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# $SO(2)$ and Hamilton-Dirac mechanics

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**Abstract**

Canonical formalism for plane rotations is developed. This group can be seen as a toy model of the Hamilton-Dirac mechanics with constraints. The Lagrangian and Hamiltonian are explicitly constructed and their physical interpretation are given. The Euler-Lagrange and Hamiltonian canonical equations coincide with the Lie equations. Consistency of the constraints is checked.

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## 1 Introduction and outline of the paper

The Lie group multiplication can be locally given as an integral of the first order partial differential equations called the Lie equations. One may ask for such a Lagrangian recapitulation of the Lie theory that the Euler-Lagrange equations coincide with the Lie equations. Based on the Lagrangian one can elaborate the corresponding canonical formalism for a Lie group.

In this paper, the canonical formalism for plane rotation group $SO(2)$ is presented. It is shown that the latter can be seen as a toy model of the *Hamilton-Dirac mechanics* with constraints [2]. The Lagrangian and Hamiltonian are explicitly constructed. The Euler-Lagrange and Hamiltonian equations coincide with the Lie equations. Consistency of the constraints is checked.

## 2 Lie equations and Lagrangian

Consider the rotation group $SO(2)$ of the real two-plane $\mathbb{R}^2$ . Rotation of $\mathbb{R}^2$ by an angle $\alpha \in \mathbb{R}$ is given by the transformation

$$\begin{cases} x' = x\cos\alpha - y\sin\alpha \\ y' = x\sin\alpha + y\cos\alpha \end{cases}$$

In matrix notations

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

By denoting $i \doteq \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$, a generic element $z \in SO(2)$ reads as a complex number

$$z = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \cos\alpha + i\sin\alpha = e^{i\alpha}$$

We consider the rotation angle $\alpha$ as a dynamical variable and $z$ as a field variable for $SO(2)$. The Lie equations read

$$\dot{z} \doteq \partial_\alpha z = iz, \qquad \dot{\overline{z}} \doteq \partial_\alpha \overline{z} = -i\overline{z}$$

**Definition 1 (Lagrangian).** The Lagrangian $L$ for $SO(2)$ can be defined by

$$L(z, \dot{z}, \overline{z}, \dot{\overline{z}}) \doteq \frac{1}{2i}(\dot{z}\overline{z} - z\dot{\overline{z}}) - z\overline{z}$$

**Theorem 2.** *The Euler-Lagrange equations of $SO(2)$ coincide with its Lie equations.*

**Proof.** Calculate

$$\frac{\partial L}{\partial \overline{z}} = \frac{\partial}{\partial \overline{z}} \left[ \frac{1}{2i}(\dot{z}\overline{z} - z\dot{\overline{z}}) - z\overline{z} \right] = \frac{1}{2i}\dot{z} - z$$
$$\frac{\partial L}{\partial \dot{\overline{z}}} = \frac{\partial}{\partial \dot{\overline{z}}} \left[ \frac{1}{2i}(\dot{z}\overline{z} - z\dot{\overline{z}}) - z\overline{z} \right] = -\frac{1}{2i}z \qquad \Longrightarrow \qquad \frac{\partial}{\partial\alpha}\frac{\partial L}{\partial \dot{\overline{z}}} = -\frac{1}{2i}\dot{z}$$

from which it follows

$$\frac{\partial L}{\partial \overline{z}} - \frac{\partial}{\partial \alpha}\frac{\partial L}{\partial \dot{\overline{z}}} = 0 \quad \Longleftrightarrow \quad \frac{1}{2i}\dot{z} - z + \frac{1}{2i}\dot{z} = 0 \quad \Longleftrightarrow \quad \dot{z} = iz$$

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# 3 Physical interpretation

By using the algebraic form of a complex number $z \doteq f + ig$, the Lie equation $\dot{z} = iz$ reads

$$\begin{cases} \dot{f} = -g \\ \dot{g} = f \end{cases}$$

and the Lagrangian of these equations reads

$$L = \frac{1}{2}(f\dot{g} - \dot{f}g) - \frac{1}{2}(f^2 + g^2)$$

It follows from the Lie equations that

$$\ddot{z} + z = 0 \quad \Longrightarrow \quad \ddot{f} + f = 0 = \ddot{g} + g$$

The Lagrangian of the latter is

$$L(f, g, \dot{f}, \dot{g}) \doteq \frac{1}{2}\left(\dot{f}^2 + \dot{g}^2\right) - \frac{1}{2}\left(f^2 + g^2\right)$$

The quantity

$$T \doteq \frac{1}{2}\left(\dot{f}^2 + \dot{g}^2\right)$$

is the *kinetic energy* of a point $(f, g) \in \mathbb{R}^2$, meanwhile

$$l \doteq f\dot{g} - g\dot{f}$$

is its *kinetic momentum* with respect to origin $(0,0) \in \mathbb{R}^2$. By using the Lie equations one can easily check that

$$\dot{f}^2 + \dot{g}^2 = f\dot{g} - g\dot{f}$$

This relation has a simple explanation in the kinematics of a rigid body [1]. The kinetic energy of a point can be represented via its kinetic momentum as follows:

$$\frac{1}{2}\left(\dot{f}^2 + \dot{g}^2\right) = T = \frac{l}{2} = \frac{1}{2}(f\dot{g} - g\dot{f})$$

This relation explains the equivalence of the Lagrangians. Both Lagrangians give rise to the same extremals. But one must remember that this relation holds only on the extremals, i.e for the Lie equations of $SO(2)$.

## 4 Hamiltonian and Hamilton equations

Our aim is to develop canonical formalism for $SO(2)$. We have already constructed such a Lagrangian $L$ that the Euler-Lagrange equations coincides with the Lie equations. According to canonical prescription, define the *canonical momenta* as

$$\begin{aligned} p &\doteq \frac{\partial L}{\partial \dot{z}} = \frac{\partial}{\partial \dot{z}}\left[\frac{1}{2i}(\dot{z}\overline{z} - z\dot{\overline{z}}) - z\overline{z}\right] = +\frac{\overline{z}}{2i} \\ s &\doteq \frac{\partial L}{\partial \dot{\overline{z}}} = \frac{\partial}{\partial \dot{\overline{z}}}\left[\frac{1}{2i}(\dot{z}\overline{z} - z\dot{\overline{z}}) - z\overline{z}\right] = -\frac{z}{2i} = \overline{p} \end{aligned}$$

Note that the canonical momenta do not depend on velocities and so we are confronted with a *constrained system* with two *constraints*

$$\varphi_1(z, \overline{z}, p, \overline{p}) \doteq p - \frac{\overline{z}}{2i} = 0, \qquad \varphi_2(z, \overline{z}, p, \overline{p}) \doteq \overline{p} + \frac{z}{2i} = 0$$

**Definition 3 (Hamiltonian).** According to Dirac theory [2] of constrained systems, the *Hamiltonian* $H$ for $SO(2)$ can be defined by

$$\begin{aligned} H &\doteq \overbrace{p\dot{z} + \overline{p}\dot{\overline{z}} - L}^{H'} + \lambda_1\varphi_1(z, \overline{z}, p, \overline{p}) + \lambda_2\varphi_2(z, \overline{z}, p, \overline{p}) \\ &= p\dot{z} + \overline{p}\dot{\overline{z}} - L + \lambda_1\left(p - \frac{\overline{z}}{2i}\right) + \lambda_2\left(\overline{p} + \frac{z}{2i}\right) \end{aligned}$$

where $\lambda_1$ and $\lambda_2$ are the *Lagrange multipliers.*

**Proposition 4.** *The Hamiltonian of $SO(2)$ can be presented as*

$$H = z\overline{z} + \lambda_1 \left( p - \frac{\overline{z}}{2i} \right) + \lambda_2 \left( \overline{p} + \frac{z}{2i} \right)$$

**Proof.** It is sufficient to calculate

$$\begin{aligned} H' &\doteq p\dot{z} + \overline{p}\,\dot{\overline{z}} - L \\ &= p\dot{z} + \overline{p}\,\dot{\overline{z}} - \frac{1}{2i}(\dot{z}\overline{z} - z\dot{\overline{z}}) + z\overline{z} \\ &= \dot{z}\left( p - \frac{\overline{z}}{2i} \right) + \dot{\overline{z}}\left( \overline{p} + \frac{z}{2i} \right) + z\overline{z} \\ &= z\overline{z} \end{aligned}$$

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**Theorem 5 (Hamiltonian equations).** *If the Lagrange multipliers*

$$\lambda_1 = iz, \qquad \lambda_2 = -i\overline{z} = \overline{\lambda_1}$$

*then the Hamiltonian equations*

$$\dot{z} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial z},$$

*coincide with the Lie equations of $SO(2)$.*

**Proof.** Really, calculate

$$\begin{aligned} \frac{\partial H}{\partial p} &= \frac{\partial}{\partial p}\left[ z\overline{z} + iz\left( p - \frac{\overline{z}}{2i} \right) - i\overline{z}\left( \overline{p} + \frac{z}{2i} \right) \right] = iz = \dot{z} \\ \frac{\partial H}{\partial z} &= \frac{\partial}{\partial z}\left[ z\overline{z} + iz\left( p - \frac{\overline{z}}{2i} \right) - i\overline{z}\left( \overline{p} + \frac{z}{2i} \right) \right] \\ &= \overline{z} + i\left( p - \frac{\overline{z}}{2i} \right) - i\overline{z}\frac{1}{2i} = \frac{\overline{z}}{2} + ip - \frac{\overline{z}}{2} = ip \\ &= -\left( \frac{\overline{z}}{2i} \right)^{\cdot} = -\dot{p} \end{aligned}$$

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**Remark 6.** One must remember that the constraints must be applied **after** the calculations of the partial derivatives of $H$.

**Corollary 7.** *The Hamiltonian of $SO(2)$ can be presented in the form*

$$H = i(zp - \overline{z}\,\overline{p})$$

*Then the Hamilton equations coincide with the Lie equations of $SO(2)$.*

## 5 Poisson brackets and constraint algebra

**Definition 8 (observables and Poisson brackets).** Sufficiently smooth functions of the canonical varibles are called *observables.* The *Poisson brackets* of the observables $F$ and $G$ are defined by

$$\{F, G\} \doteq \frac{\partial F}{\partial z}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial z} + \frac{\partial F}{\partial \overline{z}}\frac{\partial G}{\partial \overline{p}} - \frac{\partial F}{\partial \overline{p}}\frac{\partial G}{\partial \overline{z}}$$

**Example 9 (well known).** In particular, one can easily check that

$$\{z, p\} = 1 = \{\overline{z}, \overline{p}\}$$

and all other Poisson brackets between canonical variables identically vanish.

**Example 10.** In particular,

$$\{\varphi_1, H'\} = \left\{p + \frac{\overline{z}}{2i}, H'\right\} = -\frac{\partial H'}{\partial z} + \frac{1}{2i}\frac{\partial H'}{\partial \overline{p}} = -\frac{\partial}{\partial z}(z\overline{z}) = -\overline{z}$$

and similarly

$$\{\varphi_2, H'\} = \left\{\overline{p} + \frac{z}{2i}, H'\right\} = \frac{1}{2i}\frac{\partial H'}{\partial p} - \frac{\partial H'}{\partial \overline{z}} = -\frac{\partial}{\partial \overline{z}}(z\overline{z}) = -z$$

**Definition 11 (weak equality).** Observables $A$ and $B$ are called *weakly equal*, if

$$(A - B)\Big|_{\varphi_1 = 0 = \varphi_2} = 0$$

In this case we write $A \approx B$.

Using the notion of a *weak* equality one can propose the

**Theorem 12.** *The Lie equations of $SO(2)$ read*

$$\dot{z} \approx \frac{\partial H}{\partial p}, \qquad \dot{p} \approx -\frac{\partial H}{\partial z},$$

**Theorem 13.** *Lie equations of $SO(2)$ can be presented in the Poisson-Hamilton form*

$$\dot{z} \approx \{z, H\}, \qquad \dot{p} \approx \{p, H\},$$

**Proof.** As an example, check the second equation. We have

$$\{p, H\} \doteq \frac{\partial p}{\partial z}\frac{\partial H}{\partial p} - \frac{\partial p}{\partial p}\frac{\partial H}{\partial z} + \frac{\partial p}{\partial \overline{z}}\frac{\partial H}{\partial \overline{p}} - \frac{\partial p}{\partial \overline{p}}\frac{\partial H}{\partial \overline{z}} = -\frac{\partial H}{\partial z} \approx \dot{p}$$

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**Theorem 14.** *The equation of motion of an observable $F$ reads*

$$\dot{F} \approx \{F, H\}$$

**Proof.** By using the Hamilton equations, calculate

$$\begin{aligned}\dot{F} &= \frac{\partial F}{\partial z}\dot{z} + \frac{\partial F}{\partial p}\dot{p} + \frac{\partial F}{\partial \overline{z}}\dot{\overline{z}} + \frac{\partial F}{\partial \overline{p}}\dot{\overline{p}} \\ &\approx \frac{\partial F}{\partial z}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial z} + \frac{\partial F}{\partial \overline{z}}\frac{\partial H}{\partial \overline{p}} - \frac{\partial F}{\partial \overline{p}}\frac{\partial H}{\partial \overline{z}} \\ &\doteq \{F, H\}\end{aligned}$$

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**Theorem 15 (constraint algebra).** *Constraints of $SO(2)$ satsify the commutation relations*

$$\{\varphi_1, \varphi_1\} = 0 = \{\varphi_2, \varphi_2\}, \qquad \{\varphi_1, \varphi_2\} = i$$

**Proof.** First two relations are evident. To check the third one, calculate

$$\begin{aligned}4i^2\{\varphi_1, \varphi_2\} &= \{2ip - \overline{z}, 2i\overline{p} + z\} \\ &\doteq \frac{\partial(2ip - \overline{z})}{\partial z}\frac{\partial(2i\overline{p} + z)}{\partial p} - \frac{\partial(2ip - \overline{z})}{\partial p}\frac{\partial(2i\overline{p} + z)}{\partial z} \\ &+ \frac{\partial(2ip - \overline{z})}{\partial \overline{z}}\frac{\partial(2i\overline{p} + z)}{\partial \overline{p}} - \frac{\partial(2ip - \overline{z})}{\partial \overline{p}}\frac{\partial(2i\overline{p} + z)}{\partial \overline{z}} \\ &= -2i\frac{\partial(2i\overline{p} + z)}{\partial z} - \frac{\partial(2i\overline{p} + z)}{\partial \overline{p}} \\ &= -2i - 2i = -4i\end{aligned}$$

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## 6 Consistency

Now consider the dynamical behaviour of the constraints. Note that

$$\varphi_1 = 0 = \varphi_2 \quad \Longrightarrow \quad \dot{\varphi}_1 = 0 = \dot{\varphi}_2$$

To be consistent with equations of motion we must prove the

**Theorem 16 (consistency).** *The constraints of $SO(2)$ satisfy equations*

$$\{\varphi_1, H\} \approx \dot{\varphi}_1 = 0, \qquad \{\varphi_2, H\} \approx \dot{\varphi}_2 = 0$$

**Proof.** Really, first calculate

$$\begin{aligned}\{\varphi_1, H\} &\doteq \{\varphi_1, H' + \lambda_1\varphi_1 + \lambda_2\varphi_2\} \\ &\approx \{\varphi_1, H'\} + \lambda_1\{\varphi_1, \varphi_1\} + \lambda_2\{\varphi_1, \varphi_2\} \\ &= -\overline{z} + \lambda_1 \cdot 0 + \lambda_2 \cdot i \\ &= -\overline{z} + \overline{z} \\ &= 0 \\ &= \dot{\varphi}_1\end{aligned}$$

Similarly

$$
\begin{aligned}
\{\varphi_2, H\} &\doteq \{\varphi_2, H' + \lambda_1\varphi_1 + \lambda_2\varphi_2\} \\
&\approx \{\varphi_2, H'\} + \lambda_1\{\varphi_2, \varphi_1\} + \lambda_2\{\varphi_2, \varphi_2\} \\
&= -z - \lambda_1 \cdot i + \lambda_2 \cdot 0 \\
&= -z + z \\
&= 0 \\
&= \dot{\varphi}_2
\end{aligned}
$$

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## References

[1] H. Goldstein. Classical mechanics. Addison-Wesley Press, Cambridge, 1953.

[2] P. Dirac. Lectures on Quantum Mechanics. Yeshiva Univ, New York, 1964.