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Groups of Order Less Than 32 and Their Endomorphism Semigroups

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Abstract

It is proved that among the finite groups of order less than 32 only the tetrahedral group and the binary tetrahedral group are not determined by their endomorphism semigroups in the class of all groups.

1 Introduction

It is well-known that all endomorphisms of an Abelian group form a ring and many of its properties can be characterized by this ring. An excellent overview of the present situation in the theory of endomorphism rings of groups is given by P.A.Krylov, A.V.Mikhalev and A.A.Tuganbaev in their book [3]. All endomorphisms of an arbitrary group form only a semigroup. The theory of endomorphism semigroups of groups is quite modestly developed. In many of our papers we have made efforts to describe some properties of groups by the properties of their endomorphism semigroups. For example, it is shown in [4] and [7] that a direct product of groups and some semidirect products of groups can be characterized by the properties of the endomorphism semigroups of these groups. In [7] it was shown that in many cases the question of the summability of two endomorphisms of a group can be fully characterized by the properties of its endomorphism semigroup. It is also shown that groups of many well-known classes are determined by their endomorphism semigroups in the class of all groups. Some of such groups are: finite Abelian groups ([4], Theorem 4.2), generalized quaternion groups ([5], Corollary 1). On the other hand, there exist many examples of groups that are not determined by their endomorphism semigroups: the alternating group A_4 of order 12 ([11], Theorem), some semidirect products of finite cyclic groups ([9], Theorem), some Schmidt's groups ([10], Theorem 3.3). Therefore, it is useful to know much more examples of groups which are or are not determined by their endomorphism semigroups. In this paper we give the full answer to this problem for the groups of order less than 32. We will prove the following theorem.

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Theorem. *Let G be a group of order less than 32 and G^* be a group such that the endomorphism semigroups of G and G^* are isomorphic. Then*

1^0 *if $G =< a, b \mid b^3 = 1, aba = bab >$ (the binary tetrahedral group), then $G^* \cong G$ or G^* is isomorphic to the alternating group A_4 (the tetrahedral group);*
2^0 *if G is not isomorphic to the tetrahedral group or to the binary tetrahedral group, then $G^* \cong G$.*

We shall use standard notations of group theory and the following notations:
$o(g)$ – order of the element g of the group G;
End(G) – the endomorphism semigroup of the group G;
$G = H \rtimes K$ – G is a semidirect product of a normal subgroup H and a subgroup K;
C_n – the cyclic group of order n;
$D_n =< a, b \mid b^2 = a^n = 1, b^{-1}ab = a^{-1} >=< a > \rtimes < b >$ – the dihedral group of order $2n$ $(n \geq 2)$;
$Q =< a, b \mid a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} >$ – the quaternion group;
S_n – the symmetric group of degree n;
A_4 – the alternatic group of order 12 (the tetrahedral group);
$K(x) = \{ y \in \text{End}(G) \mid yx = xy = y \}, \quad (x \in \text{End}(G))$.

Let G be a fixed group and G^* an arbitrary group. We say that the group G is determined by its endomorphism semigroup in the class of all groups if the isomorphism of semigroups End(G) and End(G^*) always implies the isomorphism of groups G and G^*.

2 Preliminaries

For convenience of reference, let us recall some known facts that will be used in the proofs of our main results.

If x is an idempotent of End(G), then G decomposes into the semidirect product $G = \text{Ker}\, x \rtimes \text{Im}\, x$ and $\text{Im}\, x = \{ g \in G \mid gx = g \}$.

Lemma 2.1 ([4], Lemma 1.6). *If x is an idempotent of End(G), then*

$$K(x) = \{ y \in End(G) \mid (Im\, x)y \subset Im\, x,\ (Ker\, x)y = \langle 1 \rangle \}$$

and $K(x)$ is a subsemigroup with the unity x of End(G) which is canonically isomorphic to End$(Im\, x)$. Under this isomorphism element y of $K(x)$ corresponds to its restriction on the subgroup $Im\, x$ of G.

Lemma 2.2 ([4], Theorem 4.2). *Every finite Abelian group is determined by its endomorphism semigroup in the class of all groups.*

Lemma 2.3 ([4], Theorem 1.13). *If groups A and B are determined by their endomorphism semigroups in the class of all groups, then so is their direct product $A \times B$.*

Lemma 2.4 ([5], Corollary 1). *The quaternion group Q is determined by its endomorphism semigroup in the class of all groups.*

Lemma 2.5 ([6], Theorem 2). *The symmetric group S_n is determined by its endomorphism semigroup in the class of all groups for each $n \geq 1$.*

Lemma 2.6 ([11], Section 5). *The dihedral group D_n is determined by its endomorphism semigroup in the class of all groups.*

Lemma 2.7 ([8], Theorem). *Let G decompose into a semidirect product $G = C_{p^n} \rightthreetimes C_m$, where p is a prime, n and m are some positive integers. Then G is determined by its endomorphism semigroup in the class of all groups.*

Lemma 2.8 ([12], Theorem). *Any group of order 16 is determined by its endomorphism semigroups in the class of all groups.*

Lemma 2.9 ([11], Theorem). *Let G be a group of order 24 and G^* be another group such that the endomorphism semigroups of G and G^* are isomorphic. Then*

1^0 *if G is the binary tetrahedral group, then $G^* \cong G$ or G^* is isomorphic to the tetrahedral group A_4;*
2^0 *if G is not isomorphic to the binary tetrahedral group, then $G^* \cong G$.*

Let G be a group and G_1, G_2, K be subgroups of G such that G decomposes as follows

$$G = (G_1 \times G_2) \rightthreetimes K = G_1 \rightthreetimes (G_2 \rightthreetimes K) = G_2 \rightthreetimes (G_1 \rightthreetimes K), \tag{2.1}$$

where $< G_i, K >= G_i \rightthreetimes K \ \ (i = 1, 2)$. Denote by x, x_1 and x_2 the projections of G onto its subgroups K, $G_1 \rightthreetimes K$ and $G_2 \rightthreetimes K$, respectively. Then

$$\operatorname{Im} x = K, \ \operatorname{Im} x_1 = G_1 \rightthreetimes K, \ \operatorname{Im} x_2 = G_2 \rightthreetimes K, \tag{2.2}$$

$$\operatorname{Ker} x = G_1 \times G_2, \ \operatorname{Ker} x_1 = G_2, \ \operatorname{Ker} x_2 = G_1. \tag{2.3}$$

Assume that G^* is another group such that the endomorphism semigroups of G and G^* are isomorphic and x^*, x_1^*, x_2^* correspond to x, x_1, x_2 in this isomorphism. In [7], Theorems 2.1 and 3.1, it was proved that under these assumptions the group G^* decomposes similarly to (2.1), i.e.,

$$G^* = (G_1^* \times G_2^*) \rightthreetimes K^* = G_1^* \rightthreetimes (G_2^* \rightthreetimes K^*) = G_2^* \rightthreetimes (G_1^* \rightthreetimes K^*), \tag{2.4}$$

where $< G_i^*, K^* >= G_i^* \rightthreetimes K^* \ \ (i = 1, 2)$ and

$$\operatorname{Im} x^* = K^*, \ \operatorname{Im} x_1^* = G_1^* \rightthreetimes K^*, \ \operatorname{Im} x_2^* = G_2^* \rightthreetimes K^*, \tag{2.5}$$

$$\operatorname{Ker} x^* = G_1^* \times G_2^*, \ \operatorname{Ker} x_1^* = G_2^*, \ \operatorname{Ker} x_2^* = G_1^*. \tag{2.6}$$

3 Non-abelian groups of order < 32

All non-Abelian groups of order < 32 are described in [2] (table 1 at the end of the book). The number of these groups is 44 and they are:

- $G_1 = S_3, \ \ o(G_1) = 6$
- $G_2 = D_4, \ \ o(G_2) = 8.$
- $G_3 = Q, \ \ o(G_3) = 8.$
- $G_4 = D_5, \ \ o(G_4) = 10.$
- $G_5 = D_6, \ \ o(G_5) = 12.$
- $G_6 = A_4, \ \ o(G_6) = 12.$

- $G_7 =< a, b \mid a^3 = b^2 = (ab)^2 >, \quad o(G_7) = 12.$
- $G_8 = D_7, \quad o(G_8) = 14.$
- $G_9 = C_2 \times D_4, \quad o(G_9) = 16.$
- $G_{10} = C_2 \times Q, \quad o(G_{10}) = 16.$
- $G_{11} = D_8, \quad o(G_{11}) = 16.$
- $G_{12} = C_8 \leftthreetimes C_2 =< a, b \mid b^2 = a^8 = 1, b^{-1}ab = a^3 >, \quad o(G_{12}) = 16.$
- $G_{13} = C_8 \leftthreetimes C_2 =< a, b \mid b^2 = a^8 = 1, b^{-1}ab = a^5 >, \quad o(G_{13}) = 16.$
- $G_{14} = C_4 \leftthreetimes C_4 =< a, b \mid b^4 = a^4 = 1, b^{-1}ab = a^{-1} >, \quad o(G_{14}) = 16.$
- $G_{15} =< a, b \mid a^4 = b^4 = (ba)^2 = (b^{-1}a)^2 = 1 >, \quad o(G_{15}) = 16.$
- $G_{16} =< a, b, c \mid a^2 = b^2 = c^2 = 1, \ abc = bca = cab >, \quad o(G_{16}) = 16.$
- $G_{17} =< a, b \mid a^4 = b^2 = (ab)^2 >, \quad o(G_{17}) = 16.$
- $G_{18} = C_3 \times D_3, \quad o(G_{18}) = 18.$
- $G_{19} = D_9, \quad o(G_{19}) = 18.$
- $G_{20} =< a, b, c \mid a^2 = b^2 = c^2 = (abc)^2 = (ab)^3 = (ac)^3 = 1 >, \quad o(G_{20}) = 18.$
- $G_{21} = D_{10} \cong C_2 \times D_5, \quad o(G_{21}) = 20.$
- $G_{22} =< a, b \mid a^2baba^{-1}b = b^2 = 1 >, \quad o(G_{22}) = 20.$
- $G_{23} =< a, b \mid a^5 = b^2 = (ab)^2 >, \quad o(G_{23}) = 20.$
- $G_{24} =< a, b \mid b^3 = 1, \ b^{-1}ab = a^2 >, \quad o(G_{24}) = 21.$
- $G_{25} = D_{11}, \quad o(G_{25}) = 22.$
- $G_{26} = C_2 \times A_4, \quad o(G_{26}) = 24.$
- $G_{27} = C_2 \times D_6, \quad o(G_{27}) = 24.$
- $G_{28} = C_3 \times D_4, \quad o(G_{28}) = 24.$
- $G_{29} = C_3 \times Q, \quad o(G_{29}) = 24.$
- $G_{30} = C_4 \times D_3, \quad o(G_{30}) = 24.$
- $G_{31} = C_2 \times \mathcal{G}_7, \quad o(G_{31}) = 24.$
- $G_{32} = D_{12}, \quad o(G_{32}) = 24.$
- $G_{33} = S_4, \quad o(G_{33}) = 24.$
- $G_{34} =< a, b \mid b^3 = 1, \ aba = bab >, \quad o(G_{34}) = 24.$
- $G_{35} =< a, b \mid b^4 = a^6 = (ba)^2 = (b^{-1}a)^2 = 1 >, \quad o(G_{35}) = 24.$
- $G_{36} =< a, b \mid a^2 = b^2 = (ab)^3 >, \quad o(G_{36}) = 24.$
- $G_{37} =< a, b \mid b^4 = a^{12} = 1, \ b^2 = a^6, \ b^{-1}ab = a^{-1} >, \quad o(G_{37}) = 24.$
- $G_{38} = D_{13}, \quad o(G_{38}) = 26.$
- $G_{39} =< a, b \mid b^3 = 1, \ b^{-1}ab = a^{-2} >, \quad o(G_{39}) = 27.$
- $G_{40} = D_{14}, \quad o(G_{40}) = 28.$
- $G_{41} =< a, b \mid a^7 = b^2 = (ab)^2 >, \quad o(G_{41}) = 29.$

- $G_{42} = C_3 \times D_5, \quad o(G_{42}) = 30.$
- $G_{43} = C_5 \times D_3, \quad o(G_{43}) = 30.$
- $G_{44} = D_{15}, \quad o(G_{44}) = 30.$

The group G_{34} is called *binary tetrahedral group.*

4 Proof of the theorem

Let us now prove the theorem. Assume that G is an arbitrary group of order less than 32. We will show that G satisfies the statements of the theorem. By Lemma 2.2, we can assume that G is non-abelian, i.e., G is one of the groups $G_1, G_2, \dots, G_{44}$. In view of Lemmas 2.2–2.9, the groups $G_1 - G_5$, $G_8 - G_{19}$, G_{21}, $G_{25} - G_{33}$, $G_{35} - G_{38}$, G_{40}, $G_{42} - G_{44}$ are determined by their endomorphism semigroups in the class of all groups. Therefore, by Lemma 2.9, the theorem will be proved if we show that the groups

$$G_7, G_{20}, G_{22}, G_{23}, G_{24}, G_{39}, G_{41}$$

are determined by their endomorphism semigroups in the class of all groups. Let us do that.

Considering the group

$$G_7 =< a, b \mid a^3 = b^2 = (ab)^2 >,$$

we obtain

$$a^3 = b^2 = abab, \; b = aba, \; b^{-1}ab = a^{-1}, \; a^3b = ba^3,$$
$$b^{-1}a^3b = a^{-3}, \; a^3 = a^{-3}, \; a^6 = b^4 = 1.$$

Hence

$$c^3 = b^4 = 1, \; b^{-1}cb = c^{-1},$$

where $c = a^2$. Since $a^3 = b^2$, we have $ac = b^2, \; a = b^2c^{-1}$ and $G_7 =< a, b >=< b, c >$. Therefore,

$$G_7 =< c, b \mid c^3 = b^4 = 1, \; b^{-1}cb = c^{-1} >=< c > \leftthreetimes < b > \cong C_3 \leftthreetimes C_4.$$

By Lemma 2.7, the group G_7 is determined by its endomorphism semigroup in the class of all groups.

Next we consider the group

$$G_{22} =< a, b \mid a^2baba^{-1}b = b^2 = 1 > .$$

Step by step we conclude

$$a^2baba^{-1}b = 1 \implies baba^{-1}b = a^{-2} \implies aba^{-1} = b^{-1}a^{-2}b \implies$$
$$\implies ab^2a^{-1} = b^{-1}a^{-4}b \implies 1 = b^{-1}a^{-4}b \implies$$
$$\implies a^4 = 1, \; aba^{-1} = b^{-1}a^2b,$$
$$aba \cdot aba = a \cdot b^{-1}a^2b \cdot a = a \cdot aba^{-1} \cdot a = a^2b,$$

$$(aba)^3 = aba \cdot a^2b = aba^{-1} \cdot b = b^{-1}a^2b \cdot b = ba^2,$$
$$(aba)^4 = a^2ba^2b = a^2 \cdot b^{-1}a^2b = a^2 \cdot aba^{-1} = a^{-1}ba^{-1},$$
$$(aba)^5 = a^{-1}ba^{-1} \cdot aba = 1.$$

Denote the elements a and aba by b and a, respectively. Then

$$\mathcal{G}_{22} =< a, b \mid b^4 = a^5 = 1,\ b^{-1}ab = a^3 >=< a > \leftthreetimes < b >\cong C_5 \leftthreetimes C_4.$$

By Lemma 2.7, the group G_{22} is determined by its endomorphism semigroup in the class of all groups.

For the group

$$G_{23} =< a, b \mid a^5 = b^2 = (ab)^2 >$$

we obtain

$$b^2 = (ab)^2 = abab \implies b = aba \implies b^{-1}ab = a^{-1},$$
$$a^5 = b^2 = b^{-1}b^2b = b^{-1}a^5b = (b^{-1}ab)^5 = a^{-5} \implies a^{10} = 1,\ b^4 = 1,$$
$$\mathcal{G}_{23} =< a, b \mid a^{10} = b^4 = 1,\ b^2 = a^5,\ b^{-1}ab = a^{-1} >=$$
$$=< a^2 > \leftthreetimes < b >\cong C_5 \leftthreetimes C_4.$$

By Lemma 2.7, the group G_{23} is determined by its endomorphism semigroup in the class of all groups.

Similarly,

$$G_{24} =< a, b \mid b^3 = 1,\ b^{-1}ab = a^2 >,$$
$$b^{-2}ab^2 = b^{-1}a^2b = (b^{-1}ab)^2 = a^4,$$
$$a = b^{-3}ab^3 = b^{-1}a^4b = (b^{-1}ab)^4 = a^8,\ a^7 = 1,$$
$$\mathcal{G}_{24} =< a, b \mid b^3 = a^7 = 1,\ b^{-1}ab = a^2 >=< a > \leftthreetimes < b >\cong C_7 \leftthreetimes C_3,$$
$$G_{39} =< a, b \mid b^3 = 1,\ b^{-1}ab = a^{-2} >,$$
$$b^{-2}ab^2 = b^{-1}a^{-2}b = (b^{-1}ab)^{-2} = (a^{-2})^{-2} = a^4,$$
$$a = b^{-3}ab^3 = b^{-1}a^4b = (b^{-1}ab)^4 = (a^{-2})^4 = a^{-8},\ a^9 = 1,$$
$$\mathcal{G}_{39} =< a, b \mid b^3 = a^9 = 1,\ b^{-1}ab = a^{-2} >=< a > \leftthreetimes < b >\cong C_9 \leftthreetimes C_3,$$
$$G_{41} =< a, b \mid a^7 = b^2 = (ab)^2 >,$$
$$b^2 = abab \implies aba = b \implies b^{-1}ab = a^{-1},$$
$$a^7 = b^2 = b^{-1}b^2b = b^{-1}a^7b = a^{-7} \implies a^{14} = 1,\ b^4 = 1,$$
$$\mathcal{G}_{41} =< b, c \mid b^4 = c^7 = 1,\ b^{-1}cb = c^{-1} >=$$
$$=< c > \leftthreetimes < b >\cong C_7 \leftthreetimes C_4 \quad (c = a^2).$$

and, by Lemma 2.7, the groups G_{24}, G_{39} and G_{41} are determined by their endomorphism semigroups in the class of all groups.

Finally, let us consider the group

$$G_{20} =< a, b, c \mid a^2 = b^2 = c^2 = (abc)^2 = (ab)^3 = (ac)^3 = 1 > .$$

It follows from the defining relations of G_{20} that

$$abc \cdot abc = 1 \implies bcab = ac \implies ab \cdot ac = ab \cdot bcab =$$

$$= ac \cdot ab \implies < ab,\, ac >=< ab > \times < ac >,$$
$$ab \cdot ab \cdot ab = 1,\ a^{-1} \cdot ab \cdot a = ba \implies (ba)^3 = 1,\ (ba)^2 = ab \implies$$
$$\implies (ba)^{-1} = ab,\ ba = (ab)^{-1} \implies a^{-1} \cdot ab \cdot a = (ab)^{-1},$$
$$a^{-1} \cdot ac \cdot a = ca,\ ac \cdot ac \cdot ac = 1 \implies (ca)^3 = 1,\ (ca)^2 = ac \implies$$
$$\implies (ca)^{-1} = ac \implies a^{-1} \cdot ac \cdot a = (ac)^{-1},$$
$$\mathcal{G}_{20} =< a,\, ab,\, ac >= (< ab > \times < ac >) \leftthreetimes < a >= (C_3 \times C_3) \leftthreetimes C_2.$$

Denote the elements a, ab and ac by b, a and c, respectively. Then

$$\mathcal{G}_{20} =< a,\, b,\, c \mid b^2 = a^3 = c^3 = 1,$$

$$ac = ca,\ b^{-1}ab = a^{-1},\ b^{-1}cb = c^{-1} >=$$

$$= (< a > \times < c >) \leftthreetimes < b >\cong (C_3 \times C_3) \leftthreetimes C_2. \tag{4.1}$$

By (4.1),

$$\mathcal{G}_{20} =< a > \leftthreetimes (< c > \leftthreetimes < b >) =< c > \leftthreetimes (< a > \leftthreetimes < b >).$$

Denote the projections of G_{20} onto its subgroups $< b >$, $< a > \leftthreetimes < b >$ and $< c > \leftthreetimes < b >$ by x, x_1 and x_2, respectively. Choose another group G^* such that the endomorphism semigroups of G_{20} and G^* are isomorphic:

$$\mathrm{End}(G_{20}) \cong \mathrm{End}(G^*). \tag{4.2}$$

Since the semigroup $\mathrm{End}(G^*)$ is finite, the group G^* is finite ([1], Theorem 2). Denote the images of x, x_1 and x_2 in the isomorphism (4.2) by x^*, x_1^* and x_2^*.

Now we can use equalities (2.1)–(2.6) (take there $G = G_{20}$). By these equalities,

$$G^* = (G_1^* \times G_2^*) \leftthreetimes K^* = G_2^* \leftthreetimes (G_1^* \leftthreetimes K^*) = G_1^* \leftthreetimes (G_2^* \leftthreetimes K^*),$$

where

$$K^* = \mathrm{Im}\, x^*, \quad \mathrm{Ker}\, x^* = G_1^* \times G_2^*,$$
$$G_1^* \leftthreetimes K^* = \mathrm{Im}\, x_1^*, \quad \mathrm{Ker}\, x_1^* = G_2^*, \quad G_2^* \leftthreetimes K^* = \mathrm{Im}\, x_2^*, \quad \mathrm{Ker}\, x_2^* = G_1^*.$$

In view of Lemma 2.1,

$$\mathrm{End}(< b >) = \mathrm{End}(\mathrm{Im}\, x) \cong K(x) \cong K(x^*) \cong \mathrm{End}(\mathrm{Im}\, x^*) = \mathrm{End}(K^*).$$

Hence, by Lemma 2.2, $K^* \cong < b >\cong C_2$ and $K^* =< b^* >\cong C_2$ for some $b^* \in K^*$.

By Lemmas 2.1 and 2.7,

$$\mathrm{End}(\mathrm{Im}\, x_1) \cong K(x_1) \cong K(x_1^*) \cong \mathrm{End}(\mathrm{Im}\, x_1^*) = \mathrm{End}(G_1^* \leftthreetimes K^*),$$

$$\mathrm{Im}\, x_1^* = G_1^* \leftthreetimes K^* = G_1^* \leftthreetimes < b^* >\cong \mathrm{Im}\, x \cong C_3 \leftthreetimes C_2.$$

Therefore,

$$G_1^* =< a^* >\cong C_3, \;\; b^{*-1}a^*b^* = a^{*-1}$$

for some $a^* \in G_1^*$. Similarly,

$$G_2^* \cong C_3, \;\; G_2^* =< c^* >, \;\; b^{*-1}c^*b^* = c^{*-1}.$$

Hence

$$G^* =< a^*, b^*, c^* \mid a^{*3} = b^{*2} = c^{*3} = 1, \; a^*c^* = c^*a^*, \\ b^{*-1}a^*b^* = a^{*-1}, \; b^{*-1}c^*b^* = c^{*-1} >,$$

and the groups G_{20} and G^* are isomorphic.

The theorem is proved.

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