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Vectorial Regularization and Temporal Means in Keplerian Motion

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Abstract

We study the well-known Kepler's problem by introducing a new vectorial regularization. This helps deduce Kepler's equations by a simple and unified method. Some integral temporal means are also obtained by means of this regularization. The vectorial eccentricity plays a fundamental part in this approach.

1 Introduction

We consider the classical Cauchy problem describing Keplerian motion:

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}, \mathbf{r}(0) = \mathbf{r}_0, \dot{\mathbf{r}}(0) = \mathbf{v}_0 \qquad (*)$$

with $\mu > 0$ the gravitational parameter, $\mathbf{r}$ the position vector of the body related to the attraction center and $\mathbf{v}$ the velocity vector.

In Sec. 2, we will present some known results, such as the prime integrals of the Keplerian motion. The third of these prime integrals has been named "Laplace-Runge-Lenz vector" by some authors or "Hermann-Bernoulli-Laplace vector" by others (see [8]). We will name it **vectorial eccentricity**. This vectorial eccentricity provides an easy way of studying the characteristics of the trajectory. Its direction is identical with the main semiaxis of the conic and it has the direction of the pericenter. Its norm is the "scalar" eccentricity of the conic.

In Sec. 3, using a variable substitution in equation $(*)$, we will transform the time-variable t in a new distance-dependant time variable τ. Kepler's problem becomes an ordinary vectorial second order differential linear equation in this new variable. The solution to this equation will give new expressions of the motion's characteristics, using only this new time-variable. By using these expressions, we will suggest a simple and unitary way to deduce Kepler's equations for computing the time of motion on the trajectory in the three possible cases (ellipse, parabola, hyperbola). This simple vectorial regularization is possible only due to the vectorial character of the eccentricity we found here. The new Cauchy problem that models Keplerian motion eliminates the singularity when $r = 0$. It allows to study the rectilinear Keplerian motion as well.

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Several regularizations were given till now without using this vectorial eccentricity. Among them: that of Levi-Civita in 1920 using complex numbers in the planar case (see [16]) and the one of Kustaanheimo in 1964 for the spatial case (see [12], [13]), this one with spinors. The regularization presented in this paper has the advantage of giving an unitary and simple way to approach Kepler's problem as well as to deduce some characteristics for the planar motion only by elementary vectorial computations.

Using this vectorial regularization, we will compute some integral temporal means related to the Keplerian motion in Sec. 6, means that have the form:

$$\langle \mathbf{f} \rangle = \lim_{T\to\infty} \left(\frac{1}{T} \int_0^T \mathbf{f}\,(t)\,dt \right).$$

Some of them are known, others are presented for the first time. We generalize Laplace's formula for the integral temporal mean of r on a period in the elliptic motion, by giving a simple way of computing the temporal mean of r^n, for any integer n.

2 Kepler's Problem: A Vectorial Solution

The prime integrals of equation $(*)$ are (see [7], [8]):

1. Angular momentum conservation:

$$\mathbf{r} \times \mathbf{v} = \mathbf{r}_0 \times \mathbf{v}_0 \overset{\text{not}}{=} 2\boldsymbol{\Omega} \tag{2.1}$$

2. Energy conservation:

$$\frac{1}{2}\mathbf{v}^2 - \frac{\mu}{r} = \frac{1}{2}\mathbf{v}_0^2 - \frac{\mu}{r_0} \overset{\text{not}}{=} h \tag{2.2}$$

3. The Laplace-Runge-Lenz vector (the vectorial eccentricity):

$$\frac{\mathbf{v} \times (\mathbf{r} \times \mathbf{v})}{\mu} - \frac{\mathbf{r}}{r} = \frac{\mathbf{v}_0 \times (\mathbf{r}_0 \times \mathbf{v}_0)}{\mu} - \frac{\mathbf{r}_0}{r_0} \overset{\text{not}}{=} \mathbf{e} \tag{2.3}$$

The following denotations: $\dot{\mathbf{u}} \overset{\text{not}}{=} \frac{d}{dt}\mathbf{u}$; $\mathbf{v} \overset{\text{not}}{=} \dot{\mathbf{r}}$; $u \overset{\text{not}}{=} |\mathbf{u}|$; $\mathbf{e_u} \overset{\text{not}}{=} \frac{\mathbf{u}}{u} = vers\mathbf{u}$ will be used.

From (2.3), we get $\mathbf{v} \times (\mathbf{r} \times \mathbf{v}) = \mu\,(\mathbf{e} + \mathbf{e_r})$, and by cross-multiplying with $\mathbf{r} \times \mathbf{v}$ it results:

$$\mathbf{v} = \boldsymbol{\alpha} \times (\mathbf{e} + \mathbf{e_r}) \tag{2.4}$$

By dot-multiplying $\mathbf{v} \times (\mathbf{r} \times \mathbf{v}) = \mu\,(\mathbf{e} + \mathbf{e_r})$ with $\mathbf{r}$, it results:

$$\mathbf{r} \cdot (\mathbf{e} + \mathbf{e_r}) \overset{\text{not}}{=} p \tag{2.5}$$

In relation (2.4) $\boldsymbol{\alpha} = \frac{\mu}{2\Omega^2}\boldsymbol{\Omega}$ is a constant vector and in (2.5) $p = \frac{4\Omega^2}{\mu}$.

From equation (2.1) we get: $\mathbf{r}\cdot\mathbf{\Omega}=\mathbf{0}$, so the trajectory is situated in a plane, as $\mathbf{\Omega}$ is a constant vector. From (2.5), taking γ the angle between $\mathbf{e}$ and $\mathbf{e_r}$, it results:

$$r=\frac{p}{1+e\cos\gamma} \tag{2.6}$$

The trajectory is then a **conic** with one focus in the attraction center, the parameter $p=\frac{4\Omega^2}{\mu}$, called the **semilatus rectum,** and eccentricity e.

Vector $\mathbf{e}$ defined in (2.3) is named **eccentricity vector** and:

1. it has the orientation of the main semiaxis of the conic;
2. its sense indicates the pericenter of the conic;
3. its magnitude equals the eccentricity of the conic.

The trajectory is an ellipse if $e<1$, a parabola if $e=1$ and a hyperbola if $e>1$. The case $e=0$ leads to a circular trajectory.

Relations (2.1)-(2.3) lead to some interesting formulas by elementary vectorial computations: $\mathbf{e}\times\mathbf{r}=\frac{2(\mathbf{r}\cdot\mathbf{v})}{\mu}\mathbf{\Omega}$, $\mathbf{e}\cdot\mathbf{r}=\frac{4\Omega^2}{\mu}-r$, $\mathbf{e}\times\mathbf{v}=\frac{2}{\mu}\left(\mathbf{v}^2-\frac{\mu}{r}\right)\mathbf{\Omega}=2\left(\frac{2h}{\mu}+\frac{1}{r}\right)\mathbf{\Omega}$, $\mathbf{e}\cdot\mathbf{v}=-\mathbf{e_r}\cdot\mathbf{v}=-\dot{r}$. These relations are useful in deducing the characteristics of the motion from the initial conditions (the direction of the main semiaxis, their magnitude and the pericenter position).

Using relation (2.3), we deduce the magnitude of the eccentricity vector:

$$e=\sqrt{1+\frac{8\Omega^2 h}{\mu^2}} \tag{2.7}$$

which is indeed the formula for computing the conic's eccentricity in Keplerian motions. The conic is an ellipse, a parabola or a hyperbola, as $h<0, h=0$, respectively $h>0$.

Using (2.4) we deduce that the velocity hodograph is a **section of a circle or an entire circle,** as $\boldsymbol{\alpha}\times\mathbf{e}$ is a constant vector and $\boldsymbol{\alpha}\times\mathbf{e_r}$ is a variable vector with constant norm. The radius of the circle is $|\boldsymbol{\alpha}\times\mathbf{e_r}|=\alpha=\frac{\mu}{2\Omega}$.

Remark 1. Using complex numbers, professor A. Braier found in 1965 a replica of vector $\mathbf{e}$ with specific consequences (see [1, 2, 3]). He deduced a vectorial eccentricity (see [1]) that was again discovered by various authors in the same way between 1996-2004 (see [6], [9]-[11], [17]-[19]).

3 A Time-Regularization Method

In the Cauchy problem $(*)$ we make the substitution:

$$t=t\left(\tau\right), t\left(\tau\right)=\int_0^\tau r\left(\xi\right)d\xi+t_p \tag{3.1}$$

where t_p is the moment of time when the body is situated in the pericenter. Then $dt=rd\tau$ and we may define a differential operator:

$$\left(\ \right)'\overset{not}{=}\frac{d}{d\tau}\left(\ \right)=r\dot{\left(\ \right)} \tag{3.2}$$

We have: $\tau = 0 \Rightarrow t(0) = t_p$. The new initial conditions in the variable τ are $\mathbf{r}(0) = \mathbf{r}(t_p) = \mathbf{r}_p$ and $\mathbf{r}'(0) = (r\mathbf{v})(t_p) = r_p\mathbf{v}_p$, where $\mathbf{r}_p$ is the radius vector of the pericenter related to the focus with the attractive center and $\mathbf{v}_p$ is the velocity vector of the body in the moment of time it passes by the pericenter.

For a differentiable function $\mathbf{u} : I \subseteq \mathbb{R} \to \mathbb{R}^m$, $m \geq 1$, it comes that: $\frac{d\mathbf{u}}{d\tau} = r\frac{d\mathbf{u}}{dt}$. Applying this differential rule to $\mathbf{r}$ and r, we get the first and the second derivative:

$$\mathbf{r}' = r\dot{\mathbf{r}} = r\mathbf{v}, r\prime = r\dot{r} = \mathbf{r}\cdot\mathbf{v}, \mathbf{r}'' = (r\mathbf{v})' = r'\mathbf{v} + r\mathbf{v}' = (\mathbf{r}\cdot\mathbf{v})\,\mathbf{v} + r^2\ddot{\mathbf{r}} \tag{3.3}$$

Taking into account that $\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$ (according to equation (∗)) and that

$$\mathbf{e} = \frac{\mathbf{v}\times(\mathbf{r}\times\mathbf{v})}{\mu} - \frac{\mathbf{r}}{r} = \frac{v^2\mathbf{r} - (\mathbf{v}\cdot\mathbf{r})\,\mathbf{v}}{\mu} - \frac{\mathbf{r}}{r} = \frac{1}{\mu}\left(v^2 - \frac{2\mu}{r}\right)\mathbf{r} + \frac{\mathbf{r}}{r} - \frac{(\mathbf{v}\cdot\mathbf{r})\,\mathbf{v}}{\mu} \tag{3.4}$$

(see (2.3)), we get that:

$$\mu\mathbf{e} = 2h\mathbf{r} + \mu\frac{\mathbf{r}}{r} - (\mathbf{v}\cdot\mathbf{r})\,\mathbf{v} \Longrightarrow 2h\mathbf{r} - \mu\mathbf{e} = (\mathbf{v}\cdot\mathbf{r})\,\mathbf{v} - \mu\frac{\mathbf{r}}{r} \tag{3.5}$$

Using relations (3.3) and (3.4), we may write:

$$\mathbf{r}'' = (\mathbf{r}\cdot\mathbf{v})\,\mathbf{v} - \mu\frac{\mathbf{r}}{r} \tag{3.6}$$

Finally, from (3.5) and (3.6) we get:

$$\mathbf{r}'' - 2h\mathbf{r} = -\mu\mathbf{e} \tag{3.7}$$

With this regularization, the Cauchy problem (∗) with the new function $\mathbf{r} = \mathbf{r}(\tau)$ becomes:

$$\mathbf{r}'' - 2h\mathbf{r} = -\mu\mathbf{e}, \tag{$**$}$$

$$\begin{cases} \mathbf{r}|_{\tau=0} = \mathbf{r}_P = \begin{cases} \frac{\mathbf{\Omega}^2}{\mu(1+e)}\frac{\mathbf{e}}{e}, & \text{if } \mathbf{e} \neq \mathbf{0} \\ \mathbf{r}_0, & \text{if } \mathbf{e} = \mathbf{0} \end{cases}; \\ \mathbf{r}'|_{\tau=0} = r_P\mathbf{v}_P = \begin{cases} \frac{1}{e}\mathbf{\Omega}\times\mathbf{e}, & \text{if } \mathbf{e} \neq \mathbf{0} \\ r_0\mathbf{v}_0, & \text{if } \mathbf{e} = \mathbf{0} \end{cases}, \end{cases} \tag{3.8}$$

where $\mathbf{r}_P$ and $\mathbf{v}_P$ denote the position vector, respectively the velocity of the body in the pericenter P of the conic:

$$\mathbf{r}_p = \frac{4\Omega^2}{\mu(1+e)}\frac{\mathbf{e}}{e}, \ \mathbf{v}_p = \frac{\mu(1+e)}{2e\Omega^2}\mathbf{\Omega}\times\mathbf{e} \ \ (e \neq 0) \tag{3.9}$$

In case $\mathbf{\Omega} = \mathbf{0}$ (the rectilinear Keplerian motion), then the initial conditions (3.8) become:

$$\begin{cases} \mathbf{r}|_{\tau=0} = \mathbf{0}; \\ \mathbf{r}'|_{\tau=0} = \mathbf{0}. \end{cases} \tag{3.10}$$

The collision with the attraction center may occur in case $\mathbf{\Omega} = \mathbf{0}$, depending on the initial conditions $\mathbf{r}_0$ and $\mathbf{v}_0$. In Sec. 5 we prove that in case of collision, it occurs in a finite period of time.

The Cauchy problem $(**)$ regularizes the Cauchy problem $(*)$ by eliminating the singularity for $r = 0$.

The moment of time t_P is computed by using the solution to eq $(**)$ and Kepler's equations in each case ($e < 1$, $e = 1$, $e > 1$).

4 The Laws of Motion in the Regularized Form

4.1 The Elliptic Case

Here $e < 1, h < 0$. The regularized equation $(**)$ has the solution:

$$\mathbf{r}\left(\tau\right) = \mathbf{a}\left(\cos\omega\tau - e\right) + \mathbf{b}\sin\omega\tau \tag{4.1}$$

which is the equation of an ellipse with the center in the origin and the vectorial semiaxis:

$$\mathbf{a} = \frac{\mu}{e\omega^2}\mathbf{e} \;\; \mathbf{b} = \frac{2}{\omega e}\mathbf{\Omega}\times\mathbf{e}$$

Figure 1. The elliptic Keplerian motion: vectorial denotations

We took here $\omega = \sqrt{2\left|h\right|}$.

A remarkable fact is that in this new regularized form, the elliptic motion in the new time-variable τ is that of an **elliptic oscillator**.

We can also compute:

the norm of the radius vector:

$$r\left(\tau\right) = a\left(1 - e\cos\omega\tau\right) \tag{4.2}$$

the velocity vector:

$$\mathbf{v}\left(\tau\right) = \frac{\omega}{a\left(1 - e\cos\omega\tau\right)}\left(-\mathbf{a}\sin\omega\tau + \mathbf{b}\cos\omega\tau\right) \tag{4.3}$$

the norm of the velocity vector:

$$v\left(\tau\right) = \frac{\omega}{a\left(1 - e\cos\omega\tau\right)}\sqrt{a^2\sin^2\omega\tau + b^2\cos^2\omega\tau} \tag{4.4}$$

The motion in this case is periodic, with the main period T in time t and the main period $\frac{2\pi}{\omega}$ in new time τ. As it follows from Kepler's third law: $T = \frac{2\pi\mu}{\left(2\left|h\right|\right)^{3/2}}$. We get the relation between these two periods:

$$T = \frac{2\pi}{\omega}a. \tag{4.5}$$

4.1.1 The rectilinear situation ($\mathbf{\Omega} = \mathbf{0}$) in case $h < 0$

All computations made in the elliptic case are similar. The results may be obtained by taking $\mathbf{b} = \mathbf{0}$, $\mathbf{e} = -\frac{\mathbf{r}_0}{r_0}$. The law of motion and velocity have the expressions:

$$\begin{aligned} \mathbf{r}(\tau) &= \frac{\mu}{\omega^2}(1 - \cos\omega\tau)\frac{\mathbf{r}_0}{r_0} \\ \mathbf{v}(\tau) &= \frac{\omega\sin\omega\tau}{1-\cos\omega\tau}\frac{\mathbf{r}_0}{r_0}. \end{aligned} \tag{4.6}$$

The collision occurs when $\mathbf{r} = \mathbf{0} \Leftrightarrow \cos\omega\tau = 1$.

4.2 The Parabolic Case

Here $e = 1$, $h = 0$. From (2) we get:

$$v = \sqrt{\frac{2\mu}{r}} \tag{4.7}$$

The regularized equation (∗∗) has the solution:

$$\mathbf{r}(\tau) = \left(\frac{2\Omega^2}{\mu} - \frac{1}{2}\mu\tau^2\right)\mathbf{e} + 2\tau(\mathbf{\Omega} \times \mathbf{e}) \tag{4.8}$$

or

$$\mathbf{r}(\tau) = \frac{1}{2}\left(p - \mu\tau^2\right)\mathbf{e} + 2\tau(\mathbf{\Omega} \times \mathbf{e}) \tag{4.9}$$

where p is the *semilatus rectum* (see relation (5)). This is the vectorial equation of a parabola having the symmetry axis along the direction of $\mathbf{e}$.

Figure 2. The parabolic Keplerian motion: vectorial denotations

We can also compute:
the norm of the radius vector:

$$r(\tau) = \frac{1}{2}\left(p + \mu\tau^2\right) \tag{4.10}$$

the velocity vector:

$$\mathbf{v}(\tau) = 2\frac{-\mu\tau\mathbf{e} + 2(\mathbf{\Omega} \times \mathbf{e})}{p + \mu\tau^2} \tag{4.11}$$

the velocity vector magnitude:

$$v(\tau) = 2\sqrt{\frac{\mu}{p + \mu\tau^2}} \tag{4.12}$$

4.2.1 The rectilinear situation ($\mathbf{\Omega} = \mathbf{0}$) in case $h = 0$

All expressions above modify by taking $\mathbf{\Omega} = \mathbf{0}$, $\mathbf{e} = -\frac{\mathbf{r}_0}{r_0}$. The expressions for the law of motion and velocity are:

$$\begin{aligned} \mathbf{r}(\tau) &= \frac{\mu\tau^2}{2}\frac{\mathbf{r}_0}{r_0} \\ \mathbf{v}(\tau) &= \frac{2}{\tau}\frac{\mathbf{r}_0}{r_0} \end{aligned} \tag{4.13}$$

The collision occurs when $\tau = 0$.

4.3 The Hyperbolic Case

Here $e > 1, h > 0$.The regularized equation (∗∗) has the solution:

$$\mathbf{r}(\tau) = \mathbf{a}(e - \cosh\omega\tau) + \mathbf{b}\sinh\omega\tau \tag{4.14}$$

It is the equation of a hyperbola with the vectorial semiaxis

$$\mathbf{a} = \frac{\mu}{e\omega^2}\mathbf{e} \ \mathbf{b} = \frac{2}{\omega e}\mathbf{\Omega} \times \mathbf{e}$$

Figure 3. The hyperbolic Keplerian motion: vectorial denotations

Here $\omega = \sqrt{2h}$. There is an interesting similitude between this case and the elliptic one. Equation (4.14) is one of a **hyperbolic oscillator**.

We can also compute:

the norm of the radius vector:

$$r(\tau) = a(e\cosh\omega\tau - 1) \tag{4.15}$$

the velocity vector:

$$\mathbf{v}(\tau) = \frac{\omega}{a(e\cosh\omega\tau - 1)}(-\mathbf{a}\sinh\omega\tau + \mathbf{b}\cosh\omega\tau) \tag{4.16}$$

the norm of the velocity vector:

$$v(\tau) = \frac{\omega}{a(e\cosh\omega\tau - 1)}\sqrt{a^2\sinh^2\omega\tau + b^2\cosh^2\omega\tau} \tag{4.17}$$

The asymptotic direction of the section of the hyperbola that represent the trajectory of the body when $\tau \in [0, +\infty)$ has the unit vector:

$$\mathbf{u} = \frac{\mathbf{b} - \mathbf{a}}{ae} \tag{4.18}$$

The angle between the asymptote and the main axis of the hyperbola is:

$$\theta = \arccos\frac{1}{e} \tag{4.19}$$

4.3.1 The rectilinear situation ($\mathbf{\Omega} = \mathbf{0}$) in case $h > 0$

All expressions above modify by taking $\mathbf{\Omega} = \mathbf{0}$, $\mathbf{e} = -\frac{\mathbf{r}_0}{r_0}$. The new expressions for the law of motion and velocity become:

$$\begin{aligned} \mathbf{r}(\tau) &= \frac{\mu}{\omega^2}\left[\cosh \omega\tau - 1\right]\frac{\mathbf{r}_0}{r_0}; \\ \mathbf{v}(\tau) &= \frac{\omega \sinh \omega\tau}{\cosh \omega\tau - 1}\frac{\mathbf{r}_0}{r_0}, \end{aligned} \tag{4.20}$$

The collision occurs when $\cosh \omega\tau = 1 \Leftrightarrow \tau = 0$.

5 Kepler's Equations: A Unified Approach

We will start from the substitution we made: $t(\tau) = \int_0^\tau r(\xi)\, d\xi + t_p$.

Kepler's equations are obtained from Eqs (4.2), (4.10) and (4.15) by a simple integration.

5.1 The Elliptic Case

Starting from (4.2) we get:

$$t - t_p = \frac{a}{\omega}\left(\omega\tau - e \sin \omega\tau\right) \tag{5.1}$$

From (2.6) and (4.2) it results:
$(1 - e\cos \omega\tau)(1 + e\cos\gamma) = \frac{p}{a} = 1 - e^2$. Taking (2.7) into account, we may write:

$$\tan^2 \frac{\gamma}{2} = \frac{1+e}{1-e}\tan^2\frac{\omega\tau}{2} \tag{5.2}$$

Eq (5.2) gives the eccentric anomaly $E = \omega\tau$ in the elliptic case.

By making $t = 0$ in eqs (4.1) and (4.3), after computations it results:

$$\begin{aligned} \cos E_0 &= \frac{1}{e}\left(1 - \frac{2|h|\, r_0}{\mu}\right), \\ \sin E_0 &= \frac{\sqrt{2|h|}}{\mu e}\left(\mathbf{r}_0 \cdot \mathbf{v}_0\right). \end{aligned} \tag{5.3}$$

where $E_0 = E(0) \in [0, 2\pi)$. By making $t = 0$ in (5.1), it results:

$$t_p = -\frac{\mu}{(2h)^{\frac{3}{2}}}\left(E_0 - e\sin E_0\right), \tag{5.4}$$

with E_0 uniquely defined by eqs (5.3).

In case $\mathbf{\Omega} = \mathbf{0}$, $h < 0$, eq (5.1) becomes:

$$t - t_p = \frac{\mu}{\omega^3}\left(\omega\tau - \sin\omega\tau\right). \tag{5.5}$$

The body reaches the attraction center at the moment of time:

$$t_{\text{collision}} = t_P + \frac{2\pi\mu}{\omega^3} \tag{5.6}$$

5.2 The Parabolic Case

Starting from (4.10) we get:

$$t - t_p = \frac{1}{2}\left(\frac{\mu\tau^3}{3} + p\tau\right) \tag{5.7}$$

From (2.6) and (4.10), we get:

$$\tan\frac{\gamma}{2} = \frac{\mu\tau}{4\Omega}. \tag{5.8}$$

By making $t = 0$ in eqs (4.9) and (4.11), after computations it results:

$$\tau(0) = \frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{\mu}. \tag{5.9}$$

The moment of time t_P is computed from:

$$t_P = -\frac{1}{2}\left[p\tau(t_0) + \frac{\mu}{3}\tau^3(t_0)\right] \tag{5.10}$$

and its explicit form is:

$$t_P = -\frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{2\mu}\left[p + \frac{\mu}{3}\left(\frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{\mu}\right)^2\right]. \tag{5.11}$$

In case $\mathbf{\Omega} = \mathbf{0}$, eq (5.7) becomes:

$$t - t_p = \frac{\mu\tau^3}{6}. \tag{5.12}$$

If $\mathbf{\Omega} = \mathbf{0}$, $\mathbf{r}_0 \cdot \mathbf{v}_0 < 0$, the body reaches the attraction center at the moment of time:

$$t_{\text{collision}} = -\frac{(\mathbf{r}_0 \cdot \mathbf{v}_0)^3}{6\mu^2}. \tag{5.13}$$

In case $\mathbf{\Omega} = \mathbf{0}$, $\mathbf{r}_0 \cdot \mathbf{v}_0 > 0$, the body never reaches the attraction center, as $t_P < 0$ and the motion is not periodic.

5.3 The Hyperbolic Case

Starting from (4.15) we get:

$$t - t_p = \frac{a}{\omega}(e\sinh\omega\tau - \omega\tau) \tag{5.14}$$

From (2.6) and (4.15) we get that $(e\cosh\omega\tau - 1)(1 + e\cos\gamma) = \frac{p}{a} = e^2 - 1$. Taking (2.7) into account, it results:

$$\tan^2\frac{\gamma}{2} = \frac{e+1}{e-1}\tanh^2\frac{\omega\tau}{2} \tag{5.15}$$

Eq (5.15) gives the eccentric anomaly $E = \omega\tau$ in the hyperbolic case.

Making $t = 0$ in eqs (4.14) and (4.16), it results that $E_0 = E(0)$ is determined from:

$$E_0 = \sinh^{-1}\left[\frac{\sqrt{2h}}{\mu e}(\mathbf{r}_0 \cdot \mathbf{v}_0)\right]. \tag{5.16}$$

The moment of time t_P is then computed by making $t = t_0$ in eq (5.14); it equals to:

$$t_p = -\frac{a}{\omega}(e \sinh E_0 - E_0). \tag{5.17}$$

Another form of eq (5.17) depending only on the initial conditions is:

$$t_p = -\frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{\mu} + \frac{\mu}{(2h)^{\frac{3}{2}}} \sinh^{-1}\left[\frac{\sqrt{2h}}{\mu e}(\mathbf{r}_0 \cdot \mathbf{v}_0)\right]. \tag{5.18}$$

In case $\mathbf{\Omega} = \mathbf{0}$, eq (5.14) becomes:

$$t - t_p = \frac{\mu}{\omega^3}(\sinh \omega\tau - \omega\tau). \tag{5.19}$$

The moment of collision is determined from:

$$t_{\text{collision}} = -\frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{\mu} + \frac{\mu}{(2h)^{\frac{3}{2}}} \sinh^{-1}\left[\frac{\sqrt{2h}}{\mu}(\mathbf{r}_0 \cdot \mathbf{v}_0)\right] \tag{5.20}$$

The collision takes place in case $\mathbf{r}_0 \cdot \mathbf{v}_0 < 0$, as it results from eq (5.18). The moment of impact may be determined from eq (5.18) by taking $e = 1$. In case $\mathbf{r}_0 \cdot \mathbf{v}_0 > 0$, the body does not reach the attraction center.

Eqs. (5.1), (5.7), (5.14) represent Kepler's equations that compute the time of motion to a given position for the elliptic, parabolic and hyperbolic cases. Eqs (5.5), (5.12), (5.19) represent replicas to Kepler's equations in case the motion is rectilinear. Using the time-regularization (3.2), all six relations have been deduced. The well-known formulas for the eccentric anomaly in the elliptic and hyperbolic case have also been deduced.

The moment of time t_P was computed in each case. If $t_P > 0$, it means that the body will pass by the pericenter after the initial moment of time. If $t_P = 0$, it means that the body is situated exactly in the pericenter at the initial moment of time. If $t_P < 0$, it means the body would have passed the pericenter in a virtual time (before the launch). In the parabolic and hyperbolic case, it means the body will never pass by the pericenter, the closest point to the attraction center of the trajectory being the initial point. In the elliptic case, $t_P < 0$ means that the body will reach the pericenter after reaching the apocenter in its first revolution around the attraction point after the initial moment of time.

In case of rectilinear motion, $\mathbf{\Omega} = \mathbf{0}$, it holds:

- if $h < 0$, the collision takes place;
- if $h \geq 0$, the collision takes place iff $\mathbf{r}_0 \cdot \mathbf{v}_0 < 0$.

In each situation the collision takes place, the period of time until impact is finite.

6 Integral Temporal Means

This is the core of this paper. Using the previous results, we are ready to compute some temporal integral means in a simple and unified approach.

6.1 Mathematical Preliminaries

In the hyperbolic and parabolic case, it comes that: $\lim_{\tau\to\infty} t(\tau) = \lim_{t\to\infty} \tau(t) = +\infty$, so for a continuous function $f : \mathbb{R}_+ \to \mathbb{R}^n$ satisfying $(\exists)\lim_{t\to\infty} f(t) \in \overline{\mathbb{R}}^n$, we may write:

$$\lim_{\tau\to\infty} \mathbf{f}(t(\tau)) = \lim_{t\to\infty} \mathbf{f}(\tau(t)) \tag{6.1}$$

Lemma 1. *The following affirmations hold for any vectorial and scalar continuous map:* $\mathbf{f} : \mathbb{R}_+ \to \mathbb{R}^n$ *and* $g : \mathbb{R}_+ \to \mathbb{R}$*:*

(1°) *If* $(\exists)\lim_{t\to\infty} \mathbf{f}(t) = \mathbf{m} \in \mathbb{R}^n$, *then*

$$\lim_{T\to\infty}\left(\frac{1}{T}\int_{T_0}^{T} \mathbf{f}(t)\,dt\right) = \mathbf{m}, (\forall)\, T_0 \in \mathbb{R}_+. \tag{6.2}$$

(2°) *If* $(\exists)\lim_{t\to\infty} g(t) = m \in \overline{\mathbb{R}}$, *then*

$$\lim_{T\to\infty}\left(\frac{1}{T}\int_{T_0}^{T} g(t)\,dt\right) = m, (\forall)\, T_0 \in \mathbb{R}_+. \tag{6.3}$$

(3°) *If* $(\exists)\lim_{t\to\infty} \mathbf{f}(t) = \mathbf{m} \in \mathbb{R}^n$ *and if* $(\exists)\lim_{t\to\infty} g(t) = l \in \mathbb{R}$, *then*

$$\lim_{T\to\infty}\left(\frac{1}{T}\int_{T_0}^{T} g(t)\,\mathbf{f}(t)\,dt\right) = l\mathbf{m}, (\forall)\, T_0 \in \mathbb{R}_+. \tag{6.4}$$

(4°) *If* $\mathbf{f}(t+T) = \mathbf{f}(t), (\forall)\, t \in \mathbb{R}_+, T > 0$, *then*

$$\lim_{s\to\infty}\left(\frac{1}{s}\int_{0}^{s} \mathbf{f}(t)\,dt\right) = \frac{1}{T}\int_0^T \mathbf{f}(t)\,dt. \tag{6.5}$$

Proof. (1°) As $\mathbf{f}$ is a continuous map, then there exists $\mathbf{F} : \mathbb{R}_+ \to \mathbb{R}^n$ such as: $\mathbf{F}(t) = \int \mathbf{f}(t)\,dt$. From Leibniz-Newton formula we may write:

$\frac{1}{T}\int_{T_0}^{T} \mathbf{f}(t)\,dt = \frac{\mathbf{F}(T)-\mathbf{F}(T_0)}{T}$. Then: $\lim_{T\to\infty}\left(\frac{1}{T}\int_{T_0}^{T} \mathbf{f}(t)\,dt\right) =$

$= \lim_{T\to\infty} \frac{\mathbf{F}(T)-\mathbf{F}(T_0)}{T} = \lim_{T\to\infty} \frac{\mathbf{F}(T)}{T} =$

$= \lim_{T\to\infty} \dot{\mathbf{F}}(T) = \lim_{T\to\infty} \mathbf{f}(T) = \mathbf{m}$.

(2°) An absolute similar proof may be given here taking $G(t) = \int g(t)\,dt$.

(3°) We take $\mathbf{H}(t) = \int g(t)\,\mathbf{f}(t)\,dt$ and apply (1°)

(4°) It is known that if $\mathbf{f}:\mathbb{R}_+ \to \mathbb{R}^n$ is a periodic integrable function with the main period T, then $\int_0^{nT} \mathbf{f}(t)\,dt = n\int_0^T \mathbf{f}(t)\,dt$ for any integer $n \geq 0$. Let $s \geq 0$ be a real number.

Let $n_s \geq 0$ be an integer such as $n_s \leq \frac{s}{T} < n_s + 1$. Then $\int_0^s \mathbf{f}(t)\,dt = \int_0^{n_sT} \mathbf{f}(t)\,dt + \int_{n_sT}^{s} \mathbf{f}(t)\,dt = n_s\int_0^T \mathbf{f}(t)\,dt + \int_0^{s-n_sT} \mathbf{f}(t)\,dt$. As $0 \leq s - n_sT \leq T$ and $\mathbf{f}$ is continuous, then there exists $M > 0$ such as $\left|\int_0^{s-n_sT} \mathbf{f}(t)\,dt\right| \le M$. Making $s \to \infty$, we have: $\lim_{s\to\infty} \frac{n_s}{s} = \frac{1}{T}$ and $\lim_{s\to\infty}\left(\frac{1}{s}\int_0^{s-n_sT} \mathbf{f}(t)\,dt\right) = 0$, and it results: $\lim_{s\to\infty}\left(\frac{1}{s}\int_0^{s} \mathbf{f}(t)\,dt\right) =$

$= \lim_{s\to\infty}\left[\frac{1}{s}\left(n_s\int_0^{T} \mathbf{f}(t)\,dt + \int_0^{s-n_sT} \mathbf{f}(t)\,dt.\right)\right] =$

$= \lim_{s\to\infty}\left(\frac{n_s}{s}\int_0^{T} \mathbf{f}(t)\,dt\right) = \frac{1}{T}\int_0^{T} \mathbf{f}(t)\,dt,$ ■

Using what we presented till now, we are ready to compute in a simple and unified way some temporal integral mean formulas.

For any function $\mathbf{f} : \mathbb{R}_+ \to \mathbb{R}^n$, we denote: $\langle \mathbf{f} \rangle = \lim_{s\to\infty}\left(\frac{1}{s}\int_0^{s} \mathbf{f}(t)\,dt\right)$. According to **Lemma 1**, if $\mathbf{f}$ is a periodic bounded function with the main period T, $\langle \mathbf{f} \rangle = \frac{1}{T}\int_0^{T} \mathbf{f}(t)\,dt$. If $\mathbf{f}$ is not periodic, this is an integral that may converge or may diverge, depending on function $\mathbf{f}$.

6.2 The Elliptic Case

All functions considered here are periodic, so $\langle \mathbf{f} \rangle$ represents the temporal integral mean on a single period $[0, T]$. In the next theorem, we use:

a) $M(a, b)$ represents the arithmetic-geometric mean of a and b:
$M(a, b) = \frac{\pi}{2}\left(\int_0^{\pi/2}\left[1/\left(\sqrt{a^2\sin^2\xi + b^2\cos^2\xi}\right)\right]d\xi\right)^{-1}$.

b) φ represents the angle between the radius vector and the velocity vector: $\varphi = \sphericalangle(\mathbf{r}, \mathbf{v})$.

c) $\mathbf{a} = \frac{\mu}{e\omega^2}\mathbf{e}$, $\mathbf{b} = \frac{2}{\omega e}\mathbf{\Omega} \times \mathbf{e}$ are the vectorial semiaxis of the ellipse, $a = \frac{\mu}{\omega^2}$ and $b = \frac{2\Omega}{\omega}$ represent their magnitudes. $\omega = \sqrt{2|h|}$.

Theorem 1. *The following statements hold:*

(1°) $\langle \mathbf{e}_r \rangle = -\mathbf{e}$ (2°) $\langle \mathbf{r} \rangle = -\dfrac{3}{2}\mathbf{a}$

(3°) $\langle r \rangle = a\left(1 + \dfrac{e^2}{2}\right)$ *(Laplace)*

(4°) $\left\langle \dfrac{1}{r} \right\rangle = \dfrac{1}{a}$ (5°) $\left\langle \dfrac{1}{r^2} \right\rangle = \dfrac{1}{ab}$

(6°) $\left\langle \dfrac{\mathbf{v}}{r} \right\rangle = \dfrac{\omega^2}{\pi e a^2}\left[\mathbf{a}\ln\dfrac{1-e}{1+e} + \pi\mathbf{b}\sqrt{\dfrac{1-e}{1+e}}\right]$

(7°) $\langle r\mathbf{v} \rangle = \mathbf{e} \times \mathbf{\Omega}$ (8°) $\langle v\mathbf{r} \rangle = -\dfrac{\omega(\mathbf{a} + \mathbf{b})}{2}$

(9°) $\langle v \rangle = \dfrac{\omega(a + b)}{2a}$ (10°) $\langle v^2 \rangle = \omega^2$

(11°) $\langle rv \rangle = \dfrac{\omega(a + b)}{2}$ (12°) $\langle \cos\varphi \rangle = 0$

(13°) $\langle \sin\varphi \rangle = \dfrac{2\Omega}{\omega M(a, b)}$ (14°) $\left\langle \dfrac{1}{\sin\varphi} \right\rangle = \dfrac{\omega(a + b)}{4\Omega}$

(15°) $\langle rv^2 \rangle = \dfrac{\omega^2(a^2 + b^2)}{2a}$ (16°) $\left\langle \dfrac{1}{rv} \right\rangle = \dfrac{1}{\omega M(a, b)}$

(17°) $\left\langle \dfrac{1}{r^2 v} \right\rangle = \dfrac{1}{\omega a M(a, b)}$

(18°) $\left\langle r^{n-1}\right\rangle = a^{n-1}\left\{1+n!\sum_{k=1}^{\left[\frac{n}{2}\right]}\frac{e^{2k}}{\left[(2k)!!\right]^2(n-2k)!}\right\}, n\in\mathbb{N}, n\geqslant 2$

(19°) $\left\langle \frac{1}{r^{n+1}}\right\rangle = \frac{1}{a}\left(\frac{-1}{eb}\right)^n\sum_{k=1}^{n}\frac{(-1)^k(k+n-2)!}{(n-k)!\left[(k-1)!\right]^2}\left(\frac{a+b}{2b}\right)^{k-1},$
$n\in\mathbb{N}, n\geqslant 2$

Proof. We will use **Lemma 1**, relations (4.1)-(4.5).and:
$\frac{1}{T}\int_0^T \mathbf{f}(t)\,dt = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}} r(\tau)\,\mathbf{f}(\tau)\,d\tau.$

(1°) The integral temporal mean of the versor of the radius vector:
$\langle \mathbf{e}_r\rangle = \frac{1}{T}\int_0^T \mathbf{e}_\mathbf{r}(t)\,dt = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}}\mathbf{r}(\tau)\,d\tau = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}}\left[\mathbf{a}(\cos\omega\tau - e)+\mathbf{b}\sin\omega\tau\right]d\tau = -\mathbf{e}.$

Relation (1°) shows that the temporal integral mean of the radius vector versor has the direction of the main semiaxis of the ellipse. Its sense indicates the apocenter and its magnitude is the ellipse eccentricity.

(2°) The integral temporal mean of the radius vector has the direction of the main semiaxis and its sense indicates the apocenter:
$\langle \mathbf{r}\rangle = \frac{1}{T}\int_0^T \mathbf{r}(t)\,dt = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}} r(\tau)\,\mathbf{r}(\tau)\,d\tau = -\frac{3}{2}\mathbf{a}.$

(3°) The integral temporal mean of the radius vector magnitude:
$\langle r\rangle = \frac{1}{T}\int_0^T r(t)\,dt = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}} r^2(\tau)\,d\tau = a\left(1+\frac{e^2}{2}\right).$

(4°) The integral temporal mean of $\frac{1}{r}$ is the inverse of a:
$\left\langle\frac{1}{r}\right\rangle = \frac{1}{T}\int_0^T\frac{1}{r(t)}dt = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}} d\tau = \frac{1}{a}.$

(5°) The integral temporal mean of $\frac{1}{r^2}$ is the inverse of ab:
$\left\langle\frac{1}{r^2}\right\rangle = \frac{1}{T}\int_0^T\frac{1}{r^2(t)}dt = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}}\frac{d\tau}{1-e\cos\omega\tau} = \frac{1}{ab}.$

Here we used: $\int_0^{2\pi}\frac{d\xi}{m-\cos\xi} = \frac{2\pi}{\sqrt{m^2-1}}, m>1.$

(6°) The integral temporal mean of $\frac{\mathbf{v}}{r}$:
$\left\langle\frac{\mathbf{v}}{r}\right\rangle = \frac{1}{T}\int_0^T\frac{\mathbf{v}(t)}{r(t)}dt = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}}\mathbf{v}(\tau)\,d\tau = \frac{\omega^2}{\pi e a^2}\left[\mathbf{a}\ln\frac{1-e}{1+e}+\pi\mathbf{b}\sqrt{\frac{1-e}{1+e}}\right].$

(7°) The integral temporal mean of $r\mathbf{v}$ is a vector that has the same direction as the semiminor axis of the ellipse and opposite sense:
$\langle r\mathbf{v}\rangle = \frac{1}{T}\int_0^T r(t)\,\mathbf{v}(t)\,dt = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}} r^2(\tau)\,\mathbf{v}(\tau)\,d\tau = \mathbf{e}\times\mathbf{\Omega} = -\frac{e\omega}{2}\mathbf{b}.$

(8°) The integral temporal mean of $v\mathbf{r}$:
$\langle v\mathbf{r}\rangle = \frac{1}{T}\int_0^T v(t)\,\mathbf{r}(t)\,dt = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}} r(\tau)\,v(\tau)\,\mathbf{r}(t)\,d\tau = -\frac{\omega(\mathbf{a}+\mathbf{b})}{2}.$

(9°) The integral temporal mean of the norm of the velocity vector:
$\langle v\rangle = \frac{1}{T}\int_0^T v(t)\,dt = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}} r(\tau)\,v(\tau)\,d\tau = \frac{\omega(a+b)}{2a}.$

Relation (9°) could also be proved by taking into account that $\int_0^T v(t)\,dt$ represents the length of the trajectory covered on an interval of time $[0,T]$, which is in fact the length of the ellipse with the semiaxis a and b, $L_{\text{ellipse}} = \pi(a+b)$. Using $T = \frac{2\pi}{\omega}a$, it results $\langle v\rangle = \frac{\omega(a+b)}{2a}$.

(10°) The integral temporal mean of v^2:
$\left\langle v^2 \right\rangle = \frac{1}{T}\int_0^T v^2(t)\,dt = \frac{2}{T}\int_0^T \left(h + \frac{\mu}{r(t)}\right)dt = \omega^2.$
(11°) The integral temporal mean of rv:
$\langle rv \rangle = \frac{1}{T}\int_0^T r(t)\,v(t)\,dt = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}} r^2(\tau)\,v(\tau)\,d\tau = \frac{\omega(a+b)}{2}.$
(12°) The integral temporal mean of $\cos\varphi$:
$\langle \cos\varphi \rangle = \frac{1}{T}\int_0^T \cos\varphi(t)\,dt = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}} r\cos\varphi(\tau)\,d\tau = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}} \frac{\mathbf{r}(\tau)\cdot\mathbf{v}(\tau)}{v(\tau)}d\tau = 0.$
(13°) The integral temporal mean of $\sin\varphi$
$\langle \sin\varphi \rangle = \frac{1}{T}\int_0^T \sin\varphi(t)\,dt = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}} r(\tau)\sin\varphi(\tau)\,d\tau =$
$= \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}} \frac{2\Omega}{v(\tau)}d\tau = \frac{2\Omega}{\omega M(a,b)}.$
(14°) The integral temporal mean of $\frac{1}{\sin\varphi}$:
$\left\langle \frac{1}{\sin\varphi} \right\rangle = \frac{1}{T}\int_0^T \frac{1}{\sin\varphi(t)}dt = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}} \frac{r(\tau)}{\sin\varphi(\tau)}d\tau =$
$= \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}} \frac{r^2(\tau)\,v(\tau)}{2\Omega}d\tau = \frac{\omega(a+b)}{4\Omega}.$
(15°) The integral temporal mean of rv^2:
$\left\langle rv^2 \right\rangle = \int_0^T r(t)\,v^2(t)\,dt = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}} r^2(\tau)\,v^2(\tau)\,d\tau = \frac{\omega^2\left(a^2+b^2\right)}{2a}.$
(16°) The integral temporal mean of $\frac{1}{rv}$:
$\left\langle \frac{1}{rv} \right\rangle = \frac{1}{T}\int_0^T \frac{1}{r(t)\,v(t)}dt = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}} \frac{1}{v(\tau)}d\tau = \frac{1}{\omega M(a,b)}.$
(17°) The integral temporal mean of $\frac{1}{r^2 v}$:
$\left\langle \frac{1}{r^2 v} \right\rangle = \int_0^T \frac{1}{r^2(t)\,v(t)}dt = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}} \frac{1}{r(\tau)\,v(\tau)}d\tau = \frac{1}{\omega a M(a,b)}.$
(18°) For computing the mean of r^{n-1} on a period, $n \in \mathbb{N}, n \geqslant 2$, we will write:
$\frac{1}{T}\int_0^T r^{n-1}(t)\,dt = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}} r^n(\tau)\,d\tau = \frac{\omega a^{n-1}}{2\pi}\int_0^{\frac{2\pi}{\omega}} (1 - e\cos\omega\tau)^n\,d\tau.$
Expanding $(1 - e\cos\omega\tau)^n$ with Newton's binomial formula and computing for $k \in \mathbb{N}$:
$\int_0^{\frac{2\pi}{\omega}} (\cos\omega\tau)^{2k}\,d\tau = \frac{(2k-1)!!}{(2k)!!}\cdot\frac{2\pi}{\omega}$ and $\int_0^{\frac{2\pi}{\omega}} (\cos\omega\tau)^{2k+1}\,d\tau = 0$, we get:
$$\frac{1}{T}\int_0^T r^{n-1}(t)\,dt = a^{n-1}\left\{1 + n!\sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{e^{2k}}{\left[(2k)!!\right]^2 (n-2k)!}\right\}, n \in \mathbb{N}, n \geqslant 2.$$
(Here $[x]$ denotes the integer part of the real number x).
This is a first generalization of Laplace's formula. Indeed:
$n = 2 : \frac{1}{T}\int_0^T r(t)\,dt = a\left(1 + \frac{e^2}{2}\right)$ (Laplace) and further:
$n = 3 : \frac{1}{T}\int_0^T r^2(t)\,dt = a^2\left(1 + \frac{3e^2}{2}\right).$
$n = 4 : \frac{1}{T}\int_0^T r^3(t)\,dt = a^3\left(1 + 3e^2 + \frac{3}{8}e^4\right)$ etc.
(19°)For computing $\left\langle \frac{1}{r^{n+1}} \right\rangle$, $n \in \mathbb{N}, n \geqslant 1$, we will write:

$$\left\langle \frac{1}{r^{n+1}} \right\rangle = \frac{1}{T}\int_0^T \frac{1}{r^{n+1}(t)}dt = \frac{\omega}{2\pi a}\int_0^{\frac{2\pi}{\omega}} \frac{1}{r^n(\tau)}d\tau =$$
$$= \frac{\omega}{2\pi a^{n+1}}\int_0^{\frac{2\pi}{\omega}} \frac{d\tau}{(1-e\cos\omega\tau)^n} = \frac{1}{2\pi e^n a^{n+1}}\int_0^{\frac{2\pi}{\omega}} \frac{dx}{\left(\frac{1}{e}-\cos x\right)^n}.$$

Using the Residue Theorem (see **Appendix**), we get that:

$$\frac{1}{T}\int_0^T \frac{1}{r^{n+1}(t)}dt = \frac{1}{a}\left(\frac{-1}{eb}\right)^n \sum_{k=1}^{n} \frac{(-1)^k (k+n-2)!}{(n-k)!\,[(k-1)!]^2}\left(\frac{a+b}{2b}\right)^{k-1}, n\in\mathbb{N}, n\geq 2.$$

This formula is another generalization of Laplace's formula. With the last two relations, we gave a method for computing the integral mean of r^n, with $n \in \mathbb{Z}$. ■

6.3 The Parabolic Case

Only some significant results will be mentioned in this subsection. Using relations (4.1)-(4.7), many more integral temporal means can be easily computed. Here φ represents the angle between the radius vector and the velocity vector: $\varphi = \sphericalangle(\mathbf{r}, \mathbf{v})$. We will use relation (5.7) to make the substitution $t \to t(\tau)$ and so $\lim_{\tau\to\infty} \frac{t(\tau)}{\tau^3} = \frac{1}{6}\mu$.

Theorem 2. *The following statements hold:*

(1°) $\langle \mathbf{e_r} \rangle = \langle \mathbf{e_v} \rangle = -\mathbf{e}$ (2°) $\langle v\sqrt{r} \rangle = \sqrt{2\mu}$ (3°) $\langle v^2\mathbf{r} \rangle = -2\mu\mathbf{e}$
(4°) $\langle \sqrt{r}\mathbf{v} \rangle = -\sqrt{2\mu}\mathbf{e}$ (5°) $\langle \sin\varphi \rangle = 0$ (6°) $\langle \cos\varphi \rangle = 1$
(7°) $\left\langle \frac{1}{t}\mathbf{r} \right\rangle = \mathbf{0}$ (8°) $\left\langle \frac{r}{t} \right\rangle = 0$ (9°) $\left\langle \frac{rv}{t} \right\rangle = 0$

Proof. We will use **Lemma 1** and relations (4.7)-(4.12).

(1°) The integral mean of the versor of the radius vector on $[0,\infty)$ is a vector having the direction of the symmetry axis of the parabola. Its sense is opposite to the pericenter and its magnitude is 1:

$$\langle \mathbf{e_r} \rangle = \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T \mathbf{e_r}(t)\,dt\right) = \lim_{\tau\to\infty} \mathbf{e_r}(\tau) = -\mathbf{e}$$

We may write the same for the versor of the velocity:

$$\langle \mathbf{e_v} \rangle = \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T \mathbf{e_v}(t)\,dt\right) = \lim_{\tau\to\infty} \mathbf{e_v}(\tau) = -\mathbf{e}$$

(2°) The integral mean of $v\sqrt{r}$:

$$\langle v\sqrt{r} \rangle = \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T v(t)\sqrt{r(t)}dt\right) = \sqrt{2\mu}$$

(3°) The integral mean of $v^2\mathbf{r}$:

$$\langle v^2\mathbf{r} \rangle = \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T v^2(t)\,\mathbf{r}(t)\,dt\right) = \lim_{\tau\to\infty}\left[v^2(\tau)\,\mathbf{r}(\tau)\right] = -2\mu\mathbf{e}$$

(4°) The integral mean of $\sqrt{r}\mathbf{v}$:

$$\langle \sqrt{r}\mathbf{v} \rangle = \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T \sqrt{r(t)}\mathbf{v}(t)\,dt\right) =$$
$$= \lim_{\tau\to\infty}\left[\sqrt{r(\tau)}\mathbf{v}(\tau)\right] = -\sqrt{2\mu}\mathbf{e}$$

(5°) The integral mean of $\sin\varphi$:

$$\langle \sin\varphi \rangle = \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T \sin\varphi(t)\,dt\right) = \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T \frac{2\Omega}{r(t)\,v(t)}dt\right) =$$
$$= 2\Omega\lim_{\tau\to\infty}\left[\frac{1}{r(\tau)\,v(\tau)}\right] = 0$$

(6°) The integral mean of $\cos\varphi$:

$$\langle\cos\varphi\rangle = \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T \cos\varphi\,(t)\,dt\right) = \lim_{\tau\to\infty}\left[\cos\varphi\,(\tau)\right] = \lim_{\tau\to\infty}\left[\frac{\mathbf{r}\,(\tau)\cdot\mathbf{v}\,(\tau)}{r\,(\tau)\,v\,(\tau)}\right] = 1$$

(7°) The integral mean of $\frac{1}{t}\mathbf{r}$:

$$\left\langle\frac{1}{t}\mathbf{r}\right\rangle = \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T \frac{1}{t}\mathbf{r}\,(t)\,dt\right) = \lim_{\tau\to\infty}\left[\frac{1}{t\,(\tau)}\mathbf{r}\,(\tau)\right] =$$

$$= \lim_{\tau\to\infty}\left[\frac{\tau^3}{t\,(\tau)}\cdot\frac{1}{\tau^3}\mathbf{r}\,(\tau)\right] = \mathbf{0}$$

(8°) The integral mean of $\frac{r}{t}$:

$$\left\langle\frac{r}{t}\right\rangle = \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T \frac{r\,(t)}{t}dt\right) = \lim_{\tau\to\infty}\left[\frac{r\,(\tau)}{t\,(\tau)}\right] = \lim_{\tau\to\infty}\left[\frac{\tau^3}{t\,(\tau)}\cdot\frac{r\,(\tau)}{\tau^3}\right] = 0$$

(9°) The integral mean of $\frac{rv}{t}$:

$$\left\langle\frac{rv}{t}\right\rangle = \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T \frac{r\,(t)\,v\,(t)}{t}dt\right) = \lim_{\tau\to\infty}\left[\frac{r\,(t)\,v\,(t)}{t\,(\tau)}\right] = \lim_{\tau\to\infty}\left[\frac{\tau^3}{t\,(\tau)}\cdot\frac{r\,(t)\,v\,(t)}{\tau^3}\right] = 0$$

We remark here that the expression of $\langle\mathbf{e_r}\rangle$ is the same like in the elliptic case and $\langle\mathbf{e_r}\rangle = \langle\mathbf{e_v}\rangle$. ■

6.4 The Hyperbolic Case

Most of the results presented below involve the vectorial semiaxis of the hyperbola, as well as the unit vector of the asymptotic direction $\mathbf{u} = \frac{\mathbf{b}-\mathbf{a}}{ae}$. The following denotations are used:

a) φ represents the angle between the radius vector and the velocity vector: $\varphi = \sphericalangle\,(\mathbf{r},\mathbf{v})$.

b) $\mathbf{a} = \frac{\mu}{e\omega^2}\mathbf{e}$, $\mathbf{b} = \frac{2}{\omega e}\mathbf{\Omega}\times\mathbf{e}$ are the vectorial semiaxis of the hyperbola, $a = \frac{\mu}{\omega^2}$ and $b = \frac{2\Omega}{\omega}$ represent their magnitudes. Also, $\omega = \sqrt{2h}$.

In this case (see relation (5.14)) we may write:

$\lim_{\tau\to\infty}\frac{t(\tau)}{\sinh\omega\tau} = \lim_{\tau\to\infty}\frac{t(\tau)}{\cosh\omega\tau} = \frac{ae}{\omega}$.

Theorem 3. *The following statements hold:*

(1°) $\langle\mathbf{e_r}\rangle = \langle\mathbf{e_v}\rangle = \dfrac{\mathbf{b}-\mathbf{a}}{ae} = \mathbf{u}$ (2°) $\langle\mathbf{v}\rangle = \dfrac{\omega\,(\mathbf{b}-\mathbf{a})}{ae} = \omega\mathbf{u}$

(3°) $\langle v^\alpha\rangle = \omega^\alpha, \alpha\in\mathbb{R}$ (4°) $\left\langle\dfrac{1}{t}\mathbf{r}\right\rangle = \dfrac{\omega\,(\mathbf{b}-\mathbf{a})}{ae} = \omega\mathbf{u}$

(5°) $\left\langle\left(\dfrac{r}{t}\right)^\alpha\right\rangle = \omega^\alpha, \alpha\in\mathbb{R}$ (6°) $\langle\sin\varphi\rangle = 0$

(7°) $\langle\cos\varphi\rangle = 1$ (8°) $\left\langle\left(\dfrac{rv}{t}\right)^\alpha\right\rangle = \omega^{2\alpha}, \alpha\in\mathbb{R}$

(9°) $\langle(t\sin\varphi)^\alpha\rangle = \left(\dfrac{2\Omega}{\omega^2}\right)^\alpha, \alpha\in\mathbb{R}$

(10°) $\left\langle\dfrac{v^\alpha}{t}\mathbf{r}\right\rangle = \left\langle\left(\dfrac{r}{t}\right)^\alpha\mathbf{v}\right\rangle = \omega^{\alpha+1}\mathbf{u}$, $\alpha\in\mathbb{R}$

Proof. We will use **Lemma 1** and relations (4.15)-(4.18).

(1°) The mean of the radius vector's versor and the mean of the velocity versor are identical, equal to the versor of the asymptotic direction of the hyperbola:

$$\langle\mathbf{e_r}\rangle = \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T \mathbf{e_r}\,(t)\,dt\right) = \lim_{\tau\to\infty}\mathbf{e_r}\,(\tau) = \frac{\mathbf{b}-\mathbf{a}}{ae} = \mathbf{u}$$

$\langle \mathbf{e_v} \rangle = \lim_{T\to\infty} \left(\frac{1}{T} \int_0^T \mathbf{e_v}(t)\,dt \right) = \lim_{\tau\to\infty} \mathbf{e_v}(\tau) = \frac{\mathbf{b}-\mathbf{a}}{ae} = \mathbf{u}$

(2°) The integral mean of the velocity vector:

$\langle \mathbf{v} \rangle = \lim_{T\to\infty} \left(\frac{1}{T} \int_0^T \mathbf{v}(t)\,dt \right) = \lim_{\tau\to\infty} \mathbf{v}(\tau) = \frac{\omega(\mathbf{b}-\mathbf{a})}{ae} = \omega\mathbf{u}$

(3°) The integral mean of the power α of the norm of the velocity:

$\langle v^\alpha \rangle = \lim_{T\to\infty} \left(\frac{1}{T} \int_0^T v^\alpha(t)\,dt \right) = \lim_{\tau\to\infty} v^\alpha(\tau) = \left(\frac{\omega\sqrt{b^2+a^2}}{ae} \right)^\alpha = \omega^\alpha, \alpha \in \mathbb{R}$

Here, for $\alpha = 1$, we get the mean of the magnitude of the velocity: $\langle v \rangle = \omega$.

(4°) The integral mean of $\frac{1}{t}\mathbf{r}$:

$\left\langle \frac{1}{t}\mathbf{r} \right\rangle = \lim_{T\to\infty} \left(\frac{1}{T} \int_0^T \frac{1}{t}\mathbf{r}(t)\,dt \right) =$

$= \lim_{\tau\to\infty} \left[\frac{\cosh\omega\tau}{t(\tau)} \cdot \frac{1}{\cosh\omega\tau}\mathbf{r}(\tau) \right] = \frac{\omega}{ae}(\mathbf{b}-\mathbf{a}) = \omega\mathbf{u}$

(5°) The integral mean of $\left(\frac{r}{t}\right)^\alpha$:

$\left\langle \left(\frac{r}{t}\right)^\alpha \right\rangle = \lim_{T\to\infty} \left(\frac{1}{T} \int_0^T \left[\frac{1}{t} r(t) \right]^\alpha dt \right) = \lim_{\tau\to\infty} \left[\left(\frac{\cosh\omega\tau}{t(\tau)} \cdot \frac{1}{\cosh\omega\tau} r(\tau) \right)^\alpha \right] = \omega^\alpha$

(6°) The integral mean of $\sin\varphi$:

$\langle \sin\varphi \rangle = \lim_{T\to\infty} \left(\frac{1}{T} \int_0^T \frac{2\Omega}{r(t)\,v(t)} dt \right) = 2\Omega \lim_{\tau\to\infty} \left[\frac{1}{r(\tau)\,v(\tau)} \right] = 0$

(7°) The integral mean of $\cos\varphi$:

$\langle \cos\varphi \rangle = \lim_{T\to\infty} \left(\frac{1}{T} \int_0^T \cos\varphi(t)\,dt \right) = \lim_{\tau\to\infty} [\cos\varphi(\tau)] = \lim_{\tau\to\infty} \left[\frac{\mathbf{r}(\tau)\cdot\mathbf{v}(\tau)}{r(\tau)\,v(\tau)} \right] = 1$

(8°) The integral mean of $\left(\frac{rv}{t}\right)^\alpha$:

$\left\langle \left(\frac{rv}{t}\right)^\alpha \right\rangle = \lim_{T\to\infty} \left(\frac{1}{T} \int_0^T \left[\frac{r(t)\,v(t)}{t} \right]^\alpha dt \right) =$

$= \lim_{\tau\to\infty} \left[\left(\frac{\cosh\omega\tau}{t(\tau)} \cdot \frac{r(\tau)\,v(\tau)}{\cosh\omega\tau} \right)^\alpha \right] = \omega^{2\alpha}$

(9°) The integral mean of $(t\sin\varphi)^\alpha$:

$\langle (t\sin\varphi)^\alpha \rangle = \lim_{T\to\infty} \left(\frac{1}{T} \int_0^T \left[\frac{2\Omega t}{r(t)\,v(t)} \right]^\alpha dt \right) =$

$= (2\Omega)^\alpha \lim_{\tau\to\infty} \left[\frac{t(\tau)}{\cosh\omega\tau} \cdot \frac{\cosh\omega\tau}{r(\tau)\,v(\tau)} \right]^\alpha = \left(\frac{2\Omega}{\omega^2} \right)^\alpha.$

(10°) The integral mean of $\frac{v^\alpha}{t}\mathbf{r}$ and $\left(\frac{r}{t}\right)^\alpha \mathbf{v}$ are equal:

$\left\langle \frac{v^\alpha}{t}\mathbf{r} \right\rangle = \lim_{T\to\infty} \left(\frac{1}{T} \int_0^T \frac{v^\alpha(t)}{t}\mathbf{r}(t)\,dt \right) =$

$= \lim_{\tau\to\infty} \left[v^\alpha(\tau) \frac{\cosh\omega\tau}{t(\tau)} \cdot \frac{1}{\cosh\omega\tau}\mathbf{r}(\tau) \right] = \omega^{\alpha+1}\mathbf{u}$

Also: $\left\langle \left(\frac{r}{t}\right)^\alpha \mathbf{v} \right\rangle = \lim_{T\to\infty} \left(\frac{1}{T} \int_0^T \left[\frac{r(t)}{t} \right]^\alpha \mathbf{v}(t)\,dt \right) =$

$= \lim_{\tau\to\infty} \left[v^\alpha(\tau) \frac{\cosh\omega\tau}{t(\tau)} \cdot \frac{1}{\cosh\omega\tau}\mathbf{r}(\tau) \right] = \omega^{\alpha+1}\mathbf{u}.$ ■

Remark 2. Using that in the parabolic case $e = 1$ and in the hyperbolic case the unit vector of the asymptote is $\mathbf{u} = \frac{1}{e^2} \left[-\mathbf{e} + \sqrt{e^2-1} \frac{\mathbf{\Omega}\times\mathbf{e}}{\Omega} \right]$, we give an unitary formula for the integral temporal mean of $\mathbf{e_r}$ in the Keplerian motion:

$$\langle \mathbf{e_r} \rangle = \frac{1}{1+(e^2-1)\,\sigma(h)}\left[-\mathbf{e}+\sqrt{(e^2-1)\,\sigma(h)}\frac{\boldsymbol{\Omega}\times\mathbf{e}}{\Omega}\right] \tag{6.6}$$

where $\sigma(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ for any real number x and $h = \frac{1}{2}\mathbf{v}_0^2 - \frac{\mu}{r_0}$.

7 Conclusions

A vectorial regularization is suggested using a vectorial solution to Kepler's problem. The **vectorial eccentricity** (the Laplace -Runge-Lenz vector) plays a fundamental part in this. The law of motion $\mathbf{r}$, as well as the maps r, $\mathbf{v}$ and v, acquire an explicit form in variable τ introduced in Sec. 3, in all possible cases: elliptic, parabolic and hyperbolic. Together with the energy h (see relation (2.2)), variable τ is included in the eccentric anomaly expression $E = \sqrt{2|h|}\tau$ for elliptic and hyperbolic cases.

A unitary method of deducing Kepler's equations is given. Using this procedure, the notion eccentric anomaly acquires a natural meaning. No geometrical approaches were used, but a simple differential equation integration.

In Sec. 6, using the vectorial regularization introduced in Sec. 3, we compute some temporal integral means related to Keplerian motion. Some of them are classical, some completely new. They are all computed in a systematic way and most of them could only be computed by using this procedure.

All other regularizations of Kepler's problem do not use the vectorial eccentricity in this form. Let us remark that 40 years ago, in 1965, the same time that Kustaanheimo gave the spatial regularization using spinors, professor A. Braier gave a simple approach using complex numbers and polar coordinates. He found a replica of vector $\mathbf{e}$ with different interesting properties (see [2]). This result was found again identically by various authors 35 years after (see [6], [9]-[11], [17]-[19]). This vector $\mathbf{e}$ can be computed only from the initial conditions and gives a complete overview to the entire motion, containing all the informations about it. Moreover, it is essential in the regularized Cauchy problem that describes the motion.

Appendix

A

We compute the integral $\int_0^{2\pi}\frac{dx}{\left(\frac{1}{e}-\cos x\right)^n}$ by making the substitution $z = \exp(ix)$. We get:

$$\int_0^{2\pi}\frac{dx}{\left(\frac{1}{e}-\cos x\right)^n} = \int\limits_{|z|=1}\frac{dz}{\left[\frac{1}{e}-\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^n iz} = \int\limits_{|z|=1}\frac{i\,(-1)^{n+1}\cdot 2^n\cdot z^{n-1}dz}{\left(z^2-\frac{2}{e}z+1\right)^n} \tag{A.1}$$

Let's consider now $z_1 = \frac{1}{e} - \sqrt{\left(\frac{1}{e}\right)^2 - 1}$, $z_2 = \frac{1}{e} + \sqrt{\left(\frac{1}{e}\right)^2 - 1}$ and the function $f : D\setminus\{z_1, z_2\} \to \mathbb{C}$, $f(z) = \frac{z^{n-1}}{\left(z^2-\frac{2}{e}z+1\right)}$.

Then z_1 and z_2 are n-order poles for the function f, z_1 situated inside the circle $|z|=1$ and z_2 situated outside this circle. According to the Residue Theorem, we get:

$\int\limits_{|z|=1} f(z)\,dz = 2\pi i rez(f,z_1) = \frac{2\pi i}{(n-1)!}\lim_{z\to z_1}\frac{d^{n-1}}{dz^{n-1}}\left[(z-z_1)^n f(z)\right] \Rightarrow$

$\Rightarrow \int\limits_{|z|=1} f(z)\,dz = \frac{2\pi i}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[\frac{z^{n-1}}{(z-z_2)^n}\right]\right\}\Big|_{z=z_1}$

(It is known that if z_0 is an nth order pole for the holomorphic function $f: D\backslash\{z_0\}\subseteq\mathbb{C}\to\mathbb{C}$, then $rez(f,z_0)=\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n f(z)\right]$.)

We have to compute the $(n-1)$-th derivative of $\frac{z^{n-1}}{(z-z_2)^n}$ in $z=z_1$, which is equivalent with computing the $(n-1)$-th derivative of $\frac{(z+z_2)^{n-1}}{z^n}$ in $z=z_1-z_2$:

$$\left\{\frac{d^{n-1}}{dz^{n-1}}\left[\frac{z^{n-1}}{(z-z_2)^n}\right]\right\}\Big|_{z=z_1} = \left\{\frac{d^{n-1}}{dz^{n-1}}\left[\frac{(z+z_2)^{n-1}}{z^n}\right]\right\}\Big|_{z=z_1-z_2} \tag{A.2}$$

Expanding with Newton's binomial formula and dividing by z^n, we get:

$\left\{\frac{d^{n-1}}{dz^{n-1}}\left[\frac{(z+z_2)^{n-1}}{z^n}\right]\right\}\Big|_{z=z_1-z_2} = \left[\frac{d^{n-1}}{dz^{n-1}}\left(\sum\limits_{k=1}^{n} C_{n-1}^{n-k}\frac{z_2^{k-1}}{z^k}\right)\right]\Big|_{z=z_1-z_2}$

$= \sum\limits_{k=1}^{n} z_2^{k-1} C_{n-1}^{n-k}\left[\frac{d^{n-1}}{dz^{n-1}}\left(\frac{1}{z^k}\right)\right]\Big|_{z=z_1-z_2}$

Using that:

$$\frac{d^{n-1}}{dz^{n-1}}\left(\frac{1}{z^k}\right) = \frac{(-1)^{n-1}(k+n-2)!}{(k-1)!}\frac{1}{z^{k+n-1}} \tag{A.3}$$

we get:

$\left\{\frac{d^{n-1}}{dz^{n-1}}\left[\frac{z^{n-1}}{(z-z_2)^n}\right]\right\}\Big|_{z=z_1} = \sum\limits_{k=1}^{n} z_2^{k-1} C_{n-1}^{n-k}\left[\frac{(-1)^{n-1}(k+n-2)!}{(k-1)!}\frac{1}{z^{k+n-1}}\right]\Big|_{z=z_1-z_2}$

But $z_1 = \frac{1}{e}-\sqrt{\left(\frac{1}{e}\right)^2-1}$, $z_2=\frac{1}{e}+\sqrt{\left(\frac{1}{e}\right)^2-1}$, $C_{n-1}^{n-k}=\frac{(n-1)!}{(n-k)!(k-1)!}$, and so the expression $:\left\{\frac{d^{n-1}}{dz^{n-1}}\left[\frac{z^{n-1}}{(z-z_2)^n}\right]\right\}\Big|_{z=z_1}$ is:

$\sum\limits_{k=1}^{n}\frac{(-1)^k(n-1)!(k+n-2)!}{(n-k)![(k-1)!]^2}\left(\frac{1+\sqrt{1-e^2}}{2\sqrt{1-e^2}}\right)^{k-1}\frac{1}{\left(2\sqrt{1-e^2}\right)^n}$

Then:

$$\int\limits_{|z|=1} f(z)\,dz = 2\pi i\sum_{k=1}^{n}\frac{(-1)^k(k+n-2)!}{(n-k)![(k-1)!]^2}\left(\frac{1+\sqrt{1-e^2}}{2\sqrt{1-e^2}}\right)^{k-1}\frac{1}{\left(2\sqrt{1-e^2}\right)^n} \tag{A.4}$$

Let us remember that all these computations are made for an ellipse where e is the eccentricity and a and b are its semiaxis, so $\sqrt{1-e^2}=\frac{b}{a}$. Then:

$$\int_{|z|=1} f(z)\,dz = 2\pi i\left(\frac{a}{2b}\right)^n\sum_{k=1}^{n}\frac{(-1)^k(k+n-2)!}{(n-k)![(k-1)!]^2}\left(\frac{a+b}{2b}\right)^{k-1} \tag{A.5}$$

We use this result for the integral mean of $\frac{1}{r^{n+1}}$ for $n\in\mathbb{N}, n\geq 2$:

$$\frac{1}{T}\int_0^T\frac{1}{r^{n+1}(t)}dt = \frac{1}{a}\left(\frac{-1}{eb}\right)^n\sum_{k=1}^{n}\frac{(-1)^k(k+n-2)!}{(n-k)![(k-1)!]^2}\left(\frac{a+b}{2b}\right)^{k-1} \tag{A.6}$$

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