On the matrix $3 \times 3$ exact solvable models of the type $G_2$

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Abstract

We study the exact solvable $3 \times 3$ matrix model of the type $G_2$. We apply the construction similar to that one, which give the $2 \times 2$ matrix model. But in the studied case the construction does not give symmetric matrix potential. We conceive that the exact solvable $3 \times 3$ matrix potential model of the type $G_2$ does not exist.

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1 Introduction

In this note we continue in the study of the matrix exact solvable models [1]. We discuss the $3 \times 3$ matrix model of the type $G_2$ of the Calogero model [2]. For a comprehensive review of these systems connected with different root systems see [3].

We applied the method developed in [4] to the $2 \times 2$ matrix models of the type $A_2$ [4], $BC_2$ [5] and $G_2$ type in [6] and to the matrix $3 \times 3$ models of the type $A_2$ [4] and $BC_2$ [6]. Some general results for $N \times N$ matrix models of the type $A_2$ was obtained in [7]. It is shown that the method and especially the simplification used for $2 \times 2$ matrix models or $3 \times 3$ matrix models of the $A_2$ and $BC_2$ do not gives the symmetric $3 \times 3$ model of the $G_2$ type.

This is a reason for our conjecture that the exact solvable $3 \times 3$ matrix model of the type $G_2$ does not exist.
2 General construction

Let us consider the differential operator

\[ \mathbf{H} = \eta^{ik} \partial_{ik} - \mathbf{U}, \tag{2.1} \]

where \( \eta^{ik} \) is symmetric constant matrix, \( \partial_{ik} = \frac{\partial}{\partial x_k} \) and \( \mathbf{U} \) is matrix function of the type \( N \times N \). The aim of our general construction is to find operator (2.1), which leads after transformation

\[ \hat{\mathbf{H}} = \mathbf{G}^{-1} \mathbf{H} \mathbf{G}, \]

where \( \mathbf{G} \) is a regular matrix function, and change of variables \( y_r = y_r(x_k) \) to the differential operator

\[ \hat{\mathbf{H}} = g^{rs}(y) \partial_{rs} + 2b^r(y) \partial_y + \mathbf{V}(y), \tag{2.2} \]

for which we know finite dimensional invariant spaces. In the paper [4] are shown the conditions, which the matrix functions \( b^r \) and \( \mathbf{V} \) have to fulfill, and construction of the operator (2.1) by means of this functions. We briefly remind these conditions.

If we write

\[ \partial_k \mathbf{G} = \mathbf{GX}_k(x) \quad \text{or} \quad \partial_r \mathbf{G} = \mathbf{GY}_r(y), \tag{2.3} \]

the matrix functions \( \mathbf{X}_k(x) \) or \( \mathbf{Y}_r(y) \) must fulfil the compatibility conditions

\[ \partial_k \mathbf{X}_i - \partial_i \mathbf{X}_k = [\mathbf{X}_i \mathbf{X}_k] \quad \text{or} \quad \partial_s \mathbf{Y}_r - \partial_r \mathbf{Y}_s = [\mathbf{Y}_r \mathbf{Y}_s]. \tag{2.4} \]

The matrix functions \( b^r \) and \( \mathbf{Y}_r \) are connected by the relation

\[ b^r = g^{rs} \mathbf{Y}_s - \frac{1}{2} \Gamma^r, \]

where we denote

\[ \Gamma^r = g^{st} \Gamma^r_{st}, \quad \Gamma^r_{st} = g^{rk} \Gamma^r_{st,k}, \quad \text{and} \quad \Gamma^r_{st,k} = \frac{1}{2} (-\partial_k g_{st} + \partial_s g_{tk} + \partial_t g_{sk}), \]

and \( g_{rs} \) is inverse of the \( g^{rs} \).

If we denote \( b_r = g_{rs} b^s \) and introduce

\[ T_r = \frac{1}{N} \text{Tr} b_r, \quad \hat{b}_r = b_r - T_r, \]

we can rewrite the compatibility conditions (2.4) in the form

\[ \partial_s (T_r + \frac{1}{2} \Gamma_r) - \partial_r (T_s + \frac{1}{2} \Gamma_s) = 0 \tag{2.5} \]

\[ \partial_s \hat{b}_r - \partial_r \hat{b}_s = [\hat{b}_r, \hat{b}_s]. \tag{2.6} \]

From the equation (2.5) follows that exist function \( F(y) \) such that

\[ \partial_r F = T_r + \frac{1}{2} \Gamma_r. \tag{2.7} \]

Denoting \( \mathbf{G} = e^F \hat{\mathbf{G}} \) we can write the equation (2.3) in the form

\[ \partial_r \hat{\mathbf{G}} = \hat{\mathbf{G}} \hat{b}_r. \tag{2.8} \]
If the \( \hat{G}_0 \) is solution of the equation (2.8) and \( G_0 = e^{\xi} \hat{G}_0 \), the matrix potential \( U_0(x) \) corresponding to the matrix functions \( G_0(x) \) and \( V \) can be find from relation

\[
U_0 = \left( \eta^{ik} \partial_{ik} G_0(x) - G_0(x)V \right) G_0^{-1}(x).
\]  

As the equation (2.8) is linear their general solution can be written in the form \( \hat{G} = C \hat{G}_0 \), where \( C \) is constant matrix. The potential \( U(x) \) corresponding to such solution of (2.8) is

\[
U(x) = C U_0(x) C^{-1}.
\] 

Therefore we choose matrix functions \( b^r(y), V(y) \) and constant regular matrix \( C \) to the matrix potential (2.10) be symmetric.

3 Models of the \( G_2 \) type

We will consider matrix models with\(^1\)

\[
\eta^{11} = \eta^{22} = \frac{2}{3}, \quad \eta^{12} = \eta^{21} = -\frac{1}{3}
\]

and transformation

\[
y_1 = -x_1^2 - x_1x_2 - x_2^2, \quad y_2 = -x_1x_2(x_1 + x_2).
\]

In this case we obtain

\[
g^{11} = -2y_1, \quad g^{12} = g^{21} = -3y_2, \quad g^{22} = \frac{2}{3} y_1^2.
\]  

(3.1)

It is easy to see that the differential operator \( g^{rs} \partial_{rs} \) has invariant subspaces of two type: \( V_N^{(1)} \) spaces of polynomials generated by \( y_1^n y_2^{n_2} \), where \( n_1 + n_2 \leq N \) and \( V_N^{(2)} \) spaces of polynomials generated by \( y_1^n y_2^{2n_2} \), where \( n_1 + 2n_2 \leq N \). In the scalar case the choose of the invariant spaces \( V_N^{(1)} \) leads to the models of the \( A_2 \) type and the choose invariant spaces \( V_N^{(2)} \) to the models of the \( G_2 \) type. Matrix model of the \( A_2 \) type we study in [4]. In this paper we will study the matrix models of type \( G_2 \), i.e. we will consider invariant subspaces \( V_N^{(2)} \).

Therefore we choose matrix functions \( b^r(y) \) in the form

\[
b^1 = C_0^1 + C_1^1, \quad b^2 = C_0^2 + C_1^2 \eta_2 + y_2^{-1} (C_0^1 y_1 + C_1^2 y_1^2)
\]

and \( V \) as a constant matrix.

In this case the compatibility conditions (2.4) are

\[
\begin{align*}
[C_1^1, C_2^2] &= 0, \quad & [C_1^1, C_2^3] &= 0, \\
[C_0^1, C_2^2] + [C_1^1, C_2^2] &= 0, \quad & [C_0^1, C_2^3] &= -3C_1^2 + 2C_3^2, \\
[C_0^1, C_1^2] + [C_1^1, C_0^2] &= -2C_2^2, \quad & [C_0^1, C_0^2] &= -4C_0^2.
\end{align*}
\]  

(3.2)

\(^1\)This \( \eta^{ik} \) is connected with Laplace operator in three dimension in center of mass coordinates.
In the case of $2 \times 2$ matrix model [6] we were successful with solution of the system (3.2), where we put $C^1_0 = 0$. Therefore we choose $C^1_1 = 0$ in case $3 \times 3$ matrix, too. With this choose the conditions (3.2) gives

$$[C^1_0, C^2_0] = 0, \quad [C^1_0, C^3_0] = 2C^2_3, \quad [C^1_1, C^3_1] = -2C^2_1, \quad [C^1_0, C^3_0] = -4C^2_0. \quad (3.3)$$

It seems to sensible to chose the traceless matrix $\hat{C}^1_0$ in the (3.3) as diagonal. We will study two cases:

a) $\hat{C}^1_0 = A(e_{11} - e_{33})$ and

b) $\hat{C}^1_0 = A(e_{11} - 2e_{22} + e_{33})$,

where $A$ is a constant and $e_{rs}$ are $3 \times 3$ matrices $(e_{rs})_{ik} = \delta_{ri}\delta_{sk}$.

### 3.1 Solution in the case a)

In the case a) the general solution of (3.3) is

$$C^1_0 = -3\mu - 3\nu - 1 + 2(e_{11} - e_{33}), \quad C^1_1 = -2\omega,$$

$$C^2_0 = A^2_0 e_{31}, \quad C^2_1 = A^2_1 e_{21} + B^2_1 e_{32},$$

$$C^2_2 = 2A^2_1 e_{11} - e_{22} + B^2_2 e_{22} - e_{33}, \quad C^3_2 = -3\omega + A^3_0 e_{12} + B^3_0 e_{32} + B^3_2 e_{23}$$

or for traceless matrices $\hat{C}^r_0$

$$\hat{C}^1_0 = 2(e_{11} - e_{33}), \quad \hat{C}^1_1 = 0,$$

$$\hat{C}^2_0 = A^0_0 e_{31}, \quad \hat{C}^2_1 = A^2_1 e_{21} + B^2_1 e_{32},$$

$$\hat{C}^2_2 = A^2_1 e_{11} - e_{22} + B^2_2 e_{22} - e_{33}, \quad \hat{C}^3_2 = A^3_0 e_{12} + B e_{23}$$

The system of equations (2.8) is in this case equivalent to three systems of equations

$$
(4y_1^3 + 27y_2^2)\partial_1 X = -(4 + 9A^2_1)Y_1 X - 9A^3_0 Y_1 Z - 9A^2_0 Z
$$

$$
(4y_1^3 + 27y_2^2)\partial_1 Y = 9A^2_0 Y_1 X + 9(A^2_0 - B^2_0)Y_1 Y - 9B^2_0 Y_1 Z
$$

$$
(4y_1^3 + 27y_2^2)\partial_1 Z = -9B^2_0 Y_1 Y + (4 + 9B^2_0)Y_1 Z
$$

$$
y_2(4y_1^3 + 27y_2^2)\partial_2 X = -6(3y_2^2 + A^2_0 Y_1)X + 6A^3_0 Y_1 Y + 6A^2_0 Y_1 Z
$$

$$
y_2(4y_1^3 + 27y_2^2)\partial_2 Y = 6A^3_0 Y_1 Y + 6(A^2_0 - B^2_0)Y_1 Y + 6B^2_0 Y_1 Z
$$

$$
y_2(4y_1^3 + 27y_2^2)\partial_2 Z = 6B^2_0 Y_1 Y + 6(3y_2^2 - B^2_0 Y_1) Z
$$

where $X = \hat{G}_{k1}$, $Y = \hat{G}_{k2}$ and $Z = \hat{G}_{k3}$, $k = 1, 2, 3$.

It is easy to see that from the system (3.5) follow

$$2y_1\partial_1 X + 3y_2\partial_2 X = -2X, \quad 2y_1\partial_1 Y + 3y_2\partial_2 Y = 0, \quad 2y_1\partial_1 Z + 3y_2\partial_2 Z = 2Z,$$

which gives

$$X = y_1^{-1} F(t), \quad Y = G(t), \quad Z = y_1 H(t), \quad \frac{y_2}{y_1^3}.$$
The functions $F$, $G$ and $H$ then fulfill the system of equations

\[
\begin{align*}
(t + 27t)F' &= 3(A_2^2 - 3t)F + 3A_2^2G + 3A_2^3H, \\
(t + 27t)G' &= 3A_2^2TF - 3(A_2^2 - B_2^2)G + 3B_2^2H, \\
(t + 27t)H' &= 3B_2^2tG - 3(B_2^2 - 3t)H.
\end{align*}
\]  

To find three independent solution of system (3.6) we choose with analogy of $2 \times 2$ matrix model special value of constants $A_s^r$ and $B_s^r$, which essentially simplify the system (3.6).  

If we chose

\[
A_0^2 = \frac{4}{\pi}, \quad A_1^2 = -\frac{16}{\pi}, \quad A_2^2 = -\frac{4}{\pi}, \quad A_3^2 = -3, \\
B_0^2 = \frac{4}{\pi}, \quad B_2^2 = -\frac{8}{\pi}, \quad B_3^2 = -6,
\]

three independent solution of the system (3.6) are

\[
\begin{align*}
F_1 &= G_1 = H_1 = \frac{t^{2/3}}{4 + 27t}, \\
F_2 &= G_2 = \frac{4t^{1/3}}{4 + 27t}, \quad H_2 = -\frac{27t^{4/3}}{4 + 27t}; \\
F_3 &= \frac{4(8t + 4)}{t(4 + 27t)}, \quad G_3 = \frac{27(4 - 27t)}{4 + 27t}, \quad H_3 = -\frac{1458t}{4 + 27t}.
\end{align*}
\]

In our case the function $e^F$ is

\[
e^F = \left((x_1 - x_2)(2x_1 + x_2)(x_1 + 2x_2)\right)^{\nu} \left(x_1 x_2(x_1 + x_2)\right)^{\nu} e^{-\omega(x_1^2 + x_2^2)},
\]

which gives matrix $G_0(x)$. By direct calculation it is possible to show that corresponding matrix potential $U_0$ can not be symmetrize by any choose of constant matrices $V$ and $C$.

### 3.2 Solution in the case b)

In this case the general solution of (3.3) is

\[
\begin{align*}
C_0^1 &= -3\mu - 3\nu - 1 + \frac{2}{3} (e_{11} - 2e_{22} + e_{33}), \\
C_1^1 &= -2\omega, \\
C_2^0 &= 0, \\
C_1^2 &= A_2^0 e_{21} + B_2^0 e_{23}, \\
C_2^2 &= \frac{2}{3} \nu + \alpha (e_{11} - e_{22}) + \beta (e_{22} - e_{33}) + A_2^0 e_{13} + B_2^0 e_{31}, \\
C_3^0 &= -3\omega + A_3^2 e_{12} + B_3^2 e_{32}
\end{align*}
\]

or for traceless matrices $C^r_s$

\[
\begin{align*}
\tilde{C}_0^1 &= \frac{2}{3} (e_{11} - 2e_{22} + e_{33}), \\
\tilde{C}_1^1 &= 0, \\
\tilde{C}_2^0 &= 0, \\
\tilde{C}_1^2 &= A_2^0 e_{21} + B_2^0 e_{23}, \\
\tilde{C}_2^2 &= \alpha (e_{11} - e_{22}) + \beta (e_{22} - e_{33}) + A_2^0 e_{13} + B_2^0 e_{31}, \\
\tilde{C}_3^0 &= A_3^2 e_{12} + B_3^2 e_{32}
\end{align*}
\]

\(^2\)In the other case in the solution of (3.6) appear hypergeometric functions.
To solve the system (2.8) we have to find three independent solutions of the system

\[
\begin{align*}
(4y_1^3 + 27y_2^2)\partial_1 X &= -\frac{1}{3} (4 + 27\alpha) y_1^2 X - 9A_1^2 y_1 Y - 9B_2^2 y_1^2 Z \\
(4y_1^3 + 27y_2^2)\partial_4 Y &= -9A_2^2 y_1^2 X + \frac{1}{3} (8 + 27\alpha - 27\beta) y_1^2 Y - 9B_2^2 y_1^2 Z \\
(4y_1^3 + 27y_2^2)\partial_1 Z &= -9A_2^2 y_1^2 X - 9B_1^2 y_1 Y - \frac{1}{3} (4 - 27\beta) y_1^2 Z \\
y_2(4y_1^3 + 27y_2^2)\partial_2 X &= -6(y_2^2 - \alpha y_1^2) X + 6A_1^2 y_1^2 Y + 6B_2^2 y_1^2 Z \\
y_2(4y_1^3 + 27y_2^2)\partial_2 Y &= 6A_2^2 y_1^2 X + 6(2y_2^2 - (\alpha - \beta)y_1^2) Y + 6B_2^2 y_1^2 Z \\
y_2(4y_1^3 + 27y_2^2)\partial_1 Z &= 6A_2^2 y_1^2 X + 6B_1^2 y_1^2 Y - 6(y_2^2 + \beta y_1^2) Z \\
\end{align*}
\]

(3.10)

From the system (3.10) we obtain relations

\[
\begin{align*}
6y_1\partial_1 X + 9y_2\partial_2 X &= -2X, \\
6y_1\partial_1 Y + 9y_2\partial_2 Y &= 4Y, \\
6y_1\partial_1 Z + 9y_2\partial_2 Z &= -2Z,
\end{align*}
\]

from which follow

\[
X = y_1^{-1/3} F(t), \quad Y = y_1^{2/3} G(t), \quad Z = y_1^{-1/3} H(t), \quad t = \frac{y_2}{y_1}.
\]

Functions \( F(t), \ G(t) \) and \( H(t) \) fulfill system differential equations

\[
\begin{align*}
t(4 + 27t)F' &= 3(\alpha - t)F + 3A_1^2G + 3B_2^2H \\
t(4 + 27t)G' &= 3A_2^2tF - 3(\alpha - \beta - 2t)G + 3B_2^2tH \\
t(4 + 27t)H' &= 3A_2^2F + 3B_2^2G - 3(\beta + t)H \\
\end{align*}
\]

(3.11)

To solve (3.11) we again choose convenient constants.

First possibility is to choose

\[
A_1^2 = \frac{8}{9}, \quad A_2^2 = \frac{10}{9} (1 + p), \quad A_3^2 = -3, \\
B_1^2 = \frac{10}{9} (1 - p), \quad B_2^2 = 0, \quad B_3^2 = 0,
\]

(3.12)

In this case we have

\[
\begin{align*}
F_1 &= G_1 = H_1 = \frac{t^{7/9}}{(4 + 27t)^{8/9}} \\
F_2 &= G_2 = 0, \quad H_2 = \frac{(4 + 27t)^{7/9}}{t^{8/9}} \\
F_3 &= \frac{4t^{1/9}}{(4 + 27t)^{8/9}}, \quad G_3 = -\frac{27t^{10/9}}{(4 + 27t)^{8/9}} \quad H_3 = \frac{4 - 20p + 135(1 - p)t}{27t^{8/9}(4 + 27t)^{8/9}}.
\end{align*}
\]

(3.13)

The second possibility is to choose

\[
A_1^2 = -\frac{8}{3}, \quad A_2^2 = -\frac{2}{3} (1 + p), \quad A_3^2 = -3, \\
B_1^2 = -\frac{2}{3} (1 - p), \quad B_2^2 = 0, \quad B_3^2 = 0, \\
\alpha = \frac{4}{3}, \quad \beta = 0
\]

(3.14)
and three solutions of (3.11) are

\[ F_1 = G_1 = H_1 = \frac{(4 + 27t)^{8/9}}{t}, \]
\[ F_2 = G_2 = 0, \quad H_2 = \frac{1}{(4 + 27t)^{1/9}}, \]
\[ F_3 = \frac{4(4 + 45t)}{t(4 + 27t)^{7/9}}, \quad G_3 = \frac{16 + 180t + 405t^2}{t(4 + 27t)^{7/9}}, \quad H_3 = \frac{16 + 45(1 - p)t}{t(4 + 27t)^{7/9}}. \]

In the third case we choose

\[ A_1^2 = -\frac{4}{27}, \quad A_2^2 = -\frac{10}{27} (1 + p), \quad A_3^2 = -3, \]
\[ B_1^2 = -\frac{40}{27} (1 - p), \quad B_2^2 = 0, \quad B_3^2 = 0, \]
\[ \alpha = \frac{4}{27}, \quad \beta = -\frac{28}{27} \]

and independent solutions of (3.11) are

\[ F_1 = G_1 = H_1 = \frac{(4 + 27t)^{7/9}}{t^{8/9}}, \]
\[ F_2 = G_2 = 0, \quad H_2 = \frac{t^{7/9}}{(4 + 27t)^{8/9}}, \]
\[ F_3 = \frac{2(4 + 27t)^{1/9}}{t^{8/9}}, \quad G_3 = \frac{(4 + 27t)^{1/9}(9t + 2)}{t^{8/9}}, \quad H_3 = \frac{8 + 45(1 - p)t}{t^{8/9}(4 + 27t)^{8/9}}. \]

In the last interesting case the constants are

\[ A_1^2 = \frac{28}{27}, \quad A_2^2 = \frac{7}{27} (1 + p), \quad A_3^2 = -3, \]
\[ B_1^2 = \frac{2}{3} (1 - p), \quad B_2^2 = 0, \quad B_3^2 = 0, \]
\[ \alpha = -\frac{28}{27}, \quad \beta = \frac{1}{27} \]

and in this case the three independent solutions are, e.g.

\[ F_1 = G_1 = H_1 = \frac{t^{8/9}}{4 + 27t}, \]
\[ F_2 = G_2 = 0, \quad H_2 = t^{-1/9}, \]
\[ F_3 = \frac{2(8 + 135t)}{3t^{7/9}(4 + 27t)}, \quad G_3 = \frac{9t^{2/9}(2 - 27t)}{4 + 27t}, \quad H_3 = \frac{4(1 + p) - 27(1 - p)t}{t^{7/9}(4 + 27t)}. \]

The function \( e^F \) is in all discussed case given by relation (3.8).

### 4 Potential

To compute the corresponding potential we first use formulae (2.9).

The most interesting choose of the constant matrix \( V \) is

\[ V = -2\omega(3\mu + 3\nu + 1) + \frac{1}{3} \omega(e_{11} - 2e_{22} + e_{33}) + Ae_{12} + Be_{32}, \]

where \( A \) and \( B \) are suitable constants. The other choose of the matrix \( V \) leads to the matrix function in the potential, which must be symmetrize simultaneously with the following matrix potential.
With this choice we obtain by direct computation

\[
U_0 = \left( \eta^{ik} \partial_k G_0(x) \right) - G_0 V \right) G_0^{-1} = U_0^{(s)} + U_0^{(m)},
\]

where

\[
U_0^{(s)} = 2\omega^2 (x_1^2 + x_1 x_2 + x_2^2) + \\
+ 2(\mu^2 - \mu + \frac{5}{3}) \left( \frac{1}{(x_1 - x_2)^2} + \frac{1}{(2x_1 + x_2)^2} + \frac{1}{(x_1 + x_2)^2} \right) + \\
+ \frac{2}{9} (\nu^2 - \nu + \frac{152}{27}) \left( \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{(x_1 + x_2)^2} \right)
\]

and \(U_0^{(m)}\) is the traceless part of the potential, which is given as follows

\[
U_{11}^{(m)} = \frac{-12\mu(x_1^2 + x_1 x_2 + x_2^2)^2}{(x_1 - x_2)^4(2x_1 + x_2)^4(x_1 + 2x_2)^4} \\
\times \left( 16x_1^6 + 48x_1^5 x_2 + 69x_1^4 x_2^2 + 58x_1^3 x_2^3 + 69x_1^2 x_2^4 + 48x_1 x_2^5 + 16x_2^6 \right) + \\
\times \left( 46x_1^5 + 24x_1^4 x_2 - 87x_1^3 x_2^2 - 214x_1^2 x_2^3 - 87x_1 x_2^4 + 24x_2^5 + 8x_2^6 \right) + \\
\times \left( \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{(x_1 + x_2)^2} \right) + \\
\times \left( 2(6\nu + A + B)(x_1^2 + x_1 x_2 + x_2^2)^2 \right) \\
\times \left( (x_1 - x_2)^4(2x_1 + x_2)^4(x_1 + 2x_2)^4 \right) \\
\times \left( 8x_1^6 + 24x_1^5 x_2 - 87x_1^4 x_2^2 - 214x_1^3 x_2^3 - 87x_1^2 x_2^4 + 24x_1 x_2^5 + 8x_2^6 \right) + \\
+ \frac{8}{9} \left( x_1^2 + x_1 x_2 + x_2^2 \right) \times \\
\times \left( 94x_1^6 + 282x_1^5 x_2 - 111x_1^4 x_2^2 - 692x_1^3 x_2^3 - 111x_1^2 x_2^4 + 282x_1 x_2^5 + 94x_2^6 \right)
\]

\[
U_{22}^{(m)} = \frac{-27}{4\pi} \left( \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{(x_1 + x_2)^2} \right) - \\
- \frac{1}{4\pi} (B(3p + 7) + 48) \left( \frac{1}{(x_1 - x_2)^2} + \frac{1}{(2x_1 + x_2)^2} + \frac{1}{(x_1 + 2x_2)^2} \right)
\]

\[
U_{33}^{(m)} = -U_{11}^{(m)} - U_{22}^{(m)}
\]

\[
U_{12}^{(m)} = - \frac{3(p + 7)(6\mu + 6\nu + A + B + \frac{2}{3}) x_1^2 x_2^2 (x_1 + x_2)^2 x_1^2 + x_1 x_2 + x_2^2}{(x_1 - x_2)^4(2x_1 + x_2)^4(x_1 + 2x_2)^4}
\]

\[
U_{13}^{(m)} = - \frac{3(6\mu + 6\nu + A + B + \frac{2}{3}) x_1 x_2 (x_1 + x_2)^2}{(x_1 - x_2)^4(2x_1 + x_2)^4(x_1 + 2x_2)^4}
\]

\[
U_{23}^{(m)} = - \frac{3B(x_1 x_2 (x_1 + x_2))^4}{(x_1 - x_2)^2(2x_1 + x_2)^2(x_1 + 2x_2)^2}
\]

\[
U_{21}^{(m)} = \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{(x_1 + x_2)^2} - \\
- 18B \left( \frac{1}{(x_1 - x_2)^2} + \frac{1}{(2x_1 + x_2)^2} + \frac{1}{(x_1 + 2x_2)^2} \right)
\]
On the matrix $3 \times 3$ exact solvable models of the type $G_2$
\begin{equation}
+ \frac{8}{3} \left( 38x_1^{12} + 228x_1^{11}x_2 + 411x_1^{10}x_2^2 - 35x_1^9x_2^3 - 639x_1^8x_2^4 + 162x_1^7x_2^5 + 1128x_1^6x_2^6 + 162x_1^5x_2^7 - 639x_1^4x_2^8 - 35x_1^3x_2^9 + 411x_1^2x_2^{10} + 228x_1x_2^{11} + 38x_2^{12} \right) - \\
- \frac{2}{3} A(x_1^2 + x_1x_2 + x_2^2)^3 \left( 8x_1^6 + 24x_1^5x_2 - 87x_1^4x_2^2 - 214x_1^3x_2^3 - 87x_1^2x_2^4 + 24x_1x_2^5 + 8x_2^6 \right) + \\
+B(x_1^2 + x_1x_2 + x_2^2)^3 \left( 4x_1^6 + 12x_1^5x_2 + 51x_1^4x_2^2 + 82x_1^3x_2^3 + 12x_1^2x_2^4 + 4x_1x_2^5 + 4x_2^6 + 4px_1^6 + 12px_1^5x_2 - 3px_1^4x_2^2 - 26px_1^3x_2^3 - 3px_1^2x_2^4 + 12px_1x_2^5 + 4px_2^6 \right)
\end{equation}

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References