equation and some invariant solutions were found in [V-K-L], and [Schwarz], (see also [Sharomet]). Similarity solutions of the Z-K equation (in two dimensions) were obtained in [Korsunski], and [Z-Z-L]. It was shown in [D-L-W] that the Zabolotskaya-Khokhlov equation in two dimensions is invariant under the same group of Lie point transformations as the Kadomtsev-Pertviashvili equation.

Introducing $v = u_x$ we get the potential Zabolotskaya-Khokhlov equation

$$u_{xt} - u_x u_{xx} - u_{yy} - u_{zz} = 0, \quad (1.1)$$

The potential Zabolotskaya-Khokhlov equation can be obtained from the equation of non-stationary transonic gas flows [L-R-T] for three-dimensional non-steady motion in a compressible fluid

$$2u_{xt} + u_x u_{xx} - u_{yy} - u_{zz} = 0, \quad u = u(x, y, z, t), \quad (1.2)$$

by the following transformation: $x \rightarrow -x$, $t \rightarrow -2t$. The classical Lie point symmetry group of the equation (1.2) was studied in [Kucharczyk] and [Sukhinin]; see also [Ibragimov 83]. An infinite set of continuity equations for the equation (1.2) constructed by a formal application of First Noether Theorem to its infinite symmetry group, without regard to boundary conditions was derived in [Ibragimov 72], see also [Ibragimov 83], and the basis of these conservation laws has been obtained in [Khamitova]. It was shown in [Rosenhaus 02a] however, that symmetries with arbitrary functions of time can lead only to a finite number of essential conservation laws (with non-vanishing conserved densities), and each essential conservation law corresponds to a specific boundary conditions. Local essential conservation laws for the equation of 2-dimensional transonic gas flows were obtained in [Rosenhaus 02b].

In the present paper we generate essential conservation laws of potential Zabolotskaya-Khokhlov equation (1.1). Following the approach proposed in [Rosenhaus 02a], we construct a finite set of conserved quantities for the equation (1.1) associated with its infinite symmetry subgroups. Each conservation law corresponds to a specific choice of an arbitrary function in the group generators, which determines the appropriate boundary condition. We also find boundary conditions for conserved densities corresponding to finite Lie symmetries.

2 Infinite Symmetries and Essential Conservation Laws

By a conservation law for a differential system

$$\omega^a(x, u, u_i, \dots) = 0, \quad i = 1, \dots, m+1, \quad a = 1, \dots, n, \quad u_i^a \equiv \partial u^a / \partial x^i$$

is meant a continuity equation

$$D_i K_i \doteq 0, \quad K_i = K_i(x, u, u_j \dots), \quad i, j = 1, \dots, m+1,$$

(K_i are smooth functions) which is satisfied for any solutions of the original system. We will exclude from consideration trivial conservation laws [Olver], for which the components of the vector K_i vanish on the solutions: $K_i \doteq 0$, ($i = 1, \dots, m+1$), or the continuity equation is satisfied in the whole space: $D_i K_i = 0$. Let us further reduce the set of conservation laws to essential conservation laws. By an **essential** conservation law [Rosenhaus 03], we will mean such non-trivial conservation law $D_i K_i \doteq 0$, which gives rise to a non-vanishing conserved quantity

$$D_t \int K_t dx^1 dx^2 \dots dx^m \doteq 0, \quad K_t \not\equiv 0. \quad (2.1)$$

Let

$$S = \int L(x^i, u^a, u_i^a, \dots) d^{m+1}x$$

be the action functional, where L is the Lagrangian density, $x^i = (x^1, x^2, \dots, x^m, t)$. Then the equations of motion are

$$E^a(L) \equiv \omega^a(x, u, u_i, u_{ij}, \dots) = 0, \quad a = 1, \dots, n, \quad i, j = 1, \dots, m+1, \quad (2.2)$$

where E is the Euler-Lagrange operator

$$E^a = \frac{\partial}{\partial u^a} - \sum_i D_i \frac{\partial}{\partial u_i^a} + \sum_{i \leq j} D_i D_j \frac{\partial}{\partial u_{ij}^a} + \dots \tag{2.3}$$

Consider an infinitesimal transformation with the canonical operator

$$X_\alpha = \alpha^a \frac{\partial}{\partial u^a} + \sum_i (D_i \alpha^a) \frac{\partial}{\partial u_i^a} + \sum_{i \leq j} (D_i D_j \alpha^a) \frac{\partial}{\partial u_{ij}^a} + \dots \quad , \tag{2.4}$$

$i, j = 1, \dots, m + 1$

(summation over a is assumed). Variation of the functional S under the transformation with operator X_α is

$$\delta S = \int X_\alpha L d^{m+1}x \tag{2.5}$$

X_α is a variational (Noether) symmetry if

$$X_\alpha L = D_i M_i, \quad M_i = M_i(x, u, u_i, \dots), \quad i = 1, \dots, m + 1, \tag{2.6}$$

where M_i are smooth functions. In the future we will use the Noether identity [Rosen], see also [Ibragimov 83] or [Rosenhaus 02a] for a version used here):

$$X_\alpha = \alpha^a E^a + \sum_{i=1}^{m+1} D_i R_{\alpha i}, \tag{2.7}$$

$$R_{\alpha i} = \alpha^a \frac{\partial}{\partial u_i^a} + \left\{ \sum_{k \geq i} (D_k \alpha^a) - \alpha^a \sum_{k \leq i} D_k \right\} \frac{\partial}{\partial u_{ik}^a} + \dots \tag{2.8}$$

Applying the Noether identity (2.7) to L , and combining with (2.6) and (2.8) we will obtain

$$D_i (M_i - R_{\alpha i} L) = \alpha^a \omega^a, \quad i = 1, \dots, m + 1. \tag{2.9}$$

The equation (2.9) applied on the solution manifold ($\omega = 0, D_i \omega = 0, \dots$) leads to a continuity equation

$$D_i (M_i - R_{\alpha i} L) \doteq 0, \quad i = 1, \dots, m + 1. \tag{2.10}$$

Thus, any 1-parameter variational symmetry transformation α (2.6) leads to a conservation law (2.10) (the First Noether Theorem) with the characteristic α . The Second Noether Theorem [Noether] deals with a case of an infinite variational symmetry group where the symmetry vector α is of the form

$$\alpha = ap(x) + b_i D_i p(x) + c_{ij} D_i D_j p(x) + \dots, \tag{2.11}$$

and $p(x)$ is an arbitrary function of all base variables of the space. According to the Second Noether Theorem [Noether] infinite variational symmetries with arbitrary functions of all independent variables do not lead to non-trivial conservation laws but to a certain relation between equations of the original differential system (meaning that the system is under-determined). A general situation when $p(x)$ is an arbitrary function of not all base variables was analyzed in [Rosenhaus 02a]. In this paper, we will be mostly interested in the case of arbitrary functions of time $\gamma(t)$. Consider a variational (Noether) symmetry α of the form

$$\alpha = a\gamma(t) + b\gamma'(t) + c\gamma''(t) + \dots + h\gamma^{(l)}(t), \quad (2.12)$$

for a differential equation with the Lagrangian $L = L(x^i, u, u_i)$, $x^i = (x^1, x^2, \dots, x^m, t)$. For a Noether symmetry transformation X_α we have

$$\delta S = \int_D \delta L d^{m+1}x = \int_D X_\alpha L d^{m+1}x = \int_D D_i M_i d^{m+1}x = 0, \quad x \in D. \quad (2.13)$$

Therefore, the following conditions for M_i (Noether boundary conditions) should be satisfied [Rosenhaus 02a]

$$M_i(x, u, \dots) \Big|_{x^i \rightarrow \partial D} = 0, \quad \forall i = 1, \dots, m+1. \quad (2.14)$$

Equations (2.14) are usually satisfied for a “regular” asymptotic behavior, $u, u_i \rightarrow 0$ as $x \rightarrow \pm\infty$, or for periodic solutions. Let us consider now another type of boundary conditions related to the existence of local conserved quantities. Integrating equation (2.10) over the space (x^1, x^2, \dots, x^m) we get

$$\int dx^1 dx^2 \dots dx^m D_t(M_t - R_{\alpha t}L) \doteq \int dx^1 \dots dx^m \sum_{i=1}^m D_i(R_{\alpha i}L - M_i). \quad (2.15)$$

Applying the Noether boundary condition (2.14) and requiring the LHS of (2.15) vanish on the solution manifold we obtain the “strict” boundary conditions [Rosenhaus 02a]

$$R_{\alpha 1}L \Big|_{x^1 \rightarrow \partial D} = R_{\alpha 2}L \Big|_{x^2 \rightarrow \partial D} = \dots = R_{\alpha m}L \Big|_{x^m \rightarrow \partial D} = 0. \quad (2.16)$$

In the case $L = L(x, u, u_j)$, strict boundary conditions (2.16) take a simple form

$$\alpha \frac{\partial L}{\partial u_i} \Big|_{x^i \rightarrow \partial D} = 0, \quad \forall i = 1, \dots, m. \quad (2.17)$$

In order for the system to possess (Noether) local conserved quantities, both Noether (2.14) and strict boundary conditions (2.16) have to be satisfied. In this case corresponding Noether conservation law can be found in the form

$$\int dx^1 dx^2 \dots dx^m D_t(M_t - R_{\alpha t}L) \doteq 0. \quad (2.18)$$

Writing M_t as

$$M_t = A\gamma(t) + B\gamma'(t) + C\gamma''(t) + \dots + H\gamma^{(l)}(t), \tag{2.19}$$

from (2.18), we obtain

$$D_t \int dx^1 dx^2 \dots dx^m \left[\gamma(t)A_1 + \gamma'(t)A_2 + \dots + \gamma^{(l)}(t)A_l \right] \doteq 0, \tag{2.20}$$

where

$$A_1 = \left(A - a \frac{\partial L}{\partial u_t} \right), \quad A_2 = \left(B - b \frac{\partial L}{\partial u_t} \right), \dots \quad A_l = \left(H - h \frac{\partial L}{\partial u_t} \right).$$

Since $\gamma(t)$ is arbitrary we get

$$\begin{aligned} \int dx^1 dx^2 \dots dx^m \left(A - a \frac{\partial L}{\partial u_t} \right) &\doteq \int dx^1 dx^2 \dots dx^m \left(B - b \frac{\partial L}{\partial u_t} \right) \doteq \dots \\ &\doteq \int dx^1 dx^2 \dots dx^m \left(H - h \frac{\partial L}{\partial u_t} \right) \doteq 0. \end{aligned} \tag{2.21}$$

Obviously, equations (2.21), in general, determine not a system of conservation laws but additional constraints. Thus, Noether symmetries (empowered by strict boundary conditions) with arbitrary functions of time instead of conservation laws lead to a set of additional constraints imposed on the function u and its derivatives. Therefore, the satisfaction of the strict boundary conditions (2.16), along with the Noether boundary condition (2.14), becomes critical in the sense of avoiding additional constraints (2.21). Correspondingly, we have three possible situations:

1) Strict boundary conditions (2.16) (or (2.17)), along with the Noether boundary condition (2.14), can be satisfied for arbitrary function $\gamma(t)$. Then the system (2.21), as a consequence of an infinite symmetry (2.12), provides additional constraints that the function u and its derivatives must satisfy. No conservation laws are associated with the symmetry.

2) Strict boundary conditions (2.16) along with the Noether boundary condition (2.14), can be satisfied for some particular functions $\gamma(t)$. In this case the (finite) symmetry (2.12) will lead to the Noether conservation law (2.18) in agreement with the First Noether Theorem. Additional constraints (2.21) do not appear.

3) Strict boundary conditions cannot be satisfied for any functions $\gamma(t)$. In this case a consequence of an infinite symmetry (2.12) will be the fact that the solutions of the original differential equation with the boundary conditions (2.14), (2.16) do not exist.

Therefore, in order to avoid additional constraints (2.21) we have to find some particular functions $\gamma(t)$ (if they exist) leading to different boundary conditions than the ones in general case (when function $\gamma(t)$ is arbitrary) [Rosenhaus 02a]. Each choice of such function $\gamma(t)$ gives rise to a respective conserved quantity. Let us apply the approach above for finding non-vanishing conserved densities of the potential Zabolotskaya-Khokhlov equation (1.1) with boundary conditions on the infinity.

3 Essential Conservation Laws for Potential Z-K Equation

$$u_{xt} - u_x u_{xx} - u_{yy} - u_{zz} = 0,$$

The Lagrangian function of the equation is

$$L = -\frac{u_x u_t}{2} + \frac{u_x^3}{6} + \frac{u_y^2}{2} + \frac{u_z^2}{2}. \quad (3.1)$$

The Lie point symmetry group of the equation is formed by the following operators:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, & X_2 &= 5t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + 3y \frac{\partial}{\partial y} + 3z \frac{\partial}{\partial z} - 3u \frac{\partial}{\partial u}, \\ X_3 &= t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} - 3u \frac{\partial}{\partial u}, & X_4 &= z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z}, \\ X_5 &= 5t^2 \frac{\partial}{\partial t} + [2tx + \frac{3}{2}(y^2 + z^2)] \frac{\partial}{\partial x} + 6ty \frac{\partial}{\partial y} + 6tz \frac{\partial}{\partial z} - (x^2 + 6tu) \frac{\partial}{\partial u}, \\ X_\gamma &= -\gamma(t) \frac{\partial}{\partial x} + [\gamma'(t)x + \gamma''(t)(y^2 + z^2)/4] \frac{\partial}{\partial u}, \\ X_f &= 2f \frac{\partial}{\partial y} + f' y \frac{\partial}{\partial x} - [xyf'' + f'''(y^3/3 + z^2y)/4] \frac{\partial}{\partial u}, \\ X_h &= 2h \frac{\partial}{\partial z} + h' z \frac{\partial}{\partial x} - [xzh'' + h'''(z^3/3 + zy^2)/4] \frac{\partial}{\partial u}, \end{aligned} \quad (3.2)$$

where $\gamma(t), f(t), h(t)$ are arbitrary functions. Let us analyze infinite subgroups of the symmetry group (3.2). In order to apply the approach above we have to rewrite our symmetry operator in canonical form. For an operator

$$X = \xi^t \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + \xi^y \frac{\partial}{\partial y} + \dots + \eta \frac{\partial}{\partial u} \quad (3.3)$$

a corresponding canonical operator will take a form

$$X_\alpha = X - \xi^i D_i = \alpha \frac{\partial}{\partial u} + \zeta_i \frac{\partial}{\partial u_i} + \sigma_{ij} \frac{\partial}{\partial u_{ij}} \dots, \quad (3.4)$$

where

$$\alpha = \eta - \xi^i u_i, \quad \zeta_i = D_i \alpha, \quad \sigma_{ij} = D_{ij} \alpha, \quad \dots \quad (3.5)$$

3.1 Conserved densities associated with X_γ

We will start with the symmetry X_γ and find corresponding conserved densities ([Rosenhaus 05])

$$X_\gamma = -\gamma(t) \frac{\partial}{\partial x} + [\gamma'(t)x + \gamma''(t)(y^2 + z^2)/4] \frac{\partial}{\partial u}, \quad (3.6)$$

$\gamma(t)$ is arbitrary. We have

$$\begin{aligned} \xi^x &= -\gamma(t), & \xi^t &= \xi^y = \xi^z = 0, & \eta &= \gamma'(t)x + \gamma''(t)(y^2 + z^2)/4 \\ \alpha &= \gamma'x + \gamma''(y^2 + z^2)/4 + \gamma u_x, & X_\alpha L &= D_i M_i, & M_t &= -u\gamma'/2, \\ M_x &= \gamma L - xu\gamma''/2 - (y^2 + z^2)u\gamma'''/8, & M_y &= yu\gamma''/2, & M_z &= zu\gamma''/2, \\ a &= u_x, & b &= x, & c &= (y^2 + z^2)/4, & A &= C = 0, & B &= -u/2. \end{aligned} \tag{3.7}$$

The form of Noether and strict boundary conditions depends on the function $\gamma(t)$.

A. $\gamma(t)$ is arbitrary

Noether boundary conditions (2.14) for X_α are

$$M_i(x, u, \dots) \Big|_{x^i \rightarrow -\infty}^{x^i \rightarrow \infty} = 0, \quad \forall i = 1, \dots, 4,$$

or

$$u_i \xrightarrow{x \rightarrow \pm\infty} 0, \quad xu \xrightarrow{x \rightarrow \pm\infty} 0, \quad yu \xrightarrow{y \rightarrow \pm\infty} 0, \quad zu \xrightarrow{z \rightarrow \pm\infty} 0, \quad u \xrightarrow{t \rightarrow \pm\infty} 0. \tag{3.8}$$

Strict boundary conditions (2.17) take the form

$$xu_t, xu_x^2 \xrightarrow{x \rightarrow \pm\infty} 0, \quad y^2 u_y \xrightarrow{y \rightarrow \pm\infty} 0, \quad z^2 u_z \xrightarrow{z \rightarrow \pm\infty} 0. \tag{3.9}$$

Thus, for the case of arbitrary $\gamma(t)$ we have the following boundary conditions:

$$xu, xu_t, xu_x^2, u_y, u_z \xrightarrow{x \rightarrow \pm\infty} 0, \quad yu, y^2 u_y \xrightarrow{y \rightarrow \pm\infty} 0, \quad zu, z^2 u_z \xrightarrow{z \rightarrow \pm\infty} 0, \quad u \xrightarrow{t \rightarrow \pm\infty} 0. \tag{3.10}$$

No local conservation laws are associated with the Noether transformation X_α (3.6) for arbitrary $\gamma(t)$.

Let us consider now some specific forms of $\gamma(t)$ for which we can weaken our boundary conditions (3.8)-(3.9) in order to avoid restrictions (2.21).

B. $\gamma'(t) = 0, \gamma(t) = const.$

Noether conditions are

$$u_i \xrightarrow{x \rightarrow \pm\infty} 0, \quad i = 1, \dots, 4. \tag{3.11}$$

Strict boundary conditions, in addition to (3.11) will have

$$u_x u_y \xrightarrow{y \rightarrow \pm\infty} 0, \quad u_x u_z \xrightarrow{z \rightarrow \pm\infty} 0. \quad (3.12)$$

According to (2.18) we will get the following conservation law:

$$D_t \int u_x^2 dx dy dz \doteq 0, \quad (3.13)$$

or

$$\int u_x^2 dx dy dz \doteq \text{const.}$$

(conservation of the x -component of momentum P_x). Corresponding continuity equation (2.10) has the form

$$D_x (-2u_x^3/3 + u_y^2 + u_z^2) + D_y (-2u_x u_y) + D_z (-2u_x u_z) + D_t (u_x^2) \doteq 0. \quad (3.14)$$

C. $\gamma''(t) = 0$, $\gamma(t) = at$, $a = \text{const} \neq 0$.

Noether conditions are

$$u \xrightarrow{t \rightarrow \pm\infty} 0, \quad u_i \xrightarrow{x \rightarrow \pm\infty} 0, \quad i = 1, \dots, 4. \quad (3.15)$$

For strict boundary conditions we get

$$x u_t, x u_x^2 \xrightarrow{x \rightarrow \pm\infty} 0, \quad u_y, u_x u_y \xrightarrow{y \rightarrow \pm\infty} 0, \quad u_z, u_x u_z \xrightarrow{z \rightarrow \pm\infty} 0. \quad (3.16)$$

The conservation law associated with the boundary conditions (3.15)-(3.16) takes the form

$$D_t \int [x u_x - u + t u_x^2] dx dy dz \doteq 0. \quad (3.17)$$

As we can see, strict boundary conditions (3.16) together with Noether conditions (3.15) determined a nontrivial conservation law (3.17) with the characteristic $\alpha = x + t u_x$.

D. $\gamma''(t) \neq 0$

In this case Noether and strict boundary conditions have the same form (3.8), (3.9) as in case **A.** and lead to no essential conservation laws.

3.2 Conserved densities associated with X_f

$$X_f = 2f \frac{\partial}{\partial y} + f'y \frac{\partial}{\partial x} - [xyf'' + f'''(y^3/12 + yz^2/4)] \frac{\partial}{\partial u}, \tag{3.18}$$

$f(t)$ is arbitrary function. For a corresponding canonical operator X_α we get

$$\begin{aligned} \alpha &= 2fu_y + f'yu_x + f''xy + f'''(y^3/12 + yz^2/4), \\ \xi^x &= -f'y, \quad \xi^y = -2f, \quad \eta = f'''(y^3/12 + yz^2/4) \end{aligned} \tag{3.19}$$

Calculating $X_\alpha L$ we obtain

$$\begin{aligned} X_\alpha L &= D_i M_i, \\ M_x &= f'yL - f'''xyu/2 - f^{(IV)}(y^3/3 + yz^2)u/8, \\ M_y &= 2fL + f''xu + f'''(y^2 + z^2)u/4, \\ M_z &= f'''yzu/2, \quad M_t = -f''yu/2. \end{aligned} \tag{3.20}$$

As in the previous case the form of strict and Noether boundary conditions depend on the function $f(t)$.

A. $f(t)$ is arbitrary.

From the Noether and strict boundary conditions we will get

$$u_i, xu, x^2u_x \xrightarrow{x \rightarrow \pm\infty} 0, \quad u_i, y^2u, y^3u_y \xrightarrow{y \rightarrow \pm\infty} 0, \quad zu, z^2u_z \xrightarrow{z \rightarrow \pm\infty} 0, \quad u \xrightarrow{t \rightarrow \pm\infty} 0. \tag{3.21}$$

No local conservation laws are associated with the Noether transformation X_α (3.18) with arbitrary function $f(t)$. Let us consider now some specific forms of $f(t)$ for which we can weaken boundary conditions (3.21).

B. $f' = 0, f(t) = c = const.$

We have

$$M_x = 0, \quad M_y = 2cL, \quad M_z = M_t = 0, \quad \alpha = 2cu_y. \tag{3.22}$$

Noether conditions look as follows:

$$u_i \xrightarrow{y \rightarrow \pm\infty} 0. \tag{3.23}$$

The strict boundary conditions have a form

$$u_t u_y, u_x^2 u_y \xrightarrow{x \rightarrow \pm\infty} 0, \quad u_y \xrightarrow{y \rightarrow \pm\infty} 0, \quad u_y u_z \xrightarrow{z \rightarrow \pm\infty} 0. \tag{3.24}$$

According to (2.18), the associated conservation law has a form:

$$D_t \int u_x u_y dx dy dz \doteq 0, \quad (3.25)$$

and the characteristic $\alpha = u_y$. Expression (3.25) is a conservation of y -component of the momentum of the system, P_y with regular boundary conditions (3.23)-(3.24).

C. $f'' = 0$, $f' \neq 0$. $f(t) = at$, $a = \text{const} \neq 0$.

We have

$$M_x = ayL, \quad M_y = 2atL, \quad M_z = M_t = 0, \quad \alpha = 2atu_y + ayu_x. \quad (3.26)$$

Noether conditions are

$$u_i \xrightarrow{x \rightarrow \pm\infty} 0, \quad u_i \xrightarrow{y \rightarrow \pm\infty} 0, \quad i = 1, \dots, 4. \quad (3.27)$$

and the strict boundary conditions (2.17) read

$$u_i \xrightarrow{x \rightarrow \pm\infty} 0, \quad yu_x u_y, u_y \xrightarrow{y \rightarrow \pm\infty} 0, \quad u_z \xrightarrow{z \rightarrow \pm\infty} 0. \quad (3.28)$$

Boundary conditions (3.27)-(3.28) are weaker than the ones for the general case (3.21), and from (2.18) we obtain the following conserved quantity

$$D_t \int [2tu_x u_y + yu_x^2] dx dy dz \doteq 0. \quad (3.29)$$

D. $f''' = 0$, $f'' \neq 0$: $f(t) = bt^2/2$.

We have

$$\begin{aligned} M_x &= btyL, & M_y &= bt^2L + bxu, & M_z &= 0, & M_t &= -byu/2, \\ \alpha &= bt^2u_y + btyu_x + bxy. \end{aligned} \quad (3.30)$$

Noether boundary conditions are

$$u_i \xrightarrow{x \rightarrow \pm\infty} 0, \quad u, u_i \xrightarrow{y \rightarrow \pm\infty} 0, \quad u \xrightarrow{t \rightarrow \pm\infty} 0, \quad i = 1, \dots, 4. \quad (3.31)$$

The strict boundary conditions read

$$xu_t, xu_x^2, u_i \xrightarrow{x \rightarrow \pm\infty} 0, \quad u_x, yu_y \xrightarrow{y \rightarrow \pm\infty} 0, \quad u_z \xrightarrow{z \rightarrow \pm\infty} 0. \quad (3.32)$$

Thus, the boundary conditions for this case, (3.31)-(3.32) are weaker than in the general case (3.24). The following conservation law (2.18) is associated with the symmetry X_α (3.19):

$$D_t \int [t^2 u_x u_y + tyu_x^2 + xyu_x - yu] dx dy dz \doteq 0. \quad (3.33)$$

E. $f'''(t) \neq 0$

In this case Noether and strict boundary conditions have the same form (3.21) as in case **A.** and lead to no essential conservation laws.

3.3 Conserved densities associated with X_h

Note that the symmetry operator

$$X_h = 2h \frac{\partial}{\partial z} + h'z \frac{\partial}{\partial x} - [xzh'' + h'''(z^3/12 + zy^2/4)] \frac{\partial}{\partial u}, \tag{3.34}$$

where $h = h(t)$ is arbitrary function can be obtained from X_f by the interchange $y \leftrightarrow z$. Therefore, from the analysis of X_f we obtain the following cases:

A. $h(t)$ is arbitrary.

No local conservation laws are associated with the Noether transformation X_α (3.34) with arbitrary function $h(t)$.

B. $h' = 0, h(t) = c = const.$

The associated conservation law has a form:

$$D_t \int u_x u_z dx dy dz \doteq 0, \tag{3.35}$$

(conservation of z -component of the momentum of the system, P_z) with relatively relaxed boundary conditions

$$u_i \xrightarrow{z \rightarrow \pm\infty} 0, \quad u_t u_z, u_x^2 u_z \xrightarrow{x \rightarrow \pm\infty} 0, \quad u_y u_z \xrightarrow{y \rightarrow \pm\infty} 0, \quad i=1, \dots, 4. \tag{3.36}$$

C. $h'' = 0, h' \neq 0. h(t) = at, a = const \neq 0.$

We get

$$D_t \int [2tu_x u_z + zu_x^2] dx dy dz \doteq 0, \tag{3.37}$$

with the boundary conditions

$$u_i \xrightarrow{x \rightarrow \pm\infty} 0, \quad u_y \xrightarrow{y \rightarrow \pm\infty} 0, \quad u_z, zu_x u_z \xrightarrow{z \rightarrow \pm\infty} 0, \quad i=1, \dots, 4. \tag{3.38}$$

D. $h''' = 0, h'' \neq 0 : h(t) = bt^2/2.$

The associated essential conservation law

$$D_t \int [t^2 u_x u_z + t z u_x^2 + x z u_x - z u] dx dy dz \doteq 0, \tag{3.39}$$

corresponds to the following boundary conditions:

$$x u_t, x u_x^2, u_i \xrightarrow{x \rightarrow \pm\infty} 0, \quad u_y \xrightarrow{y \rightarrow \pm\infty} 0, \quad u, u_i, z u_z \xrightarrow{z \rightarrow \pm\infty} 0, \quad u \xrightarrow{t \rightarrow \pm\infty} 0. \tag{3.40}$$

E. $h'''(t) \neq 0$

As earlier, this case leads to no essential conservation laws.

4 Conservation Laws Associated with Finite Symmetries

Let us discuss now boundary conditions and essential conservation laws of the equation (1.1) corresponding to its finite symmetries: X_1, X_2, X_3, X_4, X_5 (3.2)

$$1) \quad X_1 = \frac{\partial}{\partial t} \tag{4.1}$$

We have

$$\xi^t = 1, \quad \xi^x = \xi^y = \xi^z = \eta = 0, \quad \alpha = \eta - \xi^i u_i = -u_t. \tag{4.2}$$

$$\delta S = \int (X L + (D_i \xi^i) L) d^{m+1}x = 0. \tag{4.3}$$

Clearly, X_1 is a Noether symmetry and

$$X_\alpha L = (X - \xi^i D_i) L = X L - D_t L = -D_t L. \tag{4.4}$$

Thus,

$$M_t = -L, \quad M_x = M_y = M_z = 0, \quad R_{\alpha i} L = \alpha \frac{\partial L}{\partial u_i}. \tag{4.5}$$

see (2.6) and (2.8). The continuity equation (2.10) here reads

$$-D_t \left(\frac{u_x^3}{6} + \frac{u_y^2 + u_z^2}{2} \right) + D_x \left(\frac{-u_t^2 + u_x^2 u_t}{2} \right) + D_y [u_t u_y] + D_z [u_t u_z] \doteq 0 \tag{4.6}$$

([Khamitova]). The associated conservation law (of energy) (2.18)

$$D_t \int dx dy dz \left(\frac{u_x^3}{6} + \frac{u_y^2 + u_z^2}{2} \right) \doteq 0. \tag{4.7}$$

corresponds to the following boundary conditions: Noether conditions (2.14)

$$L \Big|_{t \rightarrow -\infty}^{t \rightarrow \infty} = 0 : \tag{4.8}$$

$$u_x, u_y, u_z, u_t u_x \xrightarrow{t \rightarrow \pm\infty} 0, \tag{4.9}$$

and strict boundary conditions (2.17)

$$u_t, u_t u_x^2 \xrightarrow{x \rightarrow \pm\infty} 0, \quad u_t u_y \xrightarrow{y \rightarrow \pm\infty} 0, \quad u_t u_z \xrightarrow{z \rightarrow \pm\infty} 0. \tag{4.10}$$

$$2) \quad X_2 = 5t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + 3y \frac{\partial}{\partial y} + 3z \frac{\partial}{\partial z} - 3u \frac{\partial}{\partial u} \tag{4.11}$$

We have

$$\begin{aligned} \xi^t &= 5t, \quad \xi^x = x, \quad \xi^y = 3y, \quad \xi^z = 3z, \quad \eta = -3u, \\ \alpha &= \eta - \xi^i u_i = -3u - 5tu_t - xu_x - 3yu_y - 3zu_z, \end{aligned} \tag{4.12}$$

$$\delta S = \int (XL + (D_i \xi^i)L) dx dy dz dt = 0, \tag{4.13}$$

therefore, X_2 is a Noether symmetry and

$$X_\alpha L = (X - \xi^i D_i)L = -D_i(\xi^i L). \tag{4.14}$$

Thus (see (2.6)),

$$M_t = -5tL, \quad M_x = -xL, \quad M_y = -3yL, \quad M_z = -3zL. \tag{4.15}$$

The continuity equation (2.10) here is

$$D_i \left(\xi^i L + \alpha \frac{\partial L}{\partial u_i} - M_i \right) \doteq 0. \tag{4.16}$$

The associated conservation law (2.18)

$$D_t \int (5tL - \alpha u_x / 2) dx dy dz \doteq 0 \tag{4.17}$$

corresponds to the following boundary conditions: Noether conditions (2.14)

$$xL \Big|_{x \rightarrow -\infty}^{x \rightarrow \infty} = 0, \quad yL \Big|_{y \rightarrow -\infty}^{y \rightarrow \infty} = 0, \quad zL \Big|_{z \rightarrow -\infty}^{z \rightarrow \infty} = 0, \quad tL \Big|_{t \rightarrow -\infty}^{t \rightarrow \infty} = 0, \tag{4.18}$$

and (partially overlapping with them) strict boundary conditions (2.17)

$$uu_t, uu_x^2, xu_x^3, xu_x u_t \xrightarrow{x \rightarrow \pm\infty} 0, \quad uu_y, yu_y^2 \xrightarrow{y \rightarrow \pm\infty} 0, \quad uu_z, zu_z^2 \xrightarrow{z \rightarrow \pm\infty} 0. \tag{4.19}$$

$$3) \quad X_3 = t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} - 3u \frac{\partial}{\partial u}. \quad (4.20)$$

$$\delta S = \int (XL + (D_i \xi^i)L) dx dy dz dt = -6S \neq 0, \quad (4.21)$$

and X_3 is a non-Noether transformation.

$$4) \quad X_4 = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z}. \quad (4.22)$$

We have

$$\begin{aligned} \xi^y &= z, \quad \xi^z = -y, \quad \alpha = -zu_y + yu_z, \quad \delta S = 0, \\ X_\alpha L &= -zu_y + yu_z, \\ M_t &= M_x = 0, \quad M_y = -zu, \quad M_z = yu. \end{aligned} \quad (4.23)$$

The conservation law (2.18) in this case is:

$$D_t \int u_x (yu_z - zu_y) dx dy dz \doteq 0. \quad (4.24)$$

(according to [Khamitova] continuity equations corresponding to (4.24) and (4.7) form a basis of an infinite set of continuity equations constructed by a formal application of First Noether Theorem to an infinite group (3.2) [Ibragimov 83]). Corresponding Noether and strict boundary conditions are, respectively:

$$u \xrightarrow{y \rightarrow \pm\infty} 0, \quad u \xrightarrow{z \rightarrow \pm\infty} 0, \quad (4.25)$$

$$u_z u_x^2, u_z u_t, u_y u_x^2, u_y u_t \xrightarrow{x \rightarrow \pm\infty} 0, \quad u_y, yu_z u_y \xrightarrow{y \rightarrow \pm\infty} 0, \quad u_z, zu_y u_z \xrightarrow{z \rightarrow \pm\infty} 0. \quad (4.26)$$

$$5) \quad X_5 = 5t^2 \frac{\partial}{\partial t} + [2tx + \frac{3}{2}(y^2 + z^2)] \frac{\partial}{\partial x} + 6ty \frac{\partial}{\partial y} + 6tz \frac{\partial}{\partial z} - (x^2 + 6tu) \frac{\partial}{\partial u}. \quad (4.27)$$

$$\begin{aligned} \xi^x &= 2tx + \frac{3}{2}(y^2 + z^2), \quad \xi^y = 6ty, \quad \xi^z = -6tz, \quad \eta = - (x^2 + 6tu), \\ \alpha &= -(x^2 + 6tu) - u_x(2tx + \frac{3}{2}(y^2 + z^2)) - 6tyu_y - 6tzu_z, \\ M_t &= xu, \quad M_x = 3u^2/2, \quad M_y = M_z = 0. \end{aligned} \quad (4.28)$$

The continuity equation (2.10) ((4.17)) leads to the following conserved quantity:

$$D_t \int (5t^2 L - \alpha u_x / 2 - xu) dx dy dz \doteq 0. \quad (4.29)$$

Corresponding Noether and strict boundary conditions are, respectively:

$$u \xrightarrow{t \rightarrow \pm\infty} 0, \quad u^2 \xrightarrow{x \rightarrow \pm\infty} 0, \quad (4.30)$$

$$\alpha u_x^2, \alpha u_t \xrightarrow{x \rightarrow \pm\infty} 0, \quad \alpha u_y \xrightarrow{y \rightarrow \pm\infty} 0, \quad \alpha u_z \xrightarrow{z \rightarrow \pm\infty} 0, \quad (4.31)$$

out of which the most essential conditions are

$$u \xrightarrow{t \rightarrow \pm\infty} 0, \quad u, u_i, x^2 u_x^2, x^2 u_t \xrightarrow{x \rightarrow \pm\infty} 0, \\ u, u_y, y^2 u_x u_y, y u_y^2 \xrightarrow{y \rightarrow \pm\infty} 0, \quad u, u_z, z^2 u_x u_z, z u_y^2 \xrightarrow{z \rightarrow \pm\infty} 0. \quad (4.32)$$

5 Conclusions

We have generated a finite number of conserved densities for the (3+1)-dimensional potential Zabolotskaya-Khokhlov equation (1.1), associated with its infinite Lie point symmetries. We have found those continuity equations for the potential Z-K equation (out of an infinite set of all possible continuity equations, corresponding to its infinite symmetry group [Ibragimov 83]) that lead to non-vanishing conserved densities (essential conservation laws): (3.13), (3.17), (3.25), (3.29), (3.33), (3.35), (3.39), (along with (4.7), (4.17), (4.24), (4.29)). Each essential conservation law is related to a specific boundary condition. As expected ([Rosenhaus 02a]) symmetries with arbitrary functions of t lead to essential local conservation laws only in special cases when boundary conditions are weaker than those in general case. We have derived boundary conditions corresponding to our essential conservation laws and have shown that for potential Zabolotskaya-Khokhlov equation known conservation laws of momentum and energy (3.13), (3.25), (3.35), (4.7)) correspond to the weakest boundary conditions, with vanishing (or periodic) function and its derivatives on the boundary (infinity). Other essential conservation laws were demonstrated to have stricter asymptotic behavior.

References

- [1] DAVID D, LEVI D and WINTERNITZ P, Equations Invariant under the Symmetry Group of the Kadomtsev-Petviashvili Equation, *Phys. Lett. A* **129** (1988), 161–164.
- [2] IBRAGIMOV N H, Lie Groups in Some Problems of Mathematics, Novosibirsk Univ., Novosibirsk, 1972 [in Russian].
- [3] IBRAGIMOV N H, Transformation Groups Applied to Mathematical Physics, Reidel, Boston, 1985.
- [4] KHAMITOVA R S, Group Structure and the Basis of Conservation Laws, *Theor. Math. Phys.* **52** (1982), 777–781.
- [5] KORSUNSKI S V, Self-Similar Solutions of Two-Dimensional Equations of Khokhlov-Zabolotskaya Type, *Mat. Fiz. Nelin. Mekh.* **16** (1991), 81–87 [in Russian].

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- [6] KUCHARCZYK P, Teoria Grup Liego w Zastosowaniu do Rownan Rozniczkowych Czastkowych, IPPT Polish Acad. Sci., Warsaw, 1967.
- [7] LIN C C, REISSNER E and TSIEN H S, On Two-Dimensional Non-Steady Motion of a Slender Body in a Compressible Fluid, *J. Math. Phys.* **27** (1948), 220–231.
- [8] NOETHER E, Invariantenprobleme, *Nachr. Konig. Gesell. Wissen. Gttingen, Math.-Phys.* **K1** (1918), 235–257.
- [9] OLVER P J, Applications of Lie Groups to Differential Equations, Springer, New York, 1986.
- [10] ROSEN J, Some Properties of the Euler-Lagrange Operators, Preprint TAUP-269-72, Tel-Aviv University, Tel-Aviv, 1972.
- [11] ROSENHAUS V, Infinite Symmetries and Conservation Laws, *J. Math. Phys.* **43** (2002), 6129–6150.
- [12] ROSENHAUS V, On Conservation Laws for the Equation of Non-Stationary Transonic Gas Flows, *J. Dyn. Sys. Geom. Theor.* **1** (2002), 95–107.
- [13] ROSENHAUS V, On Infinite Symmetries and Essential Conservation Laws for Navier-Stokes Equations, *Proc. XXIV Intern. Conf. Group Theor. Methods Phys., Paris, 2002* (2003), 737–740.
- [14] ROSENHAUS V, Essential Conservation Laws for Equations with Infinite Symmetries, *Theor. Math. Phys.* **144** (2005), 1046–1053.
- [15] SCHWARZ F, Symmetries of the Khokhlov-Zabolotskaya equation, *J. Phys. A.* **20** (1987), 1613–1614.
- [16] SUKHININ S V, Group Property and Conservation Laws of the Equation of Transonic Motion of Gas, *Dynamika Sploshnoi Sredy* **36** (1978), 130–137 [in Russian].
- [17] SHAROMET N O, Symmetries, Invariant Solutions and Conservation Laws of the Nonlinear Acoustics Equations, *Acta Appl. Math.* **15** (1989), 83–120.
- [18] VINOGRADOV A M, KRASILSHCHIK I S and LYCHAGIN V V, Introduction to Geometry of Nonlinear Differential Equations, Nauka, Moscow, 1986.
- [19] ZABOLOTSKAYA E A and KHOKHLOV R V, Quasi-Plane Waves in the Nonlinear Acoustics of Confined Beams *Sov. Phys. Acoust.* **15** (1969), 35–40.
- [20] ZHANG J-F, ZHU Y-J and LIN J, Similarity Reductions for the Khokhlov-Zabolotskaya Equation, *Comm. Theor. Phys.* **24** (1995), 69–74.