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Interpolation of entire functions, product formula for basic sine function

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Abstract

We solve the problem of constructing entire functions where $\ln M(r; f)$ grows like $\ln^2 r$ from their values at q^{-n}, for $0 < q < 1$. As application we give a product formula for the basic sine function.

1 Introduction

In [9], Ismail and Zhang introduced the q-analogue of the exponential and trigonometric functions. They used transform formula to analytically continue to entire functions in the variable ω. Suslov (see [14]) identified a special case which leads to a comprehensive orthogonal system of functions. This opened the door for a comprehensive study of q-Fourier series, where q-analogues of some results in classical Fourier series have been proved (see [14]). In this paper we give a product formula for the basic sine function.

In this work we mostly follow the terminology of [4]. We will always assume $0 < q < 1$. We first remind the reader of the notations to be used. A q-shifted factorial is defined by (see [4])

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1}(1 - aq^k), \quad n = 1, 2, ...\infty \tag{1.1}$$

and more generally

$$(a_1, ..., a_s; q)_n = \prod_{1=0}^{s}(a_k; q)_n, \quad n = 0, 1, 2, ...\infty \tag{1.2}$$

A basic hypergeometric series is

$${}_r\varphi_s(a_1, ..., a_{r+1}; b_1, ..., b_r; q, z) = \sum_{k=0}^{\infty} \frac{(a_1, ..., a_r; q)_k}{(b_1, ..., b_s, q; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k. \tag{1.3}$$

Given a function f defined on $(-1,1)$, we set $\breve{f}(e^{i\theta}) := f(x),\ x = \cos\theta$. In other words we think of $f(\cos\theta)$ as a function of $e^{i\theta}$. In this notation the Askey-Wilson finite difference operator $\mathcal{D}_q$ is defined by

$$(\mathcal{D}_q f)(x) = \frac{\breve{f}(q^{\frac{1}{2}}e^{i\theta}) - \breve{f}(q^{-\frac{1}{2}}e^{-i\theta})}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})i\sin\theta}. \tag{1.4}$$

2 Interpolation of entire functions

In a recent work, M.E.H.Ismail and D.Stanton (see [7]) solved the problem of constructing entire functions from their values at $\frac{1}{2}[aq^n + \frac{1}{aq^n}]$, for entire functions satisfying

$$\lim_{r\longrightarrow\infty} \sup \frac{\ln M(r;f))}{\ln^2 r} = c, \tag{2.1}$$

for a particular c which depends upon q. Here $M(r;f)$ is

$$M(r;f) = \sup\{|f(z)| : |z| \le r\}.$$

In this section, we adopt their method to solve interpolation problem for the sequence

$$\{q^{-n}, n = 0, 1, ...\}.$$

Let us begin by the following lemma:

Lemma 1. *The Cauchy's kernel* $\frac{1}{y-x}$ *has the expansion*

$$\frac{1}{y-x} = \frac{(x;q)_\infty}{(y-x)(y;q)_\infty} + \sum_{k=0}^{\infty} \frac{(x;q)_k}{(y;q)_{k+1}} q^k,$$

for all y *such that* $y \neq x$ *and* $y \neq q^{-n},\ n = 0, 1, ...$

Proof. By induction on n, one proves easily that for $y \neq x$ and $y \neq q^{-n},\ n = 0, 1, ...$, we have

$$\frac{1}{y-x} = \frac{(x;q)_{n+1}}{(y-x)(y;q)_{n+1}} + \sum_{k=0}^{n} \frac{(x;q)_k}{(y;q)_{k+1}} q^k.$$

The result follows when we tend n to ∞. ■

Theorem 2. *Let* f *be an analytic function in a bounded domain* D *and let* C *be a contour within* D *and* x *belongs to the interior of* C*. If the contour* C *is at a positive distance from the set* $\{q^{-n}; n = 0, 1, ...\}$*, then*

$$f(x) = \frac{(x;q)_\infty}{2i\pi} \int_C \frac{f(y)}{(y-x)(y;q)_\infty} dy + \frac{1}{2i\pi} \sum_{k=0}^{\infty} q^k f_k (x;q)_k,$$

where

$$f_k = \int_C \frac{f(y)}{(y;q)_{k+1}} dy$$

Proof. Multiply the first expansion in Lemma 1 by $f(y)$, integrate with respect to y and interchange integration and summation, the result follows from Cauchy's theorem. ■

Theorem 3. *Any entire function f satisfying (2.1) with $c < \dfrac{1}{2\ln q^{-1}}$ has a convergent expansion*

$$f(x) = \sum_{n=0}^{\infty} f_n (x;q)_n.$$

Moreover any function f is uniquely determined by its values on $\{q^{-n} : n \geq 0\}$.

To prove Theorem 3, let us first state and prove the following lemma:

Lemma 4. *Let $-1 < \delta < 0$, and f be entire function satisfying (5) with $c < \dfrac{1}{2\ln q^{-1}}$. Then*

$$\lim_{n\longrightarrow\infty} \int_{|y|=q^{-n-\delta}} \frac{f(y)}{(y-x)(y;q)_\infty} dy = 0.$$

Moreover, the same conclusion holds if

$$\lim_{n\longrightarrow\infty} q^{n(n+2\delta+1)/2} \sup\left\{f(q^{-n-\delta}e^{i\theta}) : 0 \leq \theta \leq 2\pi\right\} = 0. \tag{2.2}$$

Proof. It is clear that $\inf\{(y;q)_\infty : |y| = r\} = |(r;q)_\infty|$. Hence for $|y| = q^{-n-\delta}$, we have

$$\begin{aligned} |(y;q)_\infty| &\geq \left|(q^{-n-\delta};q)_n (q^{-\delta};q)_\infty\right|, \\ &= q^{-n(n+2\delta+1)/2} (q^{\delta+1};q)_n (q^{-\delta};q)_\infty, \end{aligned}$$

and the result follows. ■

Instead of proving the expansion in Theorem 3 in the basis $\{(x;q)_n\}$, we shall prove the following equivalent result:

Theorem 5. *The expansion formula*

$$f(x) = \sum_{n=0}^{\infty} q^n f_n (x;q)_n,$$

with

$$f_n = \frac{1}{(q;q)_n} \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k} (-1)^k q^{\binom{k}{2}} f(q^{-k}),$$

for functions f satisfying the assumptions of Lemma 4.

Proof. Let C_m be a circle centered at $y = 0$ with radius $q^{-m-\delta}$. The Lemma 4 shows that the first integral in Theorem 2 is small if m is large. We split the remaining terms with $n > m$, and initial terms with $n \leq m$. We will show that the tail is small, leaving the initial terms. Then a residue calculation establishes the expression for f_n, because the poles of $\frac{f(y)}{(y;q)_{n+1}}$ are at $y = q^{-k},\ k = 0, 1, ..., n$.

Note that if $n > m$ then

$$\begin{aligned}
\min\{|(y;q)_{n+1}| \quad &: \quad |y| = q^{-m-\delta}\} \\
&= (q^{-m-\delta};q)_m(q^{-\delta};q)_{n+1-m} \\
&= q^{-m(m+2\delta+1)}(q^{\delta+1};q)_m(q^{-\delta};q)_{n+1-m} \\
&\geq q^{-((m+\delta)^2+1-\delta^2)/2}A,
\end{aligned}$$

where A is a positive constant independent of n and m. Therefore for sufficiently large m, and $|y| = q^{-m-\delta}$,

$$\ln[M(q^{-m-\delta}, \frac{f(y)}{(y;q)_{n+1}}] \leq [c_1 + \frac{1}{2\ln q}]\ln^2(q^{-m-\delta}) + O(m)$$

for some $c_1, c \leq c_1 \leq \frac{1}{2\ln q^{-1}}$.

This is a uniform bound of $e^{-D(\ln q^{-m-\delta})^2},\ D > 0$, for each integral for $n > m$. Since $(x;q)_n \longrightarrow (x;q)_\infty$, there is a uniform bound B for $(x;q)_n$ on compact sets. Thus the tail is bounded by

$$\sum_{n=m+1}^{\infty} Bq^n e^{-D(\ln q^{-m-\delta})^2} \leq B\frac{q^{m+1}}{1-q}e^{-D(\ln q^{-m-\delta})^2},$$

which is small for m large. ■

Theorem 6. *Let f be entire function satisfying (2.1) with $c < \frac{1}{2\ln q^{-1}}$. Then*

$$\frac{f(x)}{(x;q)_\infty} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}}}{(q;q)_n(q;q)_\infty}\frac{f(q^{-n})}{1-q^n x}.$$

Proof. Consider

$$I_m := \int_{|y|=q^{-m-\delta}} \frac{f(y)}{(y-x)(y;q)_\infty}dy.$$

¿From Lemma 4, $I_m \longrightarrow 0$ as $m \longrightarrow \infty$. On the other hand

$$I_m = \frac{f(x)}{(x;q)_\infty} - \sum_{n=0}^{m} \frac{(-1)^n q^{\binom{n+1}{2}}}{(q;q)_n(q;q)_\infty}\frac{f(q^{-n})}{1-q^n x},$$

and the Theorem follows. ■

Theorem 7. *Let the complex numbers $b_1, ...b_m$, satisfy the estimate*

$$| \, b_1 ... b_m \, | \, q^{\frac{m(1-m)}{2}} < 1, \quad m = 0, 1, ...$$

Let f be an entire function satisfying

$$M(q^{-ms-\delta}, f) \leq M(q^{ms-\delta}, (b_1 z, ..., b_m z; q^m)_\infty).$$

Then

$$\frac{f(x)}{(x;q)_\infty} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}}}{(q;q)_n (q;q)_\infty} \frac{f(q^{-n})}{1-q^n x}.$$

Proof. We have

$$\begin{aligned} M(q^{-ms-\delta}; (b_1 z, ..., b_m z; q^m)_\infty) &\leq \prod_{j=1}^{m} (-|b_j| \, q^{-ms-\delta}; q^m)_\infty, \\ &\leq \prod_{j=1}^{m} (-|b_j| \, q^{-ms-\delta}; q^m)_s (-|b_j| \, q^{-\delta}; q^m)_\infty, \end{aligned}$$

so that

$$M(q^{-ms-\delta}; f) q^{ms(ms+2\delta+1)/2} \leq C(|b_1 ... b_m| \, q^{\frac{m(1-m)}{2}})^s,$$

where C is a constant depending only on $b_1, ..., b_m, \delta$ but not on s. ■

3 Product formula for q-sine function

We start by a q-exponential function, defined in [9] as

$$\begin{aligned} \mathcal{E}_q(\cos\theta, \omega) &= \frac{(\alpha^2; q^2)_\infty}{(q\alpha^2; q^2)_\infty} \sum_{n=0}^{\infty} (-ie^{\iota\theta} q^{\frac{(1-n)}{2}}, -ie^{-\iota\theta} q^{\frac{(1-n)}{2}}; q)_n \\ & \frac{(i\omega)^n}{(q;q)_n} q^{\frac{n^2}{4}}. \end{aligned} \tag{3.1}$$

The following functions $C_q(x;\omega)$ and $S_q(x;\omega)$ given by

$$\begin{aligned} C_q(x;\omega) &= \frac{(-\omega^2; q^2)_\infty}{(-q\omega^2; q^2)_\infty} \\ & {}_2\varphi_1(-qe^{2i\theta}, -qe^{-2i\theta}; q; q^2, -\omega^2) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} S_q(x;\omega) &= \frac{(-\omega^2; q^2)_\infty}{(-q\omega^2; q^2)_\infty} \\ & \frac{2q^{\frac{1}{4}}\omega}{1-q} \cos\theta {}_2\varphi_1(-q^2 e^{2i\theta}, -q^2 e^{-2i\theta}; q^3; q^2, -\omega^2), \end{aligned} \tag{3.3}$$

were discussed recently in [14] as q-analogues of $\cos\omega x$ and $\sin\omega x$ on a q-quadratic lattice $x=\cos\theta$. The functions $C_q(x;\omega)$ and $S_q(x;\omega)$ are defined for $|\omega|<1$ only. For an analytic continuation of these functions in a large domain see [9],[14]. For example,

$$\begin{aligned} C_q(x;\omega) &= \frac{(q\omega^2e^{2i\theta},q\omega^2e^{-2i\theta};q^2)_\infty}{(q,-q\omega^2;q^2)_\infty} \\ &\times {}_2\varphi_2(-\omega^2,-q\omega^2;q\omega^2e^{2i\theta},q\omega^2e^{-2i\theta};q^2,q), \end{aligned} \tag{3.4}$$

$$\begin{aligned} S_q(x;\omega) &= \frac{(q^2\omega^2e^{2i\theta},q^2\omega^2e^{-2i\theta};q^2)_\infty}{(q^3,-q\omega^2;q^2)_\infty}\frac{2q^{\frac14}\omega}{1-q} \\ &\times {}_2\varphi_2(-\omega^2,-q\omega^2;q^2\omega^2e^{2i\theta},q^2\omega^2e^{-2i\theta};q^2,q^3), \end{aligned} \tag{3.5}$$

The notation for $S_q(x;\omega)$ is the same as the ones proposed by Suslov in [14]. The q-sine function satisfies the q-difference equation (see[14])

$$\mathcal{D}_q^2S_q(x;\omega)=-\frac{\omega^2q^{\frac12}}{(1-q)^2}S_q(x;\omega). \tag{3.6}$$

Suslov established the continuous orthogonality relations for the q-sine function(see [14]),

$$\int_0^\pi S_q(\cos\theta;\omega)S_q(\cos\theta;\omega^{'})\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(q^{\frac12}e^{2i\theta},q^{\frac12}e^{-2i\theta};q)_\infty}d\theta \quad = \quad 0$$

and

$$\begin{aligned} &\int_0^\pi S_q^2(\cos\theta;\omega)\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(q^{\frac12}e^{2i\theta},q^{\frac12}e^{-2i\theta};q)_\infty}d\theta \\ = \quad &\pi\frac{(q^{\frac12},-q^{\frac12}\omega^2;q)_\infty}{(q,-\omega^2;q)_\infty}\frac{(-\omega^2;q^2)_\infty}{(-q\omega^2;q^2)_\infty} \\ &\times{}_2\varphi_1\left(-q^{\frac12},\omega^2;-q^{\frac12}\omega^2;q,q\right). \end{aligned}$$

Here ω and $\omega^{'}$ are different solutions of the equation

$$S(\frac12(q^{\frac14}+q^{-\frac14});\omega)=0.$$

The continuous q-Hermite polynomials is defined by (see [13])

$$H_n(\cos\theta\mid q)=\sum_{k=0}^n\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}e^{i(n-2k)\theta}. \tag{3.7}$$

The continuous q-Hermite polynomials satisfy the q-difference equation (see [13])

$$\frac{1}{w(x)}\mathcal{D}_q[w(x)\mathcal{D}_qy(x)]=-4q^{-n+1}\frac{1-q^n}{(1-q)^2}y(x).$$

and the product formula (see [12])

$$H_n(x \mid q)H_n(y \mid q) = \frac{(q;q)_\infty}{2\pi t^n}\int_0^\pi K_t(\cos\theta,\cos\phi,\cos\psi) \tag{3.8}$$
$$\times H_n(\cos\psi \mid q)(e^{2i\psi},e^{-2i\psi};q)_\infty d\psi$$

where

$$K_t(\cos\theta,\cos\phi,\cos\psi) \tag{3.9}$$
$$= \frac{(t^2e^{2i\psi};q)_\infty}{(e^{-2i\psi},te^{i(\theta+\phi+\psi)},te^{i(\theta-\phi+\psi)},te^{i(\phi+\psi-\theta)},te^{i(-\theta-\phi+\psi)};q)_\infty}$$
$$\times\ {}_6\varphi_5\left(\begin{matrix} te^{i(\theta+\phi+\psi)},te^{i(\theta-\phi+\psi)},te^{i(\psi+\phi-\theta)},te^{i(-\theta-\phi+\psi)},0,0 \\ qe^{2i\psi},te^{i\psi},-te^{i\psi},\sqrt{q}te^{i\psi},-\sqrt{q}te^{i\psi}\end{matrix}\ \middle|\ q,qe^{i\psi)}\right)$$
$$+ \text{ a similar terms with } \psi \text{ replaced by } -\psi.$$

In the following proposition, we show that the function $\tilde{S}_q(x;\omega)$ defined by

$$\tilde{S}_q(x;\omega) = \frac{(1-q)\left(q^3,q\omega^2;q^2\right)_\infty S_q(x;i\omega)}{2q^{1/4}x(-q^2e^{2i\theta},-q^2e^{-2i\theta};q^2)_\infty},$$

is a nonterminating extension of the continuous q-Hermite polynomials.

Proposition 8. *For $n = 0,1,2,...$we have*

$$\tilde{S}_q(x;q^{-n}) = iq^{-n^2}H_{2n}(x \mid q).$$

Proof. ¿From (Theorem 2.2, [8]), we have

$$H_n(x \mid q) = \sum_{k=0}^n c_k\psi_k(x),$$

where

$$c_k = \frac{q^{\frac{k^2-k}{4}}(1-q)^k}{2^k(q;q)_k}(\mathcal{D}_q^k H_n(x \mid q))(0)$$

and

$$\psi_k(x) = (1+e^{2i\theta})(-q^{2-n}e^{2i\theta};q^2)_{n-1}e^{-in\theta}.$$

In the other hand

$$H_{2n+1}(0 \mid q) = 0, \quad H_{2n}(0 \mid q) = (-1)^n(q;q^2)_n$$

and

$$\mathcal{D}_q^k H_n(x \mid q) = (\frac{2}{1-q})^k q^{-\frac{1}{2}(\binom{n}{2}-\binom{n-k}{2})}\frac{(q;q)_n}{(q;q)_{n-k}}H_{n-k}(x \mid q).$$

Therefore $H_{2n}(x \mid q)$ has the q-Taylor expansion

$$H_{2n}(x \mid q) = \sum_{k=0}^n(-1)^{n-k}\frac{(q;q)_{2n}q^{2k(k-n)}}{(q;q)_{2k}(q^2;q^2)_{n-k}}\psi_{2k}(x).$$

After some computations we get the proposition. ∎

Now put

$$k\left(\omega;q\right)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2+n}}{(q^2;q^2)_n(q^2;q^2)_\infty}\frac{\left(\omega^2;q^2\right)_\infty}{1-q^n\omega^2},$$

is no difficult to see that the function $k\left(\omega;q\right)$ is entire and satisfy

$$k\left(q^{-n};q\right)=q^{n^2},\ n=0,1,\dots\ .$$

In the following proposition we establish a product formula for the basic function.

Proposition 9. *The q-sine function satisfy the product formula*

$$\begin{aligned}S_q(\cos\theta;\omega)S_q(\cos\phi;\omega) &= \int_0^\pi \Delta(\cos\theta,\cos\phi,\cos\psi)\\ &\quad\times S_q(\cos\psi;\omega)(e^{2i\psi},e^{-2i\psi};q)_\infty d\psi\end{aligned}$$

where

$$\begin{aligned}\Delta(\cos\theta,\cos\phi,\cos\psi) &= \frac{2iq^{1/4}(-q^2e^{2i\theta},-q^2e^{-2i\theta},-q^2e^{2i\phi},-q^2e^{-2i\phi};q^2)_\infty}{\pi k\left(\omega;q\right)(1-q)\left(q^3,q\omega^2;q^2\right)_\infty(-q^2e^{2i\psi},-q^2e^{-2i\psi};q^2)_\infty}\\ &\quad \frac{\cos\theta\cos\phi}{\cos\psi}K_1(\cos\theta,\cos\phi,\cos\psi).\end{aligned}$$

Proof. Put

$$\begin{aligned}g(\omega) &= k\left(i\omega^5;q^5\right)\tilde{S}_{q^5}(\cos\theta;\omega^5)\tilde{S}_{q^5}(\cos\phi;\omega^5)\\ &\quad -\int_0^\pi iK_1(\cos\theta,\cos\phi,\cos\psi,q^{10})\\ &\quad \times\tilde{S}_{q^5}(\cos\psi;\omega^5)(e^{2i\psi},e^{-2i\psi};q^{10})_\infty d\psi,\end{aligned}$$

It is easy to show that the function g is entire and from proposition 9 and the product formula (13), we have

$$g(q^{-n})=0,\ n=0,1,2,.....$$

By (3.5), we have

$$M(q^{-s-\delta/10},g)\leq CM(q^{-10s-\delta},(q^5z;q^{10})_\infty^{10}).$$

Then according to the Theorem 7, we have

$$g=0.$$

■

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