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On Some Almost Quadratic Algebras Coming from Twisted Derivations

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Abstract

This paper explores the quasi-deformation scheme devised in [1, 3] as applied to the simple Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ for specific choices of the involved parameters and underlying algebras. One of the main points of this method is that the quasi-deformed algebra comes endowed with a canonical twisted Jacobi identity. We show in the present article that when the quasi-deformation method is applied to $\mathfrak{sl}_2(\mathbb{F})$ one obtains multiparameter families of almost quadratic algebras, and by choosing parameters suitably, $\mathfrak{sl}_2(\mathbb{F})$ is quasi-deformed into three-dimensional and four-dimensional Lie algebras and algebras closely resembling Lie superalgebras and colour Lie algebras, this being in stark contrast to the classical deformation schemes where $\mathfrak{sl}_2(\mathbb{F})$ is rigid.

Key words: quasi-hom-Lie algebras, hom-Lie algebras, colour Lie algebras, quasi-deformation, σ-derivations, extensions, twisted Jacobi identities, almost quadratic algebras.
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1 Introduction

In a series of papers [1, 3, 4] two of the present authors have developed a new deformation scheme for Lie algebras. The last paper [4] is concerned with this deformation scheme when applied to the simple Lie algebra $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of zero characteristic, and it is on that paper the present one builds and elaborates on.

Let us briefly explain the aforementioned deformation procedure. By $\mathbb{F}$ we denote the underlying field of characteristic zero and $\mathfrak{g}$ is the Lie algebra we wish to deform. Let $\rho : \mathfrak{g} \to \mathrm{Der}(\mathcal{A}) \subseteq \mathfrak{gl}(\mathcal{A})$ be a representation of $\mathfrak{g}$ in terms of derivations on some commutative, associative algebra $\mathcal{A}$ with unity. The Lie structure on $\mathrm{Der}(\mathcal{A})$ is of course given by the commutator bracket, induced from the Lie algebra structure on $\mathfrak{gl}(\mathcal{A})$, the algebra of linear operators on $\mathcal{A}$. The deformation procedure now takes place on this representation by changing the involved derivations to σ-derivations, that is, linear maps $\partial_\sigma : \mathcal{A} \to \mathcal{A}$ satisfying a generalized Leibniz rule: $\partial_\sigma(ab) = \partial_\sigma(a)b + \sigma(a)\partial_\sigma(b)$, for all $a, b \in \mathcal{A}$, and for an algebra endomorphism σ on $\mathcal{A}$.

In the course of this deformation we also deform the commutator $[\cdot,\cdot]$ to a σ-deformed version $\langle\cdot,\cdot\rangle$. The deformation procedure is thus an assignment

$$\mathrm{Der}(\mathcal{A}) \ni \partial \rightsquigarrow \partial_\sigma \in \mathrm{Der}_\sigma(\mathcal{A})\ ,$$

such that $[\cdot,\cdot] \rightsquigarrow \langle\cdot,\cdot\rangle$ and where $\mathrm{Der}_\sigma(\mathcal{A})$ is the vector space of σ-derivations on $\mathcal{A}$. Remember that ∂ represents an element of $\mathfrak{g}$.

In general, the new product $\langle\cdot,\cdot\rangle$ is not closed on $\mathrm{Der}_\sigma(\mathcal{A})$. It is, however, true that it is closed on the left $\mathcal{A}$-submodule $\mathcal{A}\cdot\partial_\sigma$ of $\mathrm{Der}_\sigma(\mathcal{A})$, for $\partial_\sigma \in \mathrm{Der}_\sigma(\mathcal{A})$ subject to some (mild) conditions. This is the content of Theorem 1. This theorem also establishes a canonical Jacobi-like relation on $\mathcal{A}\cdot\partial_\sigma$ for $\langle\cdot,\cdot\rangle$, reducing to the ordinary Jacobi identity when $\sigma = \mathrm{id}$, i.e., in the "limit" case of this deformation scheme corresponding to the Lie algebra $\mathfrak{g}$. We remark that in some cases, for instance when $\mathcal{A}$ is a unique factorization domain, $\mathcal{A}\cdot\partial_\sigma = \mathrm{Der}_\sigma(\mathcal{A})$ for suitable $\partial_\sigma \in \mathrm{Der}_\sigma(\mathcal{A})$ (see [1]). In particular, this means that we have two "deformation parameters" for this scheme, namely $\mathcal{A}$ and σ. Note, however, that they are not independent. Indeed, σ certainly depends on $\mathcal{A}$.

Diagrammatically, our deformation scheme can be given as

$$\begin{tikzcd}
\mathfrak{g} \arrow[rr, "\rho"] & & \mathfrak{gl}(\mathcal{A}) \arrow[d, squiggly] \\
& \tilde{\mathfrak{g}} \arrow[ul, dotted, "\text{''limit''}"] \arrow[r, dotted] & \widetilde{\mathfrak{gl}}(\mathcal{A}). \arrow[l, bend right, "P"']
\end{tikzcd}$$

Suppose the Lie algebra $\mathfrak{g}$ is spanned as a vector space by elements $\{\mathsf{g}_i\}_{i\in I}$, where I is some index set. The representation ρ yields the assignments $\mathsf{g}_i \mapsto a_i\cdot\partial$, for $a_i \in \mathcal{A}$. This can clearly be extended linearly to the whole of $\mathfrak{g}$ by the linearity of ρ. Now the deformation is $a_i\cdot\partial \rightsquigarrow a_i\cdot\partial_\sigma \in \mathcal{A}\cdot\partial_\sigma \subseteq \mathrm{Der}_\sigma(\mathcal{A})$. Put $\tilde{\mathsf{g}}_i := a_i\cdot\partial_\sigma$. The set $\{\tilde{\mathsf{g}}_i\}_{i\in I}$ spans a linear subspace $\tilde{\mathfrak{g}}$ of $\mathcal{A}\cdot\partial_\sigma$. Restricting the bracket on $\mathcal{A}\cdot\partial_\sigma$, given by Theorem 1, to $\tilde{\mathfrak{g}}$ gives us an algebra structure on $\mathfrak{g}$. This restriction is denoted by P in the above diagram. So, forgetting that $\tilde{\mathsf{g}}_i$ is $a_i\partial_\sigma$, $\{\tilde{\mathsf{g}}_i\}_{i\in I}$ spans an abstract (i.e., not associated with some particular representation) algebra $\tilde{\mathfrak{g}}$ with multiplication $\langle\cdot,\cdot\rangle$ and structure constants given by (2.5) of Theorem 1. This algebra is then to be viewed as the deformed version of $\mathfrak{g}$. Another way to look at this is to actually compute the deformed commutator in terms of the basis elements $\tilde{\mathsf{g}}_i$ and leaving the right-hand-side as it is given by (2.5). This gives us a set of relations in degree one and two in the basis elements, which can, considered as an associative algebra given by generators and relations, be viewed as an analogue of the universal enveloping algebra $\mathbf{U}(\mathfrak{g})$ to a Lie algebra $\mathfrak{g}$, for the algebra $\tilde{\mathfrak{g}}$. Alternatively, this can be seen as a "deformation" of $\mathbf{U}(\mathfrak{g})$. In this paper we deal primarily with this "generator and relations"-approach.

The quotation marks in "limit" is to indicate that we may not actually retrieve the original $\mathfrak{g}$ by performing the appropriate (depending on the case considered) limit procedure. This is because for some "values" of the involved parameters the representation or specific operators collapse, so even taking the limit becomes meaningless in these circumstances.

This is why we choose to call our deformations *quasi-deformations*. Another complication that arises is that the pull-back P "forgets relations". That is to say that the operators in $\mathcal{A} \cdot \partial_\sigma$ may satisfy relations, for instance coming from the twisted Leibniz rules, that the abstract algebra does not satisfy.

Now, the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ can be realized as a vector space generated by elements H, E and F subject to the relations (see for instance, [5])

$$\langle H, E\rangle = 2E, \qquad \langle H, F\rangle = -2F, \qquad \langle E, F\rangle = H. \tag{1.1}$$

Our basic starting point is the following representation of $\mathfrak{sl}_2(\mathbb{F})$ in terms of first order differential operators acting on some vector space of functions in the variable t:

$$E \mapsto \partial, \qquad H \mapsto -2t\partial, \qquad F \mapsto -t^2\partial.$$

To quasi-deform $\mathfrak{sl}_2(\mathbb{F})$ means that we replace ∂ by ∂_σ in this representation. At our disposal are now the deformation parameters $\mathcal{A}$ (the "algebra of functions") and the endomorphism σ.

In this paper, in contrast to [4], we study some of the algebras appearing in the quasi-deformation scheme. In particular, we extract algebras which resemble Lie superalgebras and colour Lie algebras in that they have either commutators or anti-commutators in their relations. This is done in the case when $\mathcal{A} = \mathbb{F}[t]$ (which is a "deformation parameter" studied in [4], though not from the present aspect), and also the new interesting case $\mathcal{A} = \mathbb{F}[t]/(t^4)$. This last case leads to six relations instead of three which should be natural as $\mathfrak{sl}_2(\mathbb{F})$ only has three relations. It would be of interest to determine ring-theoretic properties of these algebras (e.g., for which values of the parameters are they domains, noetherian, PBW-algebras, Auslander-regular etc).

The paper is organized as follows. In Section 2 we recall the necessary background material and fix notation. Section 3 deals with the general quasi-deformation scheme as applied to $\mathfrak{sl}_2(\mathbb{F})$, and in Subsections 3.1 and 3.2 we explore this scheme in the particular cases of $\mathcal{A} = \mathbb{F}[t]$ and $\mathcal{A} = \mathbb{F}[t]/(t^4)$, respectively.

Let us finally comment on the title. By an "almost quadratic algebra" we mean an algebra with relations in degree at most two. Usually, the simpler term "quadratic algebra" is reserved for an algebra with homogeneous quadratic relations.

2 Qhl-algebras associated with σ-derivations

We now fix notation and state the main definitions and results from [1] and [3] needed in this paper.

Throughout we let $\mathbb{F}$ denote a field of characteristic zero and $\mathcal{A}$ be a commutative, associative $\mathbb{F}$-algebra with unity 1. Furthermore, σ will denote an endomorphism on $\mathcal{A}$. Then by a *twisted derivation* or *σ-derivation* on $\mathcal{A}$ we mean an $\mathbb{F}$-linear map $\partial_\sigma : \mathcal{A} \to \mathcal{A}$ such that a σ-twisted Leibniz rule holds:

$$\partial_\sigma(ab) = \partial_\sigma(a)b + \sigma(a)\partial_\sigma(b). \tag{2.1}$$

Among the best known σ-derivations are:

- $(\partial\, a)(t) = a'(t)$, *the ordinary differential operator* with the ordinary Leibniz rule, i.e., $\sigma = \mathrm{id}$.
- $(\partial_\sigma\, a)(t) = (D_q a)(t)$, *the Jackson q-derivation operator* with σ-Leibniz rule $(D_q\,(ab))(t) = (D_q a)(t)b(t) + a(qt)(D_q b)(t)$, where $\sigma = \mathbf{t}_q$ and $\mathbf{t}_q f(t) := f(qt)$.

In the paper [1] the notion of a hom-Lie algebra as a deformed version of a Lie algebra was introduced, motivated by some of the examples of deformations of the Witt and Virasoro algebras constructed using σ-derivations. However, finding examples of more general kinds of deformations associated to σ-derivations, prompted the introduction in [3] of quasi-hom-Lie algebras (qhl-algebras) generalizing hom-Lie algebras. Quasi-hom-Lie algebras include not only hom-Lie algebras as a subclass, but also colour Lie algebras and in particular Lie superalgebras [3].

We let $\mathrm{Der}_\sigma(\mathcal{A})$ denote the vector space of σ-derivations on $\mathcal{A}$. Fixing a homomorphism $\sigma : \mathcal{A} \to \mathcal{A}$, an element $\partial_\sigma \in \mathrm{Der}_\sigma(\mathcal{A})$ and an element $\delta \in \mathcal{A}$, we assume that these objects satisfy the following two conditions:

$$\sigma(\mathrm{Ann}(\partial_\sigma)) \subseteq \mathrm{Ann}(\partial_\sigma), \tag{2.2}$$

$$\partial_\sigma(\sigma(a)) = \delta\sigma(\partial_\sigma(a)), \quad \text{for a} \in \mathcal{A}, \tag{2.3}$$

where $\mathrm{Ann}(\partial_\sigma) := \{a \in \mathcal{A} \,|\, a \cdot \partial_\sigma = 0\}$. Let $\mathcal{A} \cdot \partial_\sigma := \{a \cdot \partial_\sigma \mid a \in \mathcal{A}\}$ denote the cyclic $\mathcal{A}$-submodule of $\mathrm{Der}_\sigma(\mathcal{A})$ generated by ∂_σ and extend σ to $\mathcal{A} \cdot \partial_\sigma$ by $\sigma(a \cdot \partial_\sigma) = \sigma(a) \cdot \partial_\sigma$. The following theorem, from [1], introducing an $\mathbb{F}$-algebra structure on $\mathcal{A} \cdot \partial_\sigma$ making it a quasi-hom-Lie algebra, is of central importance for the present paper.

Theorem 1. *If* (2.2) *holds then the map* $\langle \cdot, \cdot \rangle$ *defined by*

$$\langle a \cdot \partial_\sigma, b \cdot \partial_\sigma \rangle = (\sigma(a) \cdot \partial_\sigma) \circ (b \cdot \partial_\sigma) - (\sigma(b) \cdot \partial_\sigma) \circ (a \cdot \partial_\sigma), \tag{2.4}$$

for $a, b \in \mathcal{A}$ *and where* $\circ$ *denotes composition of maps, is a well-defined* $\mathbb{F}$*-algebra product on the* $\mathbb{F}$*-linear space* $\mathcal{A} \cdot \partial_\sigma$*. It satisfies the following identities for* $a, b, c \in \mathcal{A}$:

$$\langle a \cdot \partial_\sigma, b \cdot \partial_\sigma \rangle = (\sigma(a)\partial_\sigma(b) - \sigma(b)\partial_\sigma(a)) \cdot \partial_\sigma, \tag{2.5}$$

$$\langle a \cdot \partial_\sigma, b \cdot \partial_\sigma \rangle = -\langle b \cdot \partial_\sigma, a \cdot \partial_\sigma \rangle, \tag{2.6}$$

and if, in addition, (2.3) *holds, we have the deformed six-term Jacobi identity*

$$\circlearrowleft_{a,b,c} \big(\langle \sigma(a) \cdot \partial_\sigma, \langle b \cdot \partial_\sigma, c \cdot \partial_\sigma \rangle\rangle + \delta \cdot \langle a \cdot \partial_\sigma, \langle b \cdot \partial_\sigma, c \cdot \partial_\sigma \rangle\rangle\big) = 0, \tag{2.7}$$

where $\circlearrowleft_{a,b,c}$ *denotes cyclic summation with respect to* a, b, c.

The algebra $\mathcal{A} \cdot \partial_\sigma$ in the theorem is then a qhl-algebra with $\alpha = \sigma$, $\beta = \delta$ and $\omega = -\,\mathrm{id}_{\mathcal{A}\cdot\partial_\sigma}$. For the detailed proof of Theorem 1 see [1].

3 Quasi-deformations

Let $\mathcal{A}$ be a commutative, associative $\mathbb{F}$-algebra with unity 1, t an element of $\mathcal{A}$, and let σ denote an $\mathbb{F}$-algebra endomorphism on $\mathcal{A}$. Choose an element ∂_σ of $\mathrm{Der}_\sigma(\mathcal{A})$ and consider the $\mathbb{F}$-subspace $\mathcal{A} \cdot \partial_\sigma$ of elements of the form $a \cdot \partial_\sigma$ for $a \in \mathcal{A}$. We will usually denote

$a \cdot \partial_\sigma$ simply by $a\partial_\sigma$. Notice that $\mathcal{A} \cdot \partial_\sigma$ is a left $\mathcal{A}$-module, and by Theorem 1 there is a skew-symmetric algebra structure on $\mathcal{A} \cdot \partial_\sigma$ given by

$$\begin{aligned}\langle a \cdot \partial_\sigma, b \cdot \partial_\sigma\rangle &= \sigma(a) \cdot \partial_\sigma(b \cdot \partial_\sigma) - \sigma(b) \cdot \partial_\sigma(a \cdot \partial_\sigma)\\ &= (\sigma(a)\partial_\sigma(b) - \sigma(b)\partial_\sigma(a)) \cdot \partial_\sigma,\end{aligned} \tag{3.1}$$

where $a, b \in \mathcal{A}$. The elements $e := \partial_\sigma$, $h := -2t\partial_\sigma$ and $f := -t^2\partial_\sigma$ span an $\mathbb{F}$-linear subspace $\mathcal{S} := \mathrm{LinSpan}_\mathbb{F}\{\partial_\sigma, -2t\partial_\sigma, -t^2\partial_\sigma\} = \mathrm{LinSpan}_\mathbb{F}\{e, h, f\}$ of $\mathcal{A} \cdot \partial_\sigma$. We restrict the multiplication (3.1) to $\mathcal{S}$ without, at this point, assuming closure. Now, $\partial_\sigma(t^2) = \partial_\sigma(t \cdot t) = \sigma(t)\partial_\sigma(t) + \partial_\sigma(t)t = (\sigma(t) + t)\partial_\sigma(t)$ which, by using (3.1), leads to

$$\langle h, f\rangle = 2\langle t\partial_\sigma, t^2\partial_\sigma\rangle = 2\sigma(t)\partial_\sigma(t)t\partial_\sigma, \tag{3.2a}$$

$$\langle h, e\rangle = -2\langle t\partial_\sigma, \partial_\sigma\rangle = -2(\sigma(t)\partial_\sigma(1) - \sigma(1)\partial_\sigma(t))\partial_\sigma, \tag{3.2b}$$

$$\langle e, f\rangle = -\langle \partial_\sigma, t^2\partial_\sigma\rangle = -(\sigma(1)(\sigma(t) + t)\partial_\sigma(t) - \sigma(t)^2\partial_\sigma(1))\partial_\sigma, \tag{3.2c}$$

under the natural assumptions $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$ (see [4]), simplifying to

$$\langle h, f\rangle = 2\sigma(t)t\partial_\sigma(t)\partial_\sigma, \tag{3.3a}$$

$$\langle h, e\rangle = 2\partial_\sigma(t)\partial_\sigma, \tag{3.3b}$$

$$\langle e, f\rangle = -(\sigma(t) + t)\partial_\sigma(t)\partial_\sigma. \tag{3.3c}$$

Remark 1. *Note that when* $\sigma = \mathrm{id}$ *and* $\partial_\sigma(t) = 1$*, we retain the classical* $\mathfrak{sl}_2(\mathbb{F})$ *with relations* (1.1)*.*

3.1 Quasi-deformations with base algebra $\mathcal{A} = \mathbb{F}[t]$

Take $\mathcal{A}$ to be the polynomial algebra $\mathbb{F}[t]$, $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$. Since the set of all non-negative integer powers of t is linearly independent over $\mathbb{F}$ in $\mathbb{F}[t]$, we are in the situation of relations (3.3a), (3.3b) and (3.3c). Suppose that $\sigma(t) = q(t)$ and $\partial_\sigma(t) = p(t)$, where $p(t), q(t) \in \mathbb{F}[t]$. To have closure of (3.3a), (3.3b) and (3.3c) these polynomials are far from arbitrary. Indeed, by (3.3a) we get

$$\deg(\partial_\sigma(t)\sigma(t)) = \deg(p(t)q(t)) \leq 1.$$

If we allow $\sigma(t) = q(t)$ and $\partial_\sigma(t) = p(t)$, where p, q are arbitrary polynomials in t, then we get a deformation of $\mathfrak{sl}_2(\mathbb{F})$ which does not preserve dimension; that is, brackets of the basis elements e, f, h are not simply linear combinations in these elements but include, additionally, basis elements from the whole $\mathcal{A} \cdot \partial_\sigma$. This phenomenon will be studied further in a following subsection.

Assume $q(t) = q_0 + q_1 t$, implying that $p(t) = p_0$. Relations (3.3a), (3.3b) and (3.3c) according to (3.1) now become

$$\langle h, f\rangle :\ -2q_0 ef + q_1 hf + q_0^2 eh - q_0 q_1 h^2 - q_1^2 fh = -q_0 p_0 h - 2q_1 p_0 f, \tag{3.4a}$$

$$\langle h, e\rangle :\ -2q_0 e^2 + q_1 he - eh = 2p_0 e, \tag{3.4b}$$

$$\langle e, f\rangle :\ ef + q_0^2 e^2 - q_0 q_1 he - q_1^2 fe = -q_0 p_0 e + \frac{q_1 + 1}{2} p_0 h. \tag{3.4c}$$

We would like to show how algebras similar in kind to Lie algebras, Lie superalgebras and colour Lie algebras, in that their quadratic parts involve only commutators and anticommutators, appear in this quasi-deformation family. When $q_0 = 0$, $q_1 = 1$ and $p_0 \neq 0$, that is $\sigma(t) = q(t) = t$ and $\partial_\sigma(t) = p(t) = p_0 \neq 0$, we have

$$hf - fh = -2p_0 f, \quad he - eh = 2p_0 e, \quad ef - fe = p_0 h.$$

The special case of $p_0 = 1$ corresponds to the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$, i.e.,

$$hf - fh = -2f, \quad he - eh = 2e, \quad ef - fe = h.$$

Taking $q_0 = 0$, $q_1 = -1$ and $p_0 \neq 0$, meaning that $\sigma(t) = q(t) = -t$ and $\partial_\sigma(t) = p(t) = p_0 \neq 0$, yields

$$hf + fh = -2p_0 f, \quad he + eh = -2p_0 e, \quad ef - fe = 0.$$

Remark 2. *If we in the cases $q_0 = 0$, $q_1 = \pm 1$ and $p_0 \neq 0$ make the re-scalings $e \mapsto p_0 e$, $f \mapsto p_0 f$ and $h \mapsto p_0 h$, we get the corresponding cases when putting p_0 equal to one. Hence, in each of the cases $q_1 = 1$ and $q_1 = -1$, we obtain isomorphic algebras for all non-zero values of the parameter p_0. In the classical terminology this is a so-called "jump-deformation" (here with deformation parameter p_0) of the algebra defined by*

$$hf + fh = 0, \quad he + eh = 0, \quad ef - fe = 0.$$

We consider the case where $p_0 = -1/2$ giving rise to the relations

$$hf + fh = f, \quad he + eh = e, \quad ef - fe = 0.$$

In general, for $k \geq 0$ we have

$$\partial_\sigma(t^{k+1}) = \sum_{j=0}^{k} \sigma(t)^j t^{k-j} \partial_\sigma(t) = p(t) \sum_{j=0}^{k} q(t)^j t^{k-j}.$$

Let $q(t) = q_1 t$ and $p(t) = p_0$. Then

$$\partial_\sigma(t^{k+1}) = p_0 \sum_{j=0}^{k} (q_1 t)^j t^{k-j} = p_0 t^k \sum_{j=0}^{k} q_1^j = p_0 \{k+1\}_{q_1} t^k,$$

where $\{n\}_q = \sum_{j=0}^{n-1} q^j = \frac{1-q^n}{1-q}$ if $q \neq 1$ and $\{n\}_1 = n$ (see [2], for instance). Choosing the parameters $q_1 = -1$ and $p_0 = -1/2$, we have $p_0 \{n\}_{q_1} = -\frac{1}{4}(1 - (-1)^n)$, and hence for $n \geq 1$

$$e(t^n) = -\frac{1-(-1)^n}{4} t^{n-1}, \quad h(t^n) = \frac{1-(-1)^n}{2} t^n, \quad f(t^n) = \frac{1-(-1)^n}{4} t^{n+1}.$$

Recall that $\partial_\sigma(1) = 0$, implying $e(1) = h(1) = f(1) = 0$. It follows that we obtain two additional relations, namely $e^2 = 0$ and $f^2 = 0$. We arrive at an algebra defined by the five relations

$$hf + fh = f, \quad he + eh = e, \quad ef - fe = 0, \quad e^2 = 0, \quad f^2 = 0.$$

We suspect that this algebra is involved in some kind of duality for colour Lie algebras.

The condition (2.3) holds with $\delta = q_1$. By Theorem 1 we have a deformed Jacobi identity

$$\circlearrowleft_{x,y,z} \langle(\alpha + \mathrm{id})(x), \langle y, z\rangle\rangle = 0 \tag{3.5}$$

on $\mathcal{A} \cdot \partial_\sigma = \mathbb{F}[t] \cdot \partial_\sigma$, where $\alpha(x) := q_1^{-1}\sigma(x)$ for any $x = a\partial_\sigma \in \mathcal{A} \cdot \partial_\sigma$.

3.2 Quasi-deformations with base algebra $\mathbb{F}[t]/(t^4)$

Now, let $\mathbb{F}$ include all fourth roots of unity and take as $\mathcal{A}$ the algebra $\mathbb{F}[t]/(t^4)$. This is obviously a four-dimensional $\mathbb{F}$-vector space and a finitely generated $\mathbb{F}[t]$-module with basis $\{1, t, t^2, t^3\}$. We let, as before, $e = \partial_\sigma, h = -2t\partial_\sigma$ and $f = -t^2\partial_\sigma$. We introduce a fourth basis element $g = 2t^3\partial_\sigma$. Note that $-2t \cdot e = h$, $t \cdot h = 2f$ and $-2t \cdot f = g$. Put

$$\partial_\sigma(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3, \tag{3.6a}$$
$$\sigma(t) = q_0 + q_1 t + q_2 t^2 + q_3 t^3, \tag{3.6b}$$

considering these as elements in the ring $\mathbb{F}[t]/(t^4)$. The equalities (3.6a) and (3.6b) have to be compatible with $t^4 = 0$. This means in particular that

$$\begin{aligned}\sigma(t^4) &= \sigma(t)^4 = (q_0 + q_1 t + q_2 t^2 + q_3 t^3)^4\\ &= q_0^4 + 4q_0^3 q_1 t + 2q_0^2(2q_0q_2 + 3q_1^2)t^2 + 4q_0(q_0^2 q_3 + 3q_0q_1q_2 + q_1^3)t^3 = 0,\end{aligned}$$

implying $q_0 = 0$. Furthermore,

$$0 = \partial_\sigma(t^4) = (\sigma(t)^3 + \sigma(t)^2 t + \sigma(t)t^2 + t^3)\partial_\sigma(t). \tag{3.7}$$

Since

$$\sigma(t) = (q_1 + q_2 t + q_3 t^2)t, \quad \sigma(t)^2 = (q_1^2 + 2q_1q_2t)t^2, \quad \sigma(t)^3 = q_1^3 t^3,$$

it follows from equation (3.7) that $(q_1^3 + q_1^2 + q_1 + 1)p_0 t^3 = 0$, and hence

$$(q_1^3 + q_1^2 + q_1 + 1)p_0 = 0. \tag{3.8}$$

In other words, in case $p_0 \neq 0$, we generate deformations at the zeros of the polynomial $u^3 + u^2 + u + 1$; if $p_0 = 0$ then q_1 is a true free deformation parameter.

As before we make the assumptions that $\sigma(1) = 1$, $\partial_\sigma(1) = 0$ and so relations (3.3a), (3.3b) and (3.3c) still hold. Moreover, since

$$\partial_\sigma(t^2) = (\sigma(t) + t)\partial_\sigma(t), \quad \partial_\sigma(t^3) = (\sigma(t)^2 + \sigma(t)t + t^2)\partial_\sigma(t),$$

the introduction of the new generator g means that we obtain three additional relations using equation (3.1)

$$\langle h, g\rangle = -4\langle t\partial_\sigma, t^3\partial_\sigma\rangle = -4(\sigma(t) + t)\sigma(t)t\partial_\sigma(t)\partial_\sigma, \tag{3.9a}$$
$$\langle e, g\rangle = 2\langle \partial_\sigma, t^3\partial_\sigma\rangle = 2(\sigma(t)^2 + \sigma(t)t + t^2)\partial_\sigma(t)\partial_\sigma, \tag{3.9b}$$
$$\langle f, g\rangle = -2\langle t^2\partial_\sigma, t^3\partial_\sigma\rangle = -2\sigma(t)^2 t^2\partial_\sigma(t)\partial_\sigma. \tag{3.9c}$$

The action of $\mathcal{A}\cdot\partial_\sigma$ on $\mathcal{A}$ is now given by the values of e, h, f and g on the basis $\{1, t, t^2, t^3\}$. From $\partial_\sigma(1) = 0$ it follows that $e(1) = h(1) = f(1) = g(1) = 0$. Moreover, applying the σ-twisted Leibniz rule, we have

$$\begin{aligned} e(t) &= p_0 + p_1 t + p_2 t^2 + p_3 t^3, \\ e(t^2) &= (q_1+1)p_0 t + (q_2 p_0 + q_1 p_1 + p_1)t^2 + (q_3 p_0 + q_2 p_1 + q_1 p_2 + p_2)t^3, \\ e(t^3) &= (q_1^2 + q_1 + 1)p_0 t^2 + (q_2 p_0 + 2q_1 q_2 p_0 + q_1^2 p_1 + q_1 p_1 + p_1)t^3. \end{aligned}$$

For the action of h we have (since $h = -2t\cdot e$)

$$\begin{aligned} h(t) &= -2p_0 t - 2p_1 t^2 - 2p_2 t^3, \\ h(t^2) &= -2(q_1+1)p_0 t^2 - 2(q_2 p_0 + q_1 p_1 + p_1)t^3, \\ h(t^3) &= -2(q_1^2 + q_1 + 1)p_0 t^3. \end{aligned}$$

The values of f are easily computed using $f = \frac{1}{2}\, t\cdot h$:

$$f(t) = -p_0 t^2 - p_1 t^3, \quad f(t^2) = -(q_1+1)p_0 t^3, \quad f(t^3) = 0.$$

Finally, for g we obtain $g(t) = 2p_0 t^3$ and $g(t^2) = g(t^3) = 0$. By (2.4) the bracket can be computed abstractly on generators as

$$\langle h, f\rangle = q_1 hf + 2q_2 f^2 - q_1^2 fh + q_1 q_2 gh - q_3 gf, \tag{3.10a}$$
$$\langle h, e\rangle = q_1 he + 2q_2 fe - eh - q_3 ge, \tag{3.10b}$$
$$\langle e, f\rangle = ef - q_1^2 fe + q_1 q_2 ge, \tag{3.10c}$$
$$\langle h, g\rangle = q_1 hg + 2q_2 fg - q_3 g^2 - q_1^3 gh, \tag{3.10d}$$
$$\langle e, g\rangle = eg - q_1^3 ge, \tag{3.10e}$$
$$\langle f, g\rangle = q_1^2 fg - q_1 q_2 g^2 - q_1^3 gf. \tag{3.10f}$$

Formulas (3.6a) and (3.6b), together with the assumption that the right-hand-sides of these are elements in $\mathbb{F}[t]/(t^4)$, now yield closure. Indeed, by (3.1) we obtain with $\sigma(t) = q(t) = q_1 t + q_2 t^2 + q_3 t^3$ and $\partial_\sigma(t) = p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$

$$\begin{aligned} \langle h, f\rangle &= 2q(t)tp(t)\partial_\sigma = 2(q_1 p_0 + (q_2 p_0 + q_1 p_1)t)t^2\partial_\sigma \\ &= -2q_1 p_0 f + (q_2 p_0 + q_1 p_1)g, \\ \langle h, e\rangle &= 2p(t)\partial_\sigma = 2p_0 e - p_1 h - 2p_2 f + p_3 g, \\ \langle e, f\rangle &= -(q(t)+t)p(t)\partial_\sigma = \frac{1}{2}(1+q_1)p_0 h + (p_1 + q_1 p_1 + q_2 p_0)f \\ &\quad - \frac{1}{2}(p_2 + q_1 p_2 + q_2 p_1 + q_3 p_0)g, \\ \langle h, g\rangle &= -4(q(t)+t)q(t)tp(t)\partial_\sigma = -2(1+q_1)q_1 p_0 g, \\ \langle e, g\rangle &= 2(q(t)^2 + q(t)t + t^2)p(t)\partial_\sigma \\ &= -2(1 + q_1 + q_1^2)p_0 f + (p_1 + q_1 p_1 + q_1^2 p_1 + q_2 p_0 + 2q_1 q_2 p_0)g, \\ \langle f, g\rangle &= -2q(t)^2 t^2 p(t)\partial_\sigma = 0. \end{aligned}$$

We shall consider two cases where $q_2 = q_3 = 0$. In the first one, corresponding to $q_1 = 1$, we must have $p_0 = 0$ in order to satisfy (3.8), so we obtain six relations from (3.10a–3.10f) and the bracket expressions listed above

$$
\begin{aligned}
\langle h, f\rangle &: \quad hf - fh = p_1 g,\\
\langle h, e\rangle &: \quad he - eh = -p_1 h - 2p_2 f + p_3 g,\\
\langle e, f\rangle &: \quad ef - fe = 2p_1 f - p_2 g,\\
\langle h, g\rangle &: \quad hg - gh = 0,\\
\langle e, g\rangle &: \quad eg - ge = 3p_1 g,\\
\langle f, g\rangle &: \quad fg - gf = 0.
\end{aligned}
$$

In our specific representation of the associative algebra defined by these relations, the action of e on the set $\{t, t^2, t^3\}$ is (recall that $e(1) = h(1) = f(1) = g(1) = 0$)

$$e(t) = p_1 t + p_2 t^2 + p_3 t^3, \quad e(t^2) = 2p_1 t^2 + 2p_2 t^3, \quad e(t^3) = 3p_1 t^3,$$

while h, f and g simplify to $h(t) = -2p_1 t^2 - 2p_2 t^3$, $h(t^2) = -4p_1 t^3$, $f(t) = -p_1 t^3$ and $h(t^3) = f(t^2) = f(t^3) = g(t) = g(t^2) = g(t^3) = 0$.

In the second case, obtained by choosing $q_1 = -1$ and $q_2 = q_3 = 0$, the relations reduce to the form

$$
\begin{aligned}
\langle h, f\rangle &: \quad hf + fh = -2p_0 f + p_1 g,\\
\langle h, e\rangle &: \quad he + eh = -2p_0 e + p_1 h + 2p_2 f - p_3 g,\\
\langle e, f\rangle &: \quad ef - fe = 0,\\
\langle h, g\rangle &: \quad hg - gh = 0,\\
\langle e, g\rangle &: \quad eg + ge = -2p_0 f + p_1 g,\\
\langle f, g\rangle &: \quad fg + gf = 0.
\end{aligned}
$$

The values of e on the set $\{t, t^2, t^3\}$ are now given by

$$e(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3, \quad e(t^2) = 0, \quad e(t^3) = p_0 t^2 + p_1 t^3.$$

For the action of h, we find

$$h(t) = -2p_0 t - 2p_1 t^2 - 2p_2 t^3, \quad h(t^2) = 0, \quad h(t^3) = -2p_0 t^3,$$

whilst f and g have values $f(t) = -p_0 t^2 - p_1 t^3$, $g(t) = 2p_0 t^3$ and, as for the $q_1 = 1$ case, finally $f(t^2) = f(t^3) = g(t^2) = g(t^3) = 0$.

Twisted Jacobi identity

If $\sigma(t) = -t$ the left-hand-side of (2.3) for $a = t$ is by equation (3.6a)

$$\partial_\sigma(\sigma(t)) = \partial_\sigma(-t) = -\partial_\sigma(t) = -(p_0 + p_1 t + p_2 t^2 + p_3 t^3).$$

The right-hand-side becomes

$$\delta\sigma(\partial_\sigma(t)) = \delta \cdot \sigma(p_0 + p_1 t + p_2 t^2 + p_3 t^3) = \delta(p_0 - p_1 t + p_2 t^2 - p_3 t^3).$$

Assume that δ can be written as $\delta = \delta_0 + \delta_1 t + \delta_2 t^2 + \delta_3 t^3$. Then our relation $\partial_\sigma(\sigma(t)) = \delta\sigma(\partial_\sigma(t))$ can be written as

$$\begin{aligned} p_0 + p_1 t + p_2 t^2 + p_3 t^3 = &-\delta_0 p_0 + (\delta_0 p_1 - \delta_1 p_0)t - (\delta_0 p_2 - \delta_1 p_1 + \delta_2 p_0)t^2 \\ &+ (\delta_0 p_3 - \delta_1 p_2 + \delta_2 p_1 - \delta_3 p_0)t^3, \end{aligned}$$

keeping in mind that $t^4 = 0$. This is equivalent to the linear system of equations for $\delta_0, \delta_1, \delta_2$ and δ_3

$$\begin{cases} p_0 + \delta_0 p_0 = 0 \\ p_1 - \delta_0 p_1 + \delta_1 p_0 = 0 \\ p_2 + \delta_0 p_2 - \delta_1 p_1 + \delta_2 p_0 = 0 \\ p_3 - \delta_0 p_3 + \delta_1 p_2 - \delta_2 p_1 + \delta_3 p_0 = 0. \end{cases}$$

Assuming that $p_0 \neq 0$, then obviously $\delta_0 = -1$, and hence

$$(\delta_0, \delta_1, \delta_2, \delta_3) = (-1, -\frac{2p_1}{p_0}, -\frac{2p_1^2}{p_0^2}, -\frac{2p_1^3}{p_0^3} + \frac{2p_1 p_2}{p_0^2} - \frac{2p_3}{p_0}).$$

In the sequel, let ξ_1, ξ_2 and ξ_3 denote three free parameters taking values in $\mathbb{F}$. If $p_0 = 0$ and $p_1 \neq 0$ then we have $\delta_0 = 1$ and

$$(\delta_0, \delta_1, \delta_2, \delta_3) = (1, \frac{2p_2}{p_1}, \frac{2p_2^2}{p_1^2}, \xi_3).$$

The case $p_0 = p_1 = 0$, $p_2 \neq 0$ yields $\delta_0 = -1$, $\delta_1 = -2p_3/p_2$ and $\delta_2 = \xi_2$, $\delta_3 = \xi_3$. Finally, $p_0 = p_1 = p_2 = 0$, $p_3 \neq 0$ implies that $\delta = 1 + \xi_1 t + \xi_2 t^2 + \xi_3 t^3$. In each of these cases the twisted Jacobi identity is obtained from (2.7).

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