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Note on the evolution of compactly supported initial data under the Camassa-Holm flow

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Abstract

We clarify and extend some remarks raised in [5] [Constantin A, *J. Math. Phys.* **46** (2005), 023506] about the evolution of compactly supported initial data under the Camassa-Holm flow.

1 Introduction

The equation of Camassa and Holm [3, 4] is an approximate one-dimensional description of long waves in shallow water. It reads

$$m_t + 2mu_x + m_x u = 0\,, \quad x \in \mathbb{R}, \quad t \geq 0. \tag{1.1}$$

in which $m := u - u_{xx}$: in extenso,

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}\,, \quad x \in \mathbb{R}, \quad t \geq 0. \tag{1.2}$$

Its Eulerian form is more attractive: In terms of the Green's function $(1 - \partial_x^2)\mathrm{G} = \delta$, i.e. $\mathrm{G} := e^{|\cdot|}/2$, it reads

$$u_t + uu_x + p_x = 0 \quad \text{with the "pressure"} \quad p := \mathrm{G} * [u^2 + \frac{1}{2}u_x^2]\,, \tag{1.3}$$

i.e. a conservation law with nonlocal flux reminiscent of the three-dimensional incompressible equations

$$u_t + (u \cdot \mathrm{grad})u + \mathrm{grad}\, p = 0\,,$$

in which $-\triangle p \equiv \mathrm{trace}\{(\partial u/\partial x)^2\}$ and $\mathrm{G} = 1/(4\pi|\cdot|)$ is inverse to $-\triangle$ so that $p = \mathrm{G} * [\mathrm{trace}\{(\partial u/\partial x)^2\}]$. It is important to emphasize the Lagrangian standpoint, tracking the moving "fluid" in the natural characteristic scale $\varphi = \varphi(t,x)$ determined by

$$\varphi_t = u(t,\varphi) = u(t) \circ \varphi(t)\,, \quad \varphi(0,x) = x\,, \quad x \in \mathbb{R}, \quad t \geq 0. \tag{1.4}$$

i.e. a diffeomorphism of the real line issuing from the identity. In this language, the CH equation is expressed by a dynamical system for $(\varphi, v := u \circ \varphi)$ from which, following the seminal papers of Arnold [1] and Marsden and Ebin [13], the initial value problem is most easily studied cf. [12] and the references therein. The Lagrangian view point has also been exploited to produce explicit formulas for the updated profile in terms of initial data (cf. [16, 17, 19]).

This model was first noticed by Fokas and Fuchssteiner [14], by the method of recursion operators, as a formally integrable bi-Hamiltonian generalization of KdV, but it became the subject of serious study after it was revamped from *physical* principles by Camassa and Holm [3], see also Johnson [15]. Unlike this well-known ancestor, which is produced by approximation at the leading edge, CH was found in the course of approximating directly in the Hamiltonian for Euler's equations in the shallow water regime (more recently, (1.3) has risen also as a model for nonlinear waves in cylindrical axially symmetric hyperbolic rods, with v representing the radial stretch relative to a pre-stressed state [11]). It is a good approximation for the full inviscid water wave equation, just as consistent in the small amplitude, shallow water regime, as KdV. But more is true, the CH equation is remarkable, as compared to KdV, (a) for its peaked solitons (which are stable and thus physically observable [10]) and the simplicity of their interactions (cf. [2] for explicit formulas describing them), (b) for its equivalence to the geodesic flow in the group of compressible diffeomorphisms of the line, and (c) for the presence of breaking waves. Nonetheless, in the case where no solitons are present, KdV and CH share a deeper kindship than their respective derivations might have suggested, as was elegantly unveiled by McKean [20], who established via a Liouville-Lagrangian map how the series of CH invariants and their respective flows, alias the CH hierarchy, correspond to their KdV counterparts.

Here we will not entertain with any of these. The present note is to clarify and extend some remarks about the finite propagation speed for the Camassa-Holm equation discussed by Constantin [5]. Due to the nonlocal nature of (1.3), which is equivalent to the extended version (1.2), it is not a priori clear that a localized initial data $u_0 := u(0, \cdot)$, namely one which is compactly supported, will not spread out eventually (or even instantly) to the whole spatial domain. Below we give a concise proof of this property refining the argument of [5], and elaborate on some of its implications.

2 Main Results

As mentioned implicitly in the introduction, CH satisfies the least-action principle as it corresponds to geodesic flow, on the group of compressible diffeomorphism on the line, with respect to the right-invariant Sobolev H^1-metric assimilated as the energy. Now Noether's theorem guarantees the existence of a first integral from each one-parameter subgroup that leaves the energy functional unchanged. By right invariance, the elements of every orbit issuing from the identity constitute such a subgroup and since these are plenty (one such for each initial direction in the tangent space at the identity alias the Lie algebra associated to the group) the infinite collection of associated invariants actually corresponds to an identity, cf. [8, 9].

Theorem 1. *Let u denote a solution of the Camassa-Holm equation* (1.2)*, the latter being a re-expression of geodesic flow in the group of compressible diffeomorphisms of the line φ satisfying* (1.4)*. Then, as seen by Noether's principle, the expression $\varphi_x^2(t,x)m(t,\varphi(t,x))$ where $m = u - u_{xx}$ is time-invariant, i.e. it is identically equal to $m_0 := u_0 - (u_0)_{xx}$.*

Indeed, it follows by direct computation that $[\varphi_x^2 m \circ \varphi]_t = \varphi_x^2 \cdot \{(1.1)\} \circ \varphi = 0$, i.e. as long as the diffeomorphism of the line holds (i.e., as long as $\varphi_x > 0$), the existence of the aforementioned first integral is a direct consequence of the fact that m satisfies the Camassa-Holm equation (1.1), and conversely, every solution of (1.1) gives rises (via the diffeomorphism of the line [issuing from the identity] specified by (1.4) with $u = \mathrm{G} * m$) to such an integral of motion.

In other words,

$$\varphi_x^2(t,x)m(t,\varphi(t,x)) = m_0(x)\,, \quad x \in \mathbb{R}\,, \quad 0 \le t < T\,, \tag{2.1}$$

where $T > 0$ denotes the maximal time of existence of a smooth solution of (1.1). It is clear from (2.1) that as long as the diffeomorphism of the line holds or, what is the same as long as breaking of the wave has not occurred (cf. [18] and [19]), the support of m_0 is contained in the interval $[x_-, x_+]$ if and only if, for any $0 \le t < T$, the support of $m(t,\cdot)$ is contained in the interval $[\varphi(t,x_-), \varphi(t,x_+)]$.

Remark. *Differentiating* (1.4) *once with respect to the spatial variable, switching the order of differentiations and integrating with respect to the time variable yields*

$$\varphi_x(t,x) = e^{\int_0^t u_x(s,\varphi(s,x))ds} \quad x \in \mathbb{R}\,, \quad 0 \le t < T\,,$$

which is consistent with the fact that breakdown of the wave (in finite time) is signaled by $u_x \downarrow -\infty$ as $t \uparrow T$, cf. [18]. On the other hand, combining suitably the first couple of spatial differentiations of (1.4) *and employing the invariant* (2.1) *produces*

$$\varphi_x^2 \varphi_t - \varphi_{txx} + \varphi_{tx}\varphi_{xx}/\varphi_x = m_0\,,$$

i.e. an interesting (inhomogeneous) PDE expressing the initial data in terms of the diffeomorphism.

Amplification. *For a classical solution of* (1.2)*, singularities can only arise in the the form of breaking waves [6]. Moreover, breakdown of solutions of* (1.2) *depends on the sign disposition of the initial profile m_0 (cf. [7], [18]). Hence, the PDE of the last display offers the possibility to translate the curious sign condition signaling breakdown purely in terms of the geodesic.*

We summarize the above in the following

Corollary 1. *Let $T > 0$ denote the maximal time of existence of the smooth solution $m(t,x)$, $0 \le t < T$, $x \in \mathbb{R}$, of the initial value problem* (1.1) *with compactly supported initial data m_0, itself a real valued smooth function on the real line. Then for any $0 \le t < T$, $m(t,x)$ has compact support.*

The point is that, as long as breakdown has not occurred, the solutions of the Camassa-Holm equation (1.1) *propagate at finite speed.* Indeed, this is a direct consequence of the fact that for any given $x \in \mathbb{R}$ and $0 \le t < T$, the speed of propagation of φ is commensurate with the size of u (cf (1.4)), which is pointwise bounded by the Sobolev H^1-norm $\mathrm{E}[u] := \int_{\mathbb{R}}(u^2 + u'^2) < +\infty$, (alias the energy functional) which is preserved under the flow (1.3). Now, were a smooth velocity profile u verifying (1.3) of compact support for any given $0 \le t < T$, so would be the corresponding $m = u - u_{xx}$ solution of (1.1), i.e. for any $0 \le t < T$, $\mathrm{supp}(m(t,\cdot)) \subset \mathrm{supp}(u(t,\cdot))$. In particular, if u_0 has compact support, so would have m_0. Let us assume for simplicity that the initial data consists of a single isolated lump, i.e. that the compactly supported u_0 is such that m_0 has connected compact support on which it is sign definite, say nonnegative. For such smooth initial data, the profile never breaks down [18], so in this case $T > 0$ is arbitrarily large. So let u_0 be of compact support such that $\mathrm{supp}(m_0) = [x_-, x_+]$ where $m_0 > 0$. By corollary (1) and the fundamental identity (2.1), $\mathrm{supp}(m(t,\cdot)) = [\varphi(t,x_-), \varphi(t,x_+)]$ where $m(t,\cdot) > 0$. Hence upon spelling out the relation $u = \mathrm{G} * m$ it develops

$$2u(t,x) = e^{-x}\int_{\varphi(t,x_-)}^{x} e^{y}m(t,y)dy + e^{x}\int_{x}^{\varphi(t,x_+)} e^{-y}m(t,y)dy\,. \tag{2.2}$$

The above display implies that $u(t,\cdot)$ can not be of compact support. For suppose that $u(t,x) \equiv 0$ for $x > \alpha_+ \ge x_+$ for some $\alpha_+ < +\infty$. For this set of values, the second term on the r.h.s. of (2.2) vanishes since the domain of integration lies beyond the support of $m(t,\cdot)$ as $\mathrm{supp}(m(t,\cdot)) \subset \mathrm{supp}(u(t,\cdot))$, while the first term reduces to $e^{-x}\int_{\varphi(t,x_-)}^{\varphi(t,x_+)} e^{y}m(t,y)dy > 0$ which is a contradiction since the l.h.s. of (2.2) is zero by assumption. Hence $u(t,\cdot)$ can not vanish identically to the right. By the same reckoning we establish that $u(t,\cdot)$ can not vanish identically to the left of some $-\infty < \alpha_- \le x_-$. In short, the condition that the smooth initial profile u_0 is of compact support and such that the corresponding m_0 constitutes an isolated positive lump, implies that $u(t,\cdot)$ *is not* of compact support.

In other words, the solution $u(t,\cdot)$ of the CH equation (1.3) instantly loses the property of having compact support.

Now, it is known that the long time development of suitable class of initial data (comprising the type employed in the argument above) run by the CH flow produces a train of solitons escaping at speeds commensurate with the spectral values of the associated spectral problem, $f_{xx} = (1/4 - \lambda m_0)f$ (cf. [19] for more details), and thus becoming widely separated as they disperse. Hence it was reasonable to expect that for a suitable class of compactly supported initial data, the corresponding updates would eventually fall out of this category. What is remarkable is that for the aforementioned initial data, the spreading of the profile over the whole real line switches on immediately thereafter.

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