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# New Cellular Automata associated with the Schroedinger Discrete Spectral Problem

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**Abstract**

New Cellular Automata associated with the Schroedinger discrete spectral problem are derived. These Cellular Automata possess an infinite (countable) set of constants of motion.

## 1 Introduction

A few years ago a class of 1-dimensional ”filter” Cellular Automata (CA) [1],[2] associated with the Schroedinger discrete spectral problem (SDSP) was introduced [3]. These CA are endowed with a number of interesting properties, namely:

- they possess an infinite (countable) set of constants of motion
- they are time reversible
- they exhibit an interesting dynamic (solitons)

However the authors of [3] were not able to fully develop the usual integration scheme (solving the direct and inverse spectral problem) for these CA; this is due to two technical reasons:

1. these CA are ”filter” CA [2]: i.e. their evolved state at a given time $t$ at a given lattice position $n$ depends not only on the state of these CA at time $t-1$ but also on the ”evolved” state of the CA themselves at the positions $j < n$; thus, although these CA are computable, their evolution is not given in an explicit form;
2. it was impossible to find a Jost solution of the related SDSP at $-\infty$, due to the difficulty of projecting rational expressions on a finite field.

In this paper we introduce three new CA associated to SDSP, and therefore endowed with an infinite (countable) set of constants of motion, which (separately) overcome these two difficulties: indeed the first one (CA1) is a standard CA (not a "filter" one) and the second and the third ones (CA2a, CA2b) are direct projections of the Backlund Transformation (BT) of the Kac-van Moerbeke Lattice [4] on a finite field. This very natural way to obtain a CA associated to a discrete spectral problem, namely to project the related BT on a finite field, is usually not exploited, due to the presence of denominators in the BT itself: in this paper we show how this difficulty can be overcome, by selecting properly the modulus of the finite field and the parameterization of the fields to be projected. We believe that these results are important as a necessary (if preliminary) step in order to develop a fully integration scheme in finite fields (work is in progress in this direction).

## 2 Spectral problem, associated CA and their constants of motion

First we show how to construct CA from the Schroedinger discrete spectral problem, namely:

$$\psi(n-1,z)+u(n)\psi(n+1,z)=(z+z^{-1})\psi(n,z); \tag{2.1a}$$

where

$$n\in\mathbb{Z};\ \ z\in\mathbb{C}; \tag{2.1b}$$

$$u(n)\underset{|n|\longrightarrow\infty}{\longrightarrow}1\ . \tag{2.1c}$$

Assuming that the eigenfunction $\psi$ as well as the field $u$ depend also parametrically on the the discrete time variable $t$ ($t\in\mathbb{Z}$), we look for a compatible evolution of $\psi$ in the form:

$$\psi(n,z,t+1)=\sum_{j=0}^{M}U^{(j)}\left(n,t\right)\psi(n+2j,z,t)\ . \tag{2.2}$$

Even if the results exhibited in the following could be easily extended to the general case, for the sake of simplicity we will restrict considerations only to the case $M=1, U^{(0)}=1, U^{(1)}=U$ ; in this simple case compatibility between (2.1) and (2.2) yields:

$$u(n,t+1)+U(n-1,t)=u(n,t)+U(n,t)\ , \tag{2.3a}$$

$$u(n,t+1)U(n+1,t)=u(n+2,t)U(n,t)\ . \tag{2.3b}$$

The ordinary solution of (2.3) obtains solving f.i. eq. (2.3a ) in terms of $U$ and then inserting the obtained expression of $U$ in eq. (2.3b), or (equivalently, even if the explicit solutions looks quite different), solving first eq. (2.3b) and then eq. (2.3a). These two solutions are the simplest Backlund Transformations of the Kac-van Moerbeke lattice [4] :

$$u(n,t+1) = u(n,t) + \left( \frac{u(n+1,t)}{u(n-1,t+1)} - 1 \right) \left( b + \sum_{j=-\infty}^{n-1} (u(j,t+1) - u(j,t)) \right), \quad (2.4a)$$

$$u(n,t+1) = u(n,t) + b \left( \frac{u(n+1,t)}{u(n-1,t+1)} - 1 \right) \prod_{j=-\infty}^{n-1} \frac{u(j+1,t)}{u(j-1,t+1)}. \quad (2.4b)$$

where $b$ is an arbitrary constant (the BT parameter).

In order to view eqs. (2.4) as defining a CA, one should project them on a finite ring, say $Z_m = Z/m = \{0,1,..,m-1\}$, thus the problem to project possibly null denominators arises. In [3] a strategy to overcome this difficulty was developed: eqs. (2.3) were considered as " modulo congruences" an in particular eq.(2.3a) was solved in terms of $U$ in the finite ring $Z_m$ equating each member of the equation itself to zero. Such peculiar solution reads (here and in the following the symbol $\stackrel{m}{=}$ denotes a $m$-modulo congruence):

$$U(n,t) \stackrel{m}{=} h\ \delta\left(u(n-1,t+1)\right) \delta\left(u(n+2,t)\right), \quad (2.5)$$

where $h$ , although assumed to be a constant in the following, could be in principle an arbitrary function of the field $u$, while the modulo_delta function is defined as :

$$\delta(x) \stackrel{m}{=} 0 \ \ iff\ x \neq 0 \quad \mod\ m,\ x \in Z_m. \quad (2.6)$$

If the modulus $m$ is a prime number (but also in other cases), it admits the following simple representation, :

$$\delta(x) \stackrel{m}{=} (m-1) \prod_{k=1}^{m-1} (x+k), \quad (2.7)$$

Note that, if the modulus $m$ is a prime number, then

$$\delta(0) \stackrel{m}{=} 1. \quad (2.8)$$

Inserting (2.5) into (2.3a) an interesting "filter" CA obtains (CA0):

$$\begin{aligned} u(n,t+1) &\stackrel{m}{=} u(n,t) + h\ \delta\left(u(n-1,t+1)\right) \delta\left(u(n+2,t)\right) + \\ &+ h(m-1)\ \delta\left(u(n-2,t+1)\right) \delta\left(u(n+1,t)\right), \end{aligned} \quad (2.9)$$

whose properties were investigated in [3]. However the " filter" and thus rather implicit nature of CA0 is disturbing in order to fully investigate the mathematical properties of the CA itself.

As a first result of this paper we show that, within the same strategy adopted in [3], there is an alternative solution of eqs. (2.3) that allows to construct a new CA (CA1) which is "standard", i.e. a non "filter" one. Indeed inserting (2.3a) into (2.3b), one gets:

$$(u(n,t) + U(n,t) - U(n-1,t))\, U(n+1,t) = u(n+2,t) U(n,t). \quad (2.10)$$

It is straightforward to see that the above equation admits, in the finite ring $Z_m$, the following solution:

$$U(n,t) \overset{m}{=} h\ u(n,t)u(n+1,t)\delta\left(u(n-1,t)\right)\delta\left(u(n+2,t)\right), \tag{2.11}$$

where $\delta(x)$ is given by (2.6,2.7) and $h$ is again, in principle, an arbitrary function of the field (however, for the sake of simplicity, we will assume it to be a constant). Inserting (2.11) in (2.3a) , the following "non filter" CA obtains (CA1):

$$u(n,t+1) \overset{m}{=} u(n,t) + h\ u(n,t)u(n+1,t)\delta\left(u(n-1,t)\right)\delta\left(u(n+2,t)\right) +$$

$$+h\ (m-1)u(n-1,t)u(n,t)\delta\left(u(n-2,t)\right)\delta\left(u(n+1,t)\right). \tag{2.12}$$

Note that CA1 loses the symmetry

$$u(n+j,t) \leftrightarrow u(n-j,t+1), \tag{2.13}$$

which implied the "time reversibility of CA0 (see [3]); however CA1 is parity-invariant and moreover it possess a countable set of constants of motion (see below). For the sake of completeness, let us exhibit the explicit form of CA1 in the simplest non trivial case $(h=1,\ m=2)$:

$$u(n,t+1) \overset{2}{=} u(n,t) + u(n,t)u(n+1,t)\left(1+u(n-1,t)\right)\left(1+u(n+2,t)\right) +$$

$$+u(n-1,t)u(n,t)\left(1+u(n-2,t)\right)\left(1+u(n+1,t)\right). \tag{2.14}$$

Now, as a second result of this paper, we show how to project directly the BT (2.4) on a finite field $Z_m$ obtaining two new "filter" automata (CA2a, CA2b). In our opinion this direct projection should preserve all the properties of the integrability scheme (2.1,2.2). To do so, we need the following *Theorem* (it should be well known to the experts in number-theory, however, for the sake of completeness and for the convenience of the reader, we give also a neat proof of the theorem itself).

**Theorem 1.** *Consider the finite ring* $Z_m = Z/m = \{0,1,..,m-1\}$ *with:*

$$m = p^k,\ k \in \mathbb{N},\ p = prime. \tag{2.15}$$

*Consider also the set* $Y \subset Z_m$ *of all the elements:*

$$y_s \overset{m}{=} s \cdot\ p\ ,\ s = 0,1,... \tag{2.16}$$

*Then:*

*1)* $Y$ *is a (sub)ring (* $\mod\ m$);

*2) its complementary* $X = Z_m - Y$ *is a group;*

*3)* $(x + y \cdot z) \in X,$ *for any* $x \in X,\ y \in Y,\ z \in Z_m$.

**Proof.** 1) It is evident that $Y$ is closed with respect to addition and multiplication, see (2.16).

2) Obviously $1 \in X$ (see again (2.16)). Moreover we know from elementary number theory that the modulo congruence $a \cdot z \overset{m}{=} c$ admits exactly $(a, m)$ different solutions $z \in Z_m$ $iff$ $(a, m)$ divides $c$ (here and in the following $(a, b)$ denotes the *h.f.c.* of $a$ and $b$). Since obviously $(x, m) = 1$ for any $x \in X$, it follows that any element of $X$ has a unique inverse in $Z_m$. On the contrary, no element of $Y$ can have an inverse ($(y, m) = \left(s \cdot p, p^k\right) \neq 1$): thus, since $X \cap Y = 0,\ X \cup Y = Z_m$, all elements of $Z_m$ having an inverse must belong to $X$ . It is also obvious that $X$ is closed (with respect to multiplication).

3) Since it is evident that $(x + y \cdot z)$ cannot belong to $Y$, then it must belong to $X$ (again from $X \cap Y = 0,\ X \cup Y = Z_m$) ■

Let us now parametrize the field $u$ in the following way:

$$u(n, t) = c_1 + c_2\, q(n, t), \tag{2.17}$$

where $c_1,\ c_2$ are (up to now) arbitrary constants.

As a consequence of the above *Theorem*, we can project directly the BTs (2.4) in the finite ring $Z_m, m = p^k,\ k \in \mathbb{N},\ p = prime$, provided that we choose $c_1 \in X,\ c_2 \in Y$ . However, due to (2.1c), we have to assume:

$$c_1 = 1,\ q(n, t) \underset{|n| \longrightarrow \infty}{\longrightarrow} 0\ . \tag{2.18}$$

In terms of the new field $q$, we obtain from (2.4a) the new cellular automaton CA2a:

$$q(n, t+1) \overset{m}{=} q(n, t) + $$
$$+ \left( \frac{q(n+1, t) - q(n-1, t+1)}{1 + c_2\, q(n-1, t+1)} \right) \left( b + c_2 \sum_{j=-\infty}^{n-1} (q(j, t+1) - q(j.t) \right), \tag{2.19}$$

and, from (2.4b), the new cellular automaton CA2b

$$q(n, t+1) \overset{m}{=} q(n, t) + $$
$$+ b \left( \frac{q(n+1, t) - q(n-1, t+1)}{1 + c_2\, q(n-1, t+1)} \right) \prod_{j=-\infty}^{n-1} \frac{1 + c_2\, q(j+1, t)}{1 + c_2\, q(j-1, t+1)}. \tag{2.20}$$

Note that these two new cellular automata are again "filter" CA. However, looking at the above definitions of these two CA and noting that the evolved states $q(j, t+1)$ appear only for $j < n$ and taking into account the condition $q(n, t) \underset{n \longrightarrow -\infty}{\longrightarrow} 0$, it is clear that $q(n, t+1)$ can be computed for any $n$: thus the above CA are computable (starting from $-\infty$) at least until the condition $q(n, t) \underset{n \longrightarrow +\infty}{\longrightarrow} 0$ is fulfilled. However a deeper investigation of the dynamics of the CA introduced in this paper is postponed to a subsequent paper.

CA1, CA2a, CA2b (as well as CA0) are related to the same linear problem (2.1,2.2), indeed they solve the compatibility conditions (2.3), thus they share the same infinite (countable) set of constants of motion. This is well known from the literature and it has also been proved in [3], [5], where the set of constants of motion was explicitly derived for CA0; here we report just the first conserved quantities (in terms of the above parametrization):

$$K_1 \stackrel{m}{=} \sum_{s=-\infty}^{+\infty} q(s,0), \tag{2.21}$$

$$K_2 \stackrel{m}{=} \sum_{s=-\infty}^{+\infty} q(s,0) \sum_{r=s+2}^{+\infty} q(r,0), \tag{2.22}$$

$$K_3 \stackrel{m}{=} \sum_{s=-\infty}^{+\infty} q(s,0) \left( c_2 \sum_{r=s+2}^{+\infty} q(r,0) \sum_{l=r+2}^{+\infty} q(l,0) - \sum_{r=s+3}^{+\infty} q(r,0) \right). \tag{2.23}$$

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