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Riemann Invariants and Rank-k Solutions of Hyperbolic Systems

A M GRUNDLAND [a,b] *and B HUARD* [a]

[a] *Centre de Recherches Mathématiques, Université de Montréal, C. P. 6128, Succ. Centre-ville, Montréal, (QC) H3C 3J7, Canada*

[b] *Université du Québec, Trois-Rivières CP500 (QC) G9A 5H7, Canada*
E-mails: grundlan@crm.umontreal.ca; benoit.huard@dms.umontreal.ca

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Abstract

In this paper we employ a "direct method" to construct rank-k solutions, expressible in Riemann invariants, to hyperbolic system of first order quasilinear differential equations in many dimensions. The most important feature of our approach is the analysis of group invariance properties of these solutions and applying the conditional symmetry reduction technique to the initial equations. We discuss in detail the necessary and sufficient conditions for existence of these type of solutions. We demonstrate our approach through several examples of hydrodynamic type systems; new classes of solutions are obtained in a closed form.

1 Introduction

This work has been motivated by a search for new ways of constructing multiple Riemann waves for nonlinear hyperbolic systems. Riemann waves and their superpositions were first studied two centuries ago in connection with differential equations describing a compressible isothermal gas flow, by D. Poisson [19] and later by B. Riemann [20]. Since then many different approaches to this topic have been developed by various authors with the purpose of constructing solutions to more general hydrodynamic-type systems of PDEs. For a classical presentation we refer reader to a treatise by R. Courant and D. Hilbert [2] and for a modern approach to the subject, see e.g. [12, 17, 21] and references therein. A review of most recent developments in this area can be found in [3, 5, 13].

The task of constructing multiple Riemann waves has been approached so far mainly through the method of characteristics. It relies on treating Riemann invariants as new independent variables (which remain constant along appropriate characteristic curves of the basic system). This leads to the reduction of the dimensionality of the initial system which has to be subjected however to the additional differential constraints, limiting the scope of resulting solutions.

We propose here a new (though a very natural) way of looking at solutions expressible in terms of Riemann invariants, namely from the point of view of their group invariance properties. We show that this approach (initiated in [4, 11]) leads to the larger classes of solutions, extending beyond Riemann multiple waves.

We are looking for the rank-k solutions of first order quasilinear hyperbolic system of PDEs in p independent variables x^i and q unknown functions u^α of the form

$$\Delta^{\mu i}_{\alpha}(u)\, u^\alpha_i = 0, \quad \mu = 1, \ldots, l. \tag{1.1}$$

We denote by U and X the spaces of dependent variables $u = (u^1, \ldots, u^q) \in \mathbb{R}^q$ and independent variables $x = (x^1, \ldots, x^p) \in \mathbb{R}^p$, respectively. The functions $\Delta^{\mu i}_{\alpha}$ are assumed to be real valued functions on U and are components of the tensor products $\Delta^{\mu i}_{\alpha}\partial_i \otimes du^\alpha$ on $X \times U$. Here, we denote the partial derivatives by $u^\alpha_i = \partial_i u^\alpha \equiv \partial u^\alpha / \partial x^i$ and we adopt the convention that repeated indices are summed unless one of them is in a bracket. For simplicity we assume that all considered functions and manifolds are at least twice continuously differentiable in order to justify our manipulations. All our considerations have a local character. For our purposes it suffices to search for solutions defined on a neighborhood of the origin $x = 0$. In order to solve (1.1), we look for a map $f : X \to J^1(X \times U)$ annihilating the contact 1-forms, i.e.

$$f^*(du^\alpha - u^\alpha_i\, dx^i) = 0. \tag{1.2}$$

The image of f is in a submanifold of the first jet space J^1 over X given by (1.1) for which J^1 is equipped with coordinates $x^i, u^\alpha, u^\alpha_i$.

This paper is organized as follows. Section 2 contains a detailed account of the construction of rank-1 solutions of PDEs (1.1). In section 3 we discuss the construction of rank-k solutions, using geometric and group invariant properties of the system (1.1). Section 4 deals with a number of examples of hydrodynamic type systems which illustrate the theoretical considerations. Several new classes of solutions in implicit and explicit form are obtained. Section 5 contains a comparison of our results with the generalized method of characteristics for multi-dimensional systems of PDEs.

2 The rank-1 solutions

It is well known [2] that any hyperbolic system (1.1) admits rank-1 solutions

$$u = f(r), \quad r(x, u) = \lambda_i(u)\, x^i, \tag{2.1}$$

where $f = (f^\alpha)$ are some functions of r and a wave vector is a nonzero function

$$\lambda(u) = (\lambda_1(u), \ldots, \lambda_p(u)) \tag{2.2}$$

such that

$$\ker(\Delta^i \lambda_i) \neq 0. \tag{2.3}$$

Solution (2.1) is called a Riemann wave and the scalar function $r(x)$ is the Riemann invariant associated with the wave vector λ.

The function f is a solution of (1.1) if and only if the condition

$$\left(\Delta^{\mu i}_{\alpha}(f)\lambda_i(f)\right) f'^{\alpha} = 0, \quad f'^{\alpha} = \frac{df^{\alpha}}{dr} \tag{2.4}$$

holds, i.e. if and only if f' is an element of $\ker(\Delta^i \lambda_i)$. Note that equation (2.4) is an underdetermined system of the first order ordinary differential equations (ODEs) for f. The image of a solution (2.1) is a curve in U space defined by the map $f : \mathbb{R} \to \mathbb{R}^q$ satisfying the set of ODEs (2.4). The extent to which expresion (2.4) constrains the function f depends on the dimension of $\ker(\Delta^i \lambda_i)$. For example, if $\Delta^i \lambda_i = 0$ then there is no constraint on the function f at all and no integration is involved. The rank-1 solutions have the following common properties :

1. The Jacobian matrix is decomposable (in matrix notation)

$$\partial u = \left(1 - \frac{\partial f}{\partial r}\frac{\partial r}{\partial u}\right)^{-1} \frac{\partial f}{\partial r}\,\lambda, \tag{2.5}$$

or equivalently

$$\partial u = \frac{\partial f}{\partial r}\left(1 - \frac{\partial r}{\partial u}\frac{\partial f}{\partial r}\right)^{-1} \lambda, \tag{2.6}$$

where we have

$$\begin{aligned} &\partial u = (u^{\alpha}_i) \in \mathbb{R}^{q\times p}, \quad \frac{\partial f}{\partial r} = \left(\frac{\partial f^{\alpha}}{\partial r}\right) \in \mathbb{R}^q, \\ &\frac{\partial r}{\partial u} = \left(\frac{\partial r}{\partial u^{\alpha}}\right) = \frac{\partial \lambda_i}{\partial u^{\alpha}} x^i \in \mathbb{R}^q, \quad \lambda = (\lambda_i) \in \mathbb{R}^p. \end{aligned} \tag{2.7}$$

This property follows directly from differentiation of (2.1). The inverses $\left(1 - \frac{\partial f}{\partial r}\frac{\partial r}{\partial u}\right)^{-1}$ or $\left(1 - \frac{\partial r}{\partial u}\frac{\partial f}{\partial r}\right)^{-1}$ are scalar functions and are defined, since $\partial r/\partial u = 0$ at $x = 0$. From equations (2.5) or (2.6), it can be noted that $u(x)$ has rank at most equal to 1.

2. The graph of the rank-1 solution $\Gamma = \{x, u(x)\}$ is (locally) invariant under the linearly independent vector fields

$$X_a = \xi^i_a(u)\partial_i, \quad a = 1, \dots, p-1 \tag{2.8}$$

acting on $X \times U$ space. Here the vectors

$$\xi_a(u) = \left(\xi^1_a(u), \dots, \xi^p_a(u)\right)^T \tag{2.9}$$

satisfy the orthogonality conditions

$$\lambda_i\, \xi^i_a = 0, \quad a = 1, \dots, p-1 \tag{2.10}$$

for a fixed wave vector λ for which (2.3) holds. The vector fields (2.8) span a Lie vector module g over functions on U which constitutes an infinite-dimensional Abelian Lie algebra. The algebra g uniquely defines a module Λ (over the functions on U) of 1-forms $\lambda_i(u)\,dx^i$ annihilating all elements of g. A basis of Λ is given by

$$\lambda = \lambda_i(u)\,dx^i, \quad \xi^i_a\,\lambda_i = 0 \tag{2.11}$$

for all indices $a = 1, \ldots, p-1$. The set $\{r = \lambda_i(u)x^i, u^1, \ldots, u^q\}$ is the complete set of invariants of the vector fields (2.8).

3. It should be noted that rescaling the wave vector λ produces the same solution due to the homogeneity of the original system (1.1).

4. Due to the orthogonality conditions (2.10), together with property (2.5) or (2.6), any rank-1 solution is a solution of the overdetermined system of equations composed of system (1.1) and the differential constraints

$$\xi_a^i(u)\, u_i^\alpha = 0, \quad a = 1, \ldots, p-1. \tag{2.12}$$

The side equations (2.12) mean that the characteristics of the vector fields (2.8) are equal to zero.

5. One can always find nontrivial solutions of (2.4) if (1.1) is an underdetermined system ($l < q$) or if it is properly determined ($l = q$) and hyperbolic. Here, a weaker assumption can be imposed on the system (1.1). Namely, it is sufficient to require that eigenvalues of the matrix $(\Delta^i \lambda_i)$ are real functions.

The method of construction of rank-1 solutions to (1.1) can be summarized as follows. First, we seek a wave vector $\lambda = (\lambda_1, \ldots, \lambda_p)$ such that

$$\operatorname{rank}\left(\Delta^{\mu i}_{\alpha} \lambda_i\right) < l. \tag{2.13}$$

For each such choice of λ_i we look for the solutions γ^α of the wave relations

$$\left(\Delta^{\mu i}_{\alpha}\, \lambda_i\right) \gamma^\alpha = 0, \quad \mu = 1, \ldots, l. \tag{2.14}$$

Functions $f^\alpha(r)$ are required to satisfy the ODEs

$$f'^\alpha(r) = \gamma^\alpha(f(r)). \tag{2.15}$$

Alternatively, the system of equations (2.4) is linear in the variables λ_i. Nonzero solutions λ_i exist if and only if

$$\operatorname{rank}\left(\Delta^{\mu i}_{a}\,(f(r))\, f'^\alpha(r)\right) < p. \tag{2.16}$$

If (2.16) is satisfied for some function $f(r)$ then one can easily find $\lambda_i(r)$ satisfying equations (2.4). Using $u = f(r)$ one can define $\lambda_i(u)$ (not uniquely in general). If $l < p$ then (2.16) is identically satisfied for any function $f(r)$ and this approach does not require any integration.

3 The rank-k solutions

This section is devoted to the construction of rank-k solutions of a multi-dimensional system of PDEs (1.1). These solutions may be considered as nonlinear superpositions of rank-1 solutions.

Suppose that we fix k linearly independent wave vectors $\lambda^1, \ldots, \lambda^k$, $1 \leq k < p$ with Riemann invariant functions

$$r^A(x, u) = \lambda_i^A(u)x^i, \quad A = 1, \ldots, k. \tag{3.1}$$

The equation

$$u = f\left(r(x,u)\right), \quad r(x,u) = \left(r^1(x,u), \ldots, r^k(x,u)\right) \tag{3.2}$$

then defines a unique function $u(x)$ on a neighborhood of $x = 0$. The Jacobian matrix of (3.2) is given by

$$\partial u = \left(\mathbb{I} - \frac{\partial f}{\partial r}\frac{\partial r}{\partial u}\right)^{-1} \frac{\partial f}{\partial r}\lambda, \tag{3.3}$$

or equivalently

$$\partial u = \frac{\partial f}{\partial r}\left(\mathbb{I} - \frac{\partial r}{\partial u}\frac{\partial f}{\partial r}\right)^{-1} \lambda, \tag{3.4}$$

where $f = (f^\alpha)$, f^α are arbitrary functions of $r = (r^A)$ and

$$\begin{aligned} &\partial u = (u_i^\alpha) \in \mathbb{R}^{q\times p}, \quad \frac{\partial f}{\partial r} = \left(\frac{\partial f^\alpha}{\partial r^A}\right) \in \mathbb{R}^{q\times k}, \\ &\lambda = \left(\lambda_i^A\right) \in \mathbb{R}^{k\times p}, \quad \frac{\partial r}{\partial u} = \left(\frac{\partial r^A}{\partial u^\alpha}\right) = \frac{\partial \lambda_i^A}{\partial u^\alpha} x^i \in \mathbb{R}^{k\times q}. \end{aligned} \tag{3.5}$$

We assume here that the inverse matrices appearing in expressions (3.3) or (3.4), denoted by

$$\Phi^1 = \left(\mathbb{I} - \frac{\partial f}{\partial r}\frac{\partial r}{\partial u}\right) \in \mathbb{R}^{q\times q}, \quad \Phi^2 = \left(\mathbb{I} - \frac{\partial r}{\partial u}\frac{\partial f}{\partial r}\right) \in \mathbb{R}^{k\times k} \tag{3.6}$$

respectively, are invertible in some neighborhood of the origin $x = 0$. This assumption excludes the gradient catastrophe phenomenon for the function u.

Note that the rank of the Jacobian matrix (3.3) or (3.4) is at most equal to k. Hence the image of the rank-k solution is a k-dimensional submanifold $\mathcal{S}$ which lies in a submanifold of J^1.

If the set of vectors

$$\xi_a(u) = \left(\xi_a^1(u), \ldots, \xi_a^p(u)\right)^T, \quad a = 1, \ldots, p-k, \tag{3.7}$$

satisfies the orthogonality conditions

$$\lambda_i^A \, \xi_a^i = 0 \tag{3.8}$$

for $A = 1, \ldots, k$, $a = 1, \ldots, p-k$ then by virtue of (3.3) or (3.4) we have

$$Q_a^\alpha(x, u^{(1)}) \equiv \xi_a^i\,(u)\, u_i^\alpha = 0, \quad a = 1, \ldots, p-k, \quad \alpha = 1, \ldots, q. \tag{3.9}$$

Therefore rank-k solutions, given by (3.2), are obtained from the overdetermined system (1.1) subjected to differential constraints (DCs) (3.9)

$$\Delta_\alpha^{\mu i}(u) u_i^\alpha = 0, \quad \xi_a^i(u) u_i^\alpha = 0, \quad a = 1, \ldots, p-k. \tag{3.10}$$

Note that the conditions (3.9) are more general than the one required for the existence of Riemann k-wave solutions (see expression (5.1) and discussion in Section 5).

Let us note also that there are different approaches to the overdetermined system (3.10) employed in different versions of Riemann invariant method for multi-dimensional PDEs. The essence of our approach lies in treating the problem from the point of view of the conditional symmetry method (for description see e.g. [15]). Below we proceed with the adaptation of this method for our purpose.

The graph of the rank-k solution $\Gamma = \{x, u(x)\}$ of (3.9) is invariant under the vector fields

$$X_a = \xi^i_a(u)\partial_i, \quad a = 1, \ldots, p-k \tag{3.11}$$

acting on $X \times U \subset \mathbb{R}^p \times \mathbb{R}^q$. The functions $\{r^1, \ldots, r^k, u^1, \ldots, u^q\}$ constitute a complete set of invariants of the Abelian Lie algebra $\mathcal{A}$ generated by the vector fields (3.11).

In order to solve the overdetermined system (3.10) we subject it to several transformations, based on the set of invariants of $\mathcal{A}$, which simplify its structure considerably. To achieve this simplification we choose an appropriate system of coordinates on $X \times U$ space which allows us to rectify the vector fields X_a, given by (3.11). Next, we show how to find the invariance conditions in this system of coordinates which guarantee the existence of rank-k solutions in the form (3.2).

Let us assume that the k by k matrix

$$\Pi = \left(\lambda^A_i\right), \quad 1 \leq A, i \leq k < p \tag{3.12}$$

built from the components of the wave vectors λ^A is invertible. Then the linearly independent vector fields

$$\begin{aligned} X_{k+1} &= \partial_{k+1} - \sum_{A,j=1}^{k} \left(\Pi^{-1}\right)^j_A \lambda^A_{k+1}\partial_j, \\ &\vdots \\ X_p &= \partial_p - \sum_{A,j=1}^{k} \left(\Pi^{-1}\right)^j_A \lambda^A_p\partial_j, \end{aligned} \tag{3.13}$$

have the required form (3.11) for which the orthogonality conditions (3.8) are satisfied. The change of independent and dependent variables

$$\bar{x}^1 = r^1(x,u), \ldots, \bar{x}^k = r^k(x,u), \quad \bar{x}^{k+1} = x^{k+1}, \ldots, \bar{x}^p = x^p, \bar{u}^1, \ldots, \bar{u}^q = u^q \tag{3.14}$$

permits us to rectify the vector fields X_a and get

$$X_{k+1} = \partial_{\bar{x}^{k+1}}, \ldots, X_p = \partial_{\bar{x}^p}. \tag{3.15}$$

Note that a p-dimensional submanifold is transverse to the projection $(x,u) \to x$ at $x = 0$ if and only if it is transverse to the projection $(\bar{x},\bar{u}) \to \bar{x}$ at $\bar{x} = 0$. The transverse p-dimensional submanifolds invariant under $X_{k+1}, \ldots, X_p$ are defined by the implicit equation of the form

$$\bar{u} = f(\bar{x}^1, \ldots, \bar{x}^k). \tag{3.16}$$

Hence, expression (3.16) is the general integral of the invariance conditions

$$\bar{u}_{\bar{x}^{k+1}} = 0, \ldots, \bar{u}_{\bar{x}^p} = 0. \tag{3.17}$$

The system (1.1) is subjected to the invariance conditions (3.17) and, when written in terms of new coordinates $(\bar{x}, \bar{u}) \in X \times U$, takes the form

$$\Delta^{\mu} \left(\Phi^1\right)^{-1} \frac{\partial \bar{u}}{\partial \bar{x}} \lambda = 0, \quad , \bar{u}_{\bar{x}^{k+1}} = 0, \ldots, \bar{u}_{\bar{x}^p} = 0, \tag{3.18}$$

or

$$\Delta^{\mu} \frac{\partial \bar{u}}{\partial \bar{x}} \left(\Phi^2\right)^{-1} \lambda = 0, \quad , \bar{u}_{\bar{x}^{k+1}} = 0, \ldots, \bar{u}_{\bar{x}^p} = 0, \tag{3.19}$$

where the matrices Φ^1 and Φ^2 are given by

$$\left(\Phi^1\right)^A_i = \delta^A_i - \bar{u}^{\alpha}_i \frac{\partial r^A}{\partial \bar{u}^{\alpha}}, \quad \left(\Phi^2\right)^A_i = \delta^A_i - \frac{\partial r^A}{\partial \bar{u}^{\alpha}} \bar{u}^{\alpha}_i. \tag{3.20}$$

The above considerations characterize geometrically the solutions of the overdetermined system (3.10) in the form (3.2). Let us illustrate these considerations with some examples.

Example 1. Let us assume that there exist k independent relations of dependence for the matrices $\Delta^1, \ldots, \Delta^p$ such that the conditions

$$\Delta^{\mu i}_{\alpha} \lambda^A_i = 0, \quad A = 1, \ldots, k \tag{3.21}$$

hold. Suppose also that the original system (1.1) has the evolutionary form and each of the q by q matrices $A^1, \ldots, A^n$ is scalar, i.e.

$$\Delta^0 = \mathbb{I}, \quad \Delta^{i\alpha}_{\beta} = a^i(u) \delta^{\alpha}_{\beta}, \quad i = 1, \ldots, n \tag{3.22}$$

for some functions $a^1, \ldots, a^n$ defined on U, where $p = n+1$ and for convenience we denote the independent variables by $x = (t = x^0, x^1, \ldots, x^n) \in X$. Then the system (1.1) is particularly simple and becomes

$$u_t + a^1(u) u_1 + \ldots + a^n(u) u_n = 0. \tag{3.23}$$

The corresponding wave vectors

$$\begin{aligned} &\lambda^1 = (-a^1(u), 1, 0, \ldots, 0), \\ &\vdots \\ &\lambda^n = (-a^n(u), 0, \ldots, 0, 1) \end{aligned} \tag{3.24}$$

are linearly independent and satisfy conditions (3.21). A vector function $u(x, t)$ is a solution of (3.23) if and only if the vector field

$$X = \partial_t + a^i(u) \partial_i$$

defined on $\mathbb{R}^{n+q+1}$ is tangent to the $(n+1)$-dimensional submanifold $\mathcal{S} = \{u = u(x,t)\} \subset \mathbb{R}^{n+q+1}$. The solution is thus identified with the $(n+1)$-dimensional submanifold $\mathcal{S} \subset$

$\mathbb{R}^{n+q+1}$ which is transverse to $\mathbb{R}^{n+q+1} \to \mathbb{R}^{n+1} : (x,t,u) \to (x,t)$ and is invariant under the vector field X. The functions $\{r(x,t,u) = (r^1 = x^1 - a^1(u)t, \dots, r^n = x^n - a^n(u)t), u^1, \dots, u^q\}$ are invariants of X, such that $dr^1 \wedge \dots \wedge dr^n \wedge du^1 \wedge \dots \wedge du^q \neq 0$. If we define $\bar{t} = t, \bar{u} = u$, then $(r, \bar{t}, \bar{u})$ are coordinates on $\mathbb{R}^{n+q+1}$ and the vector field X can be rectified

$$X = \partial_{\bar{t}}.$$

The general solution is

$$\mathfrak{S} = \{F(r, \bar{u}) = 0\}$$

where $F : \mathbb{R}^{n+q} \to \mathbb{R}^q$ satisfies the condition

$$\det\left(\frac{\partial F}{\partial r}\frac{\partial r}{\partial \bar{u}} + \frac{\partial F}{\partial \bar{u}}\right) \neq 0$$

but is otherwise arbitrary. Note that it may be assumed that

$$\frac{\partial r}{\partial u}(x_0, t_0, u_0) = 0,$$

in which case the transversality condition is

$$\det\left(\frac{\partial F}{\partial \bar{u}}(x_0, t_0, u_0)\right) \neq 0.$$

Hence the general solution of (3.23) near (x_0, t_0, u_0) is

$$\mathfrak{S} = \{\bar{u} = f(r)\},$$

where $f : \mathbb{R}^n \to \mathbb{R}^q$ is arbitrary. Thus the equation

$$u = f(x^1 - a^1(u)t, \dots, x^n - a^n(u)t), \tag{3.25}$$

defines a unique function $u(x,t)$ on a neighborhood of the point (x_0, t_0, u_0) for any f. Note that

$$t = 0, \quad u(x, 0) = f(x^1, \dots, x^n),$$

so the function f is simply the Cauchy data on $\{t = 0\}$.

Example 2. Another interesting case to consider is when the matrix Φ^1 (or Φ^2) is a scalar matrix. Then system (3.18) is equivalent to the quasilinear system in k independent variables $\bar{x}^1, \dots, \bar{x}^k$ and q dependent variables $\bar{u}^1, \dots, \bar{u}^q$. So, we have

$$B^A(\bar{u})\bar{u}^\alpha_A = 0, \tag{3.26}$$

where

$$B^A = \Delta^i \lambda^A_i. \tag{3.27}$$

If $k \geq 2$ then Φ^1 is a scalar if and only if

$$\frac{\partial r^1}{\partial u} = 0, \ldots, \frac{\partial r^k}{\partial u} = 0 \tag{3.28}$$

and consequently, if and only if the vector fields $\lambda^1, \ldots, \lambda^k$ are constant wave vectors.

Finally, a more general situation occurs when the matrix Φ^1 (or Φ^2) satisfies the conditions

$$\frac{\partial \Phi^1}{\partial \bar{x}^{k+1}} = 0, \ldots, \frac{\partial \Phi^1}{\partial \bar{x}^p} = 0. \tag{3.29}$$

Then the system (3.18) is independent of variables $\bar{x}^{k+1}, \ldots, \bar{x}^p$. The conditions (3.29) hold if and only if

$$\frac{\partial^2 r}{\partial u \partial \bar{x}^{k+1}} = 0, \ldots, \frac{\partial^2 r}{\partial u \partial \bar{x}^p} = 0. \tag{3.30}$$

Using (3.1) and (3.12) we get

$$\frac{\partial \lambda_i^A}{\partial u} = \sum_{l,B=1}^{k} \frac{\partial \Pi_l^A}{\partial u} \left(\Pi^{-1}\right)^l_B \lambda_i^B. \tag{3.31}$$

Equation (3.31) can be rewritten in the simpler form

$$\frac{\partial}{\partial u} \left(\sum_{B=1}^{k} \left(\Pi^{-1}\right)^l_B \lambda_i^B \right) = 0, \quad 1 \leq l \leq k < i \leq p. \tag{3.32}$$

Thus system (3.18) is independent of variables $\bar{x}^{k+1}, \ldots, \bar{x}^p$ if the k by $p-k$ matrix $\left(\lambda_i^B\right)$, $1 \leq B \leq k < i \leq p$ is equal to the matrix ΠC, where C is a constant k by $(p-k)$ matrix. In this case (3.18) is a system not necessarily quasilinear, in k independent variables $\bar{x}^1, \ldots, \bar{x}^k$ and q dependent variables $\bar{u}^1, \ldots, \bar{u}^q$.

Let us now derive the neccesary and sufficient conditions for existence of solutions in the form (3.2) of the overdetermined system (3.10). Substituting (3.3) or (3.4) into (1.1) yields

$$\operatorname{Tr}\left[\Delta^\mu \left(\mathbb{I} - \frac{\partial f}{\partial r} \frac{\partial r}{\partial u} \right)^{-1} \frac{\partial f}{\partial r} \lambda \right] = 0, \tag{3.33}$$

or equivalently

$$\operatorname{Tr}\left[\Delta^\mu \frac{\partial f}{\partial r} \left(\mathbb{I} - \frac{\partial r}{\partial u} \frac{\partial f}{\partial r} \right)^{-1} \lambda \right] = 0, \tag{3.34}$$

respectively, where

$$\Delta^\mu = \left(\Delta^{\mu i}_{\alpha}\right) \in \mathbb{R}^{p \times q}, \quad \mu = 1, \ldots, l. \tag{3.35}$$

Given the system of PDEs (1.1) (i.e. functions $\Delta^{\mu i}_{\alpha}(u)$) it follows that equations (3.33) (or (3.34)) are conditions on the functions $f^{\alpha}(r)$ and $\lambda^A_i(u)$ (or $\xi^i_a(u)$). Since $\partial r/\partial u$ depends explicitly on x it may happen that these conditions have only trivial solutions (i.e. f=const) for some values of k. We discuss a set of conditions following from (3.33) or (3.34) which allow the system (3.10) to possess the nontrivial rank-k solutions.

Let g be a (p-k)-dimensional Lie vector module over $C^{\infty}(X \times U)$ with generators X_a given by (3.11). Let Λ be a k-dimensional module generated by $k < p$ linearly independent 1-forms

$$\lambda^A = \lambda^A_i(u)dx^i, \quad A = 1, \ldots, k$$

which are annihilated by $X_a \in g$. It is assumed here that the vector fields X_a and λ^A are related by the orthogonality conditions (3.8) and form a basis of g and Λ, respectively. For $k > 1$, it is always possible to choose a basis λ^A of the module Λ of the form

$$\lambda^A = dx^{i_A} + \lambda^A_{i_a}\, dx^{i_a}, \quad A = 1, \ldots, k \tag{3.36}$$

where (i_A, i_a) is a permutation of $(1, \ldots, p)$. Here we split the coordinates x^i into x^{i_A} and x^{i_a}. Then from (3.1) we obtain the relation

$$\frac{\partial r^A}{\partial u^{\alpha}} = \frac{\partial \lambda^A_{i_a}}{\partial u^{\alpha}} x^{i_a}. \tag{3.37}$$

Substituting (3.37) into equations (3.33) or (3.34) yields, respectively

$$\mathrm{Tr}\left(\Delta^{\mu}(\mathbb{I} - Q_a x^{i_a})^{-1}\frac{\partial f}{\partial r}\lambda\right) = 0, \tag{3.38}$$

or

$$\mathrm{Tr}\left(\Delta^{\mu}\frac{\partial f}{\partial r}(\mathbb{I} - K_a x^{i_a})^{-1}\lambda\right) = 0, \tag{3.39}$$

where we use the following notation

$$Q_a = \frac{\partial f}{\partial r}\eta_a \in \mathbb{R}^{q\times q}, \quad K_a = \eta_a\frac{\partial f}{\partial r} \in \mathbb{R}^{k\times k}, \tag{3.40}$$

$$\eta_a = \left(\frac{\partial \lambda^A_{i_a}}{\partial u^{\alpha}}\right) \in \mathbb{R}^{k\times q}, \quad i_a = 1, \ldots, p-1. \tag{3.41}$$

The functions r^A and x^{i_a} are all independent in the neighborhood of the origin $x = 0$. The functions Δ^{μ}, $\frac{\partial f}{\partial r}, \lambda, Q_a$ and K_a depend on r only. For these specific functions, equations (3.38) (or (3.39)) must be satisfied for all values of coordinates x^{i_a}. In order to find appropriate conditions for $f(r)$ and $\lambda(u)$ let us notice that, according to the Cayley-Hamilton theorem, for any n by n invertible matrice M, $(M^{-1}\det M)$ is a polynomial in M of order $(n-1)$. Hence, one can replace equation (3.38) by

$$\mathrm{Tr}\left(\Delta^{\mu}\, Q\, \frac{\partial f}{\partial r}\,\lambda\right) = 0, \tag{3.42}$$

where we introduce the following notation

$$Q = (\mathbb{I} - Q_a x^{i_a})^{-1} \det(\mathbb{I} - Q_a x^{i_a}).$$

Taking equation (3.42) and all its x^{i_a} derivatives (with r=const) at $x^{i_a} = 0$, yields

$$\operatorname{Tr}\left(\Delta^\mu \frac{\partial f}{\partial r}\lambda\right) = 0, \tag{3.43}$$

$$\operatorname{Tr}\left(\Delta^\mu Q_{(a_1} \dots Q_{a_s)} \frac{\partial f}{\partial r}\lambda\right) = 0, \tag{3.44}$$

where $s = 1, \dots, q-1$ and $(a_1, \dots, a_s)$ denotes symmetrization over all indices in the bracket. A similar procedure for equation (3.39) yields (3.43) and the trace condition

$$\operatorname{Tr}\left(\Delta^\mu \frac{\partial f}{\partial r} K_{(a_1}, \dots, K_{a_s)}\lambda\right) = 0, \tag{3.45}$$

where now $s = 1, \dots, k-1$.
Equation (3.43) represents an initial value condition on a surface in X space given by $x^{i_a} = 0$. Equations (3.44) (or (3.45)) correspond to the preservation of (3.43) by flows represented by the vector fields (3.11). Note that X_a can be put into the form

$$X_a = \partial_{i_a} - \lambda^A_{i_a}\partial_A, \quad \xi^i_a \cdot \lambda^A_i = 0, \quad A = 1, \dots, k. \tag{3.46}$$

By virtue of (3.40), (3.41), equations (3.44) or (3.45) take the unified form

$$\operatorname{Tr}\left(\Delta^\mu \frac{\partial f}{\partial r}\eta_{(a_1}\frac{\partial f}{\partial r} \dots \eta_{a_s)}\frac{\partial f}{\partial r}\lambda\right) = 0, \tag{3.47}$$

where either $\max s = q-1$ or $\max s = k-1$.

The vector fields X_a and the Lie module g spanned by the vector fields $X_1, \dots, X_{p-k}$ are called the conditional symmetries and the conditional symmetry module of (1.1), respectively if X_a are Lie point symmetries of the original system (1.1) supplemented by the DCs (3.9) [15].

Let us now associate the system (1.1) and the conditions (3.9) with the subvarieties of the solution spaces

$$\mathcal{B}_\Delta = \{(x, u^{(1)}) : \Delta^{\mu i}_\alpha(u)\, u^\alpha_i = 0, \quad \mu = 1, \dots, l\},$$

and

$$\mathcal{B}_Q = \{(x, u^{(1)}) : \xi^i_a(u) u^\alpha_i = 0, \quad a = 1, \dots, p-k, \quad \alpha = 1, \dots, q\},$$

respectively. We have the following.

Proposition 1. *A nondegenerate first order hyperbolic system of PDEs (1.1) admits a (p-k)-dimensional Lie vector module g of conditional symmetries if and only if (p-k) linearly independent vector fields $X_1, \dots, X_{p-k}$ satisfy the conditions (3.43) and (3.47) on some neighborhood of (x_0, u_0) of $\mathcal{B} = \mathcal{B}_\Delta \cap \mathcal{B}_Q$.*

Proof. The vector fields X_a constitute the conditional symmetry module g for the system (1.1) if they are Lie point symmetries of the overdetermined system (3.10). This means that the first prolongation of X_a has to be tangent to the system (3.10). Hence g is a conditional symmetry module of (1.1) if and only if the equations

$$\mathrm{pr}^{(1)} X_a(\Delta^{\mu i}_{\alpha}(u)\, u^{\alpha}_i) = 0, \quad \mathrm{pr}^{(1)} X_a\left(\xi^i_b(u) u^{\alpha}_i\right) = 0, \quad a, b = 1, \ldots, p-k \tag{3.48}$$

are satisfied on J^1 whenever the equations (3.10) hold. Now we show that if the conditions (3.43) and (3.47) are satisfied then the symmetry criterion (3.48) is identically equal to zero.

In fact, applying the first prolongation of the vector fields X_a

$$\mathrm{pr}^{(1)} X_a = X_a + \xi^i_{a,u^\beta} u^\beta_j u^\alpha_i \frac{\partial}{\partial u^\alpha_j}$$

to the original system (1.1) yields

$$\mathrm{pr}^{(1)} X_a\left(\Delta^{\mu i}_{\alpha} u^{\alpha}_i\right) = \Delta^{\mu i}_{\alpha} \xi^j_{a,u^\beta} u^\beta_i u^\alpha_j = 0, \tag{3.49}$$

whenever equations (3.10) hold. On the other hand, carrying out the differentiations of (3.8) gives

$$\xi^j_{a,u^\beta} \lambda^B_j = -\xi^j_a \lambda^B_{j,u^\beta}. \tag{3.50}$$

Comparing (3.49) and (3.50) leads to

$$\Omega^{\mu}{}^A_B \xi^j_a Z_A(\lambda^B_j) = 0, \tag{3.51}$$

where we introduce the following notation

$$\Omega^{\mu}{}^A_B = \Delta^{\mu i}_{\alpha} Z^\alpha_B \lambda^A_i. \tag{3.52}$$

Here the new vector fields Z_B are defined on U

$$Z_A = Z^\alpha_A \frac{\partial}{\partial u^\alpha} \in T_u U. \tag{3.53}$$

It is convenient to write equation (3.51) in the equivalent form

$$\mathrm{Tr}(\Delta^\mu Z \theta_a Z \lambda) = 0, \quad \mu = 1, \ldots, l \tag{3.54}$$

where the following notation has been used

$$\theta_a = \lambda^A_{i,u^\beta} \xi^i_a. \tag{3.55}$$

The assumption that system (1.1) is hyperbolic implies that there exist the real-valued vector fields λ^A and γ_A defined on U for which the wave relation

$$\left(\Delta^{\mu\, i}_{\alpha} \lambda^A_i\right) \gamma^\alpha_{(A)} = 0, \quad A = 1, \ldots, k \tag{3.56}$$

is satisfied and that the U space is spanned by the linearly independent vector fields

$$\gamma_A = \gamma^\alpha_A\, \partial_{u^\alpha} \in T_u U. \tag{3.57}$$

Hence, one can represent the vector fields Z_A through the basis generated by the vector fields $\{\gamma_1, \ldots, \gamma_k\}$, i.e.

$$Z_A = h_A^B \gamma_B. \tag{3.58}$$

Using equations (3.3) and (3.6) we find the coefficients

$$h_A^B = ((\Phi^1)^{-1})_A^B.$$

This means that the submanifold $\mathcal{S}$, given by (3.2), can be represented parametrically by

$$\frac{\partial f^\alpha}{\partial r^A} = h_A^B \gamma_B^\alpha. \tag{3.59}$$

On the other hand, comparing (3.3) and (3.58) gives

$$u_i^\alpha = Z_A^\alpha \lambda_i^A. \tag{3.60}$$

Applying the invariance criterion (3.48) to the side conditions (3.9) we obtain

$$\mathrm{pr}^{(1)} X_a(Q_b^\alpha) = \xi^i_{[b} \xi^j_{a],u^\beta} u_i^\beta u_j^\alpha. \tag{3.61}$$

The bracket $[a, b]$ denotes antisymmetrization with respect to the indices a and b. By virtue of equations (3.50) and (3.60), the right side of (3.61) is identically equal to zero. Substituting (3.58) into equation (3.54) and taking into account equation (3.36) and (3.59) we obtain that for any value of $x \in X$ the resulting formulae coincide with equations (3.43) and (3.47). Hence, the infinitesimal symmetry criterion (3.48) for the overdetermined system (3.10) is identically satisfied whenever conditions (3.43) and (3.47) hold.

The converse also holds. The assumption that the system (1.1) is nondegenerate means that it is locally solvable and takes a maximal rank at every point $(x_0, u_0^{(1)}) \in \mathcal{B}_\Delta$. Therefore [14] the infinitesimal symmetry criterion is a necessary and sufficient condition for the existence of symmetry group G of the overdetermined system (3.10). Since the vector fields X_a form an Abelian distribution, it follows that the conditions (3.43) and (3.47) hold. That ends the proof since the solutions of the original system (1.1) are invariant under the Lie algebra generated by $(p-k)$ vector fields $X_1, \ldots, X_{p-k}$. ∎

Note that the set of solutions of the determining equations obtained by applying the symmetry criterion to the overdetermined system (3.10) is different than the set of solutions of the determining equations for the initial system (1.1). Thus the system (3.10) admits other symmetries than the original system (1.1). So, new reductions for the system (1.1) can be constructed, since each solution of system (3.10) is a solution of system (1.1).

In our approach the construction of solutions of the original system (1.1) requires us to solve first the system (3.47) for λ_i^A as functions of u^α and then find $u = f(r)$ by solving (3.43). Note that the functions $f^*(\lambda_i^A)$ are the functions $\lambda_i^A(f)$ pulled back to the surface $\mathcal{S}$. The $\lambda_i^A(f)$ then become functions of the parameters $r^1, \ldots, r^k$ on $\mathcal{S}$. For simplicity of notation we denote $f^*(\lambda_i^A)$ by $\lambda_i^A(r^1, \ldots, r^k)$.

The system composed of (3.43) and (3.47) is, in general, nonlinear. So, we cannot expect to solve it in a closed form, except in some particular cases. But nevertheless, as we show in section 4, there are physically interesting examples for which solutions of (3.43) and (3.47) lead to the new solutions of (1.1) which depend on some arbitrary functions. These particular solutions of (3.43) and (3.47) are obtained by expanding each function λ_i^A into a polynomial in the dependent variables u^α and requiring that the coefficients of the successive powers of u^α vanish. We then obtain a system of first order PDEs for the coefficients of the polynomials. Solving this system allows us to find some particular classes of solutions of the initial system (1.1) which can be constructed by applying the symmetry reduction technique.

4 Examples of applications

We start with considering the case of rank-2 solutions of the system (1.1) with two dependent variables ($q = 2$). Then (3.47) adopts the simplified form.

$$\mathrm{Tr}\left(\Delta^\mu \frac{\partial f}{\partial r}\,\eta_a\,\frac{\partial f}{\partial r}\lambda\right) = 0. \tag{4.1}$$

By virtue of (3.43), equation (4.1) can be transformed to

$$\mathrm{Tr}\left[\Delta^\mu \frac{\partial f}{\partial r}\left(\eta_a \frac{\partial f}{\partial r} - \mathbb{I}\mathrm{Tr}\left(\eta_a \frac{\partial f}{\partial r}\right)\right)\lambda\right] = 0. \tag{4.2}$$

Using the Cayley Hamilton identity, we get the relation

$$AB - \mathbb{I}\,\mathrm{Tr}AB = (B - \mathbb{I}\,\mathrm{Tr}B)(A - \mathbb{I}\,\mathrm{Tr}A) \tag{4.3}$$

for any 2 by 2 matrices $A, B \in \mathbb{R}^{2\times 2}$. Now we can rewrite (4.2) in the equivalent form

$$-\mathrm{Tr}\left[\Delta^\mu \frac{\partial f}{\partial r}\left(\frac{\partial f}{\partial r} - \mathbb{I}\mathrm{Tr}\frac{\partial f}{\partial r}\right)(\eta_a - \mathbb{I}\mathrm{Tr}\eta_a)\lambda\right] = 0. \tag{4.4}$$

So we have

$$\det\left(\frac{\partial f}{\partial r}\right)\mathrm{Tr}[\Delta^\mu(\eta_a - \mathbb{I}\mathrm{Tr}\eta_a)\lambda] = 0. \tag{4.5}$$

The rank-2 solutions require that the condition $\det \partial f/\partial r \neq 0$ be satisfied (otherwise $q = 2$ can be reduced to $q = 1$). As a consequence of this, we obtain the following condition

$$\mathrm{Tr}[\Delta^\mu(\eta_a - \mathbb{I}\mathrm{Tr}\eta_a)\lambda] = 0, \quad \mu = 1, \ldots, l, \tag{4.6}$$

which coincides with the result obtained earlier for this specific case [11]. One can look first for solutions $\lambda(u)$ of (4.6) and then find $f(r)$ by solving (3.43). Note that equations (4.6) form a system of $l(p-2)$ equations for $2(p-2)$ functions $\lambda_{i_a}^A(u)$. This indicates that they should have solutions (say, for generic systems) if (1.1) is not overdetermined.

Example 3. We are looking for rank-2 solutions of the (2+1) hydrodynamic type equations

$$u_t^i + u^j u_j^i + A_k^{ij} u_j^k = 0, \quad i, j, k = 1, 2 \tag{4.7}$$

where A^i are some matrix functions of u^1 and u^2. Using the condition representing the tracelessness of the matrices $\Delta^{1i}_{\alpha} u^{\alpha}_i$ and $\Delta^{2i}_{\alpha} u^{\alpha}_i$, it is convenient to rewrite the system (4.7) in the following form

$$\begin{aligned}
&\mathrm{Tr}\left[\begin{pmatrix} 1 & u^1 + A^{11}_1 & u^2 + A^{12}_1 \\ 0 & A^{11}_2 & A^{12}_2 \end{pmatrix}\begin{pmatrix} u^1_t & u^2_t \\ u^1_x & u^2_x \\ u^1_y & u^2_y \end{pmatrix}\right] = 0, \\
&\mathrm{Tr}\left[\begin{pmatrix} 0 & u^1 & u^2 \\ 1 & u^1 + A^{21}_2 & u^2 + A^{22}_2 \end{pmatrix}\begin{pmatrix} u^1_t & u^2_t \\ u^1_x & u^2_x \\ u^1_y & u^2_y \end{pmatrix}\right] = 0.
\end{aligned} \tag{4.8}$$

Let $\mathcal{F}$ be a smooth orientable surface immersed in 3-dimensional Euclidean $(x, y, t) \in X$ space. Suppose that $\mathcal{F}$ can be written in the following parametric form

$$u = f(r^1, r^2) = (u^1(r^1, r^2), u^2(r^1, r^2)), \tag{4.9}$$

such that the Jacobian matrix is different from zero

$$J = \det\left(\frac{\partial f^{\alpha}}{\partial r^A}\right) = \det\begin{pmatrix} \partial u^1/\partial r^1 & \partial u^1/\partial r^2 \\ \partial u^2/\partial r^1 & \partial u^2/\partial r^2 \end{pmatrix} \neq 0. \tag{4.10}$$

Without loss of generality, it is possible to choose a basis λ^A of module Λ of the form

$$\lambda^A_i = \begin{pmatrix} \lambda^1_1 & \lambda^1_2 & \lambda^1_3 \\ \lambda^2_1 & \lambda^2_2 & \lambda^2_3 \end{pmatrix}^T = \begin{pmatrix} \varepsilon & a^1 & b^1 \\ \varepsilon & a^2 & b^2 \end{pmatrix}^T, \quad \varepsilon = \pm 1, \tag{4.11}$$

where a^A and b^A are functions of u^1 and u^2 to be determined.

The rank-2 solution can be constructed from the most general solution of equations (4.6) for $\lambda^A = (-1, a^A, b^A)$, $A = 1, 2$. These equations lead to a system of four PDEs with four dependent variables $a^A, b^A, A = 1, 2$ and two independent variables u^1 and u^2,

$$\begin{aligned}
&-(A^{11}_2 a^2 + A^{12}_2 b^2)\frac{\partial a^1}{\partial u^1} + ((u^1 + A^{11}_1)a^2 + (u^2 + A^{12}_1)b^2 - 1)\frac{\partial a^1}{\partial u^2} \\
&\quad + (A^{11}_2 a^1 + A^{12}_2 b^1)\frac{\partial a^2}{\partial u^1} - ((u^1 + A^{11}_1)a^1 + (u^2 + A^{12}_1)b^1 - 1)\frac{\partial a^2}{\partial u^2} = 0,
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
&-(A^{11}_2 a^2 + A^{12}_2 b^2)\frac{\partial b^1}{\partial u^1} + ((u^1 + A^{11}_1)a^2 + (u^2 + A^{12}_1)b^2 - 1)\frac{\partial b^1}{\partial u^2} \\
&\quad + (A^{11}_2 a^1 + A^{12}_2 b^1)\frac{\partial b^2}{\partial u^1} - ((u^1 + A^{11}_1)a^1 + (u^2 + A^{12}_1)b^1 - 1)\frac{\partial b^2}{\partial u^2} = 0,
\end{aligned}$$

$$\begin{aligned}
&(1 - (u^1 + A^{21}_2)a^2 - (u^2 + A^{22}_2)b^2)\frac{\partial a^1}{\partial u^1} + (A^{21_1} a^2 + A^{22}_1 b^2)\frac{\partial a^1}{\partial u^2} \\
&- (1 - (u^1 + A^{21}_2)a^1 - (u^2 + A^{22}_2)b^1)\frac{\partial a^2}{\partial u^1} - (A^{21}_1 a^1 + A^{22}_1 b^1)\frac{\partial a^2}{\partial u^2} = 0,
\end{aligned}$$

$$\begin{aligned}
&(1 - (u^1 + A^{21}_2)a^2 - (u^2 + A^{22}_2)b^2)\frac{\partial b^1}{\partial u^1} + (A^{21_1} a^2 + A^{22}_1 b^2)\frac{\partial b^1}{\partial u^2} \\
&- (1 - (u^1 + A^{21}_2)a^1 - (u^2 + A^{22}_2)b^1)\frac{\partial b^2}{\partial u^1} - (A^{21}_1 a^1 + A^{22}_1 b^1)\frac{\partial b^2}{\partial u^2} = 0.
\end{aligned}$$

Finally, a rank-2 solution of (4.8) is obtained from the explicit parametrization of the surface $\mathcal{F}$ in terms of the parameters r^1 and r^2, by solving equations (3.43) in which λ^A adopt the form (4.11)

$$\begin{aligned}
&((u^1+A_1^{11})a^1+(u^2+A_1^{12})b^1-1)\frac{\partial u^1}{\partial r^1}+((u^1+A_1^{11})a^2+(u^2+A_1^{12})b^2-1)\frac{\partial u^1}{\partial r^2}\\
&\qquad+(A_2^{11}a^1+A_2^{12}b^1)\frac{\partial u^2}{\partial r^1}+(A_2^{11}a^2+A_2^{12}b^2)\frac{\partial u^2}{\partial r^2}=0, \qquad (4.13)\\
&(A_1^{21}a^1+A_1^{22}b^1)\frac{\partial u^1}{\partial r^1}+(A_1^{21}a^2+A_1^{22}b^2)\frac{\partial u^1}{\partial r^2}+((u^1+A_2^{21})a^1+(u^2+A_2^{22})b^1-1)\frac{\partial u^2}{\partial r^1}\\
&\qquad+((u^1+A_2^{21})a^2+(u^2+A_2^{22})b^2-1)\frac{\partial u^2}{\partial r^2}=0,
\end{aligned}$$

while the quantities r^1 and r^2 are implicitly defined as functions of y, x, t by equation (3.1) with λ^A given by (4.11).

In the case when equation (4.8) does admit two linearly independent vector fields λ^A with $\varepsilon=-1$, there exists a class of rank-2 solutions of equations (4.12) and (4.13) invariant under the vector fields

$$X_1=\partial_t+u^1\partial_x, \quad X_2=\partial_t+u^2\partial_y. \qquad (4.14)$$

Following the procedure outlined in Section 3 we assume that the functions f^1 and f^2 appearing in equation (3.2) are linear in u^2. Then the invariance conditions take the form

$$x-u^1t=g(u^1)+u^2h(u^1), \quad y-u^2t=a(u^1)+u^2b(u^1), \qquad (4.15)$$

where a, b, g and h are some functions of u^1.

One can show that if $h=0$, then the solution of the system (4.12), (4.13) is defined implicitly by the relations

$$x-u^1t=g(u^1), \quad y-u^2t=a(u^1)+u^2g_{,u^1}, \qquad (4.16)$$

where a and g are arbitrary functions of u^1. Note that in this case the functions u^1 and u^2 satisfy the following system of equations

$$\begin{aligned}
&u_t^1+u^1u_x^1+u^2u_y^1+A_1^{11}(u_x^1-u_y^2)+A_1^{12}u_y^1=0,\\
&u_t^2+u^1u_x^2+u^2u_y^2+A_1^{21}(u_x^1-u_y^2)+A_1^{22}u_y^1=0,
\end{aligned} \qquad (4.17)$$

for any functions A_k^{ij} of two variables u^1 and u^2.

If the function h of u^1 does not vanish anywhere ($h\neq 0$) then the rank-2 solution is defined implicitly by equations (4.15) and satisfies the following system of PDEs

$$\begin{aligned}
&u_t^1+u^1u_x^1+u^2u_y^1+A_2^{12}[u_y^2-u_x^1+l(u^1)u_x^2+m(u^1)u_y^1]=0,\\
&u_t^2+u^1u_x^2+u^2u_y^2+A_2^{22}[u_y^2-u_x^1+l(u^1)u_x^2+m(u^1)u_y^1]=0,
\end{aligned} \qquad (4.18)$$

where A_2^{12} and A_2^{22} are any functions of two variables u^1 and u^2. Given the functions l and m of u^1, we can prescribe the functions a and b in expression (4.15) to find

$$h=\int l\, b_{,u^1}du^1, \quad g=\int[b-hm-a_{,u^1}]du^1. \qquad (4.19)$$

For instance, consider a rank-2 solution of equations (4.12) and (4.13) invariant under the vector fields

$$X_1 = \partial_t + u^1\partial_x, \quad X_2 = \partial_t - u^2\partial_y \tag{4.20}$$

with the wave vectors λ^A which are the nonzero multiples of $\lambda^1 = (u^1, -1, 0)$ and $\lambda^2 = (u^2, 0, -1)$. Then the solution is defined by the implicit relations

$$\begin{aligned} &x - u^1 t = g(u^1), \\ &y + u^2 t = h(u^1) + u^2 g_{,u^1}. \end{aligned} \tag{4.21}$$

and satisfies the following system of equations

$$\begin{aligned} &u^1_t + u^1 u^1_x + u^2 u^1_y + b(u^1, u^2) u^1_y = 0, \\ &u^2_t + u^1 u^2_x + u^2 u^2_y + c(u^1, u^2) u^1_y = 0, \end{aligned} \tag{4.22}$$

where b and c are arbitrary functions of u^1 and u^2.

Thus, putting it all together, we see that the constructed solutions correspond to superpositions of two rank-1 solutions (i.e. simple waves) with local velocities u^1 and u^2, respectively. According to [10], if we choose the initial data ($t = 0$) for the functions u^1 and u^2 sufficiently small and such that their first derivatives with respect to x and y will have compact and disjoint supports, then asymptotically there exists a finite time $t = T > 0$ for which rank-2 solution decays in the exact way in two rank-1 solutions, being of the same type as in the initial moment.

Example 4. Consider the overdetermined hyperbolic system in $(2 + 1)$ dimensions $(p = 3)$

$$\begin{aligned} &\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} + ka \operatorname{grad} a = 0 \\ &\frac{\partial a}{\partial t} + (\vec{u} \cdot \nabla) a + k^{-1} a \operatorname{div} \vec{u} = 0, \\ &\frac{\partial a}{\partial x} = 0, \quad \frac{\partial a}{\partial y} = 0, \end{aligned} \tag{4.23}$$

describing the nonstationary isentropic flow of a compressible ideal fluid. Here we use the following notations : $\vec{u} = (u^1, u^2)$ is the flow velocity, $a(t) = \left(\frac{\gamma p}{\rho}\right)^{1/2} \neq 0$ is the sound velocity which depends on t only, $k = 2(\gamma - 1)^{-1}$ and γ is the polytropic exponent.

The system (4.23) can be written in an equivalent form as

$$\begin{aligned} &\operatorname{Tr}\left[\begin{pmatrix} 1 & u^1 & u^2 \\ 0 & 0 & 0 \\ 0 & ka & 0 \end{pmatrix}\begin{pmatrix} u^1_t & u^2_t & a_t \\ u^1_x & u^2_x & 0 \\ u^1_y & u^2_y & 0 \end{pmatrix}\right] = 0, \\ &\operatorname{Tr}\left[\begin{pmatrix} 0 & 0 & 0 \\ 1 & u^1 & u^2 \\ 0 & 0 & ka \end{pmatrix}\begin{pmatrix} u^1_t & u^2_t & a_t \\ u^1_x & u^2_x & 0 \\ u^1_y & u^2_y & 0 \end{pmatrix}\right] = 0, \\ &\operatorname{Tr}\left[\begin{pmatrix} 0 & k^{-1}a & 0 \\ 0 & 0 & k^{-1}a \\ 1 & u^1 & u^2 \end{pmatrix}\begin{pmatrix} u^1_t & u^2_t & a_t \\ u^1_x & u^2_x & 0 \\ u^1_y & u^2_y & 0 \end{pmatrix}\right] = 0. \end{aligned} \tag{4.24}$$

We are interested here in the rank-2 solutions of (4.24). So, we require that conditions (3.43) and (3.47) be satisfied. This demand constitutes the necessary and sufficient condition for the existence of a surface $\mathcal{F}$ written in a parametric form (4.9) for which equation (4.10) holds. In our case, $p = q = 3$ and $k = 2$, conditions (3.43) and (3.47) become

$$\mathrm{Tr}\left(\Delta^\mu \frac{\partial f}{\partial r}\lambda\right) = 0, \quad \mu = 1, 2, 3, \tag{4.25}$$

and

$$\mathrm{Tr}\left(\Delta^\mu \frac{\partial f}{\partial r}\left(\eta_1 \frac{\partial f}{\partial r}\eta_2 + \eta_2 \frac{\partial f}{\partial r}\eta_1\right)\frac{\partial f}{\partial r}\lambda\right) = 0, \tag{4.26}$$

respectively. Here, we assume the following basis for the wave vectors

$$\lambda_i^A = \begin{pmatrix} \lambda_0^1 & \lambda_1^1 & \lambda_2^1 \\ \lambda_0^2 & \lambda_1^2 & \lambda_2^2 \end{pmatrix}^T = \begin{pmatrix} -1 & v^1 & w^1 \\ -1 & v^2 & w^2 \end{pmatrix}^T, \tag{4.27}$$

where v^A and w^A are some functions of u^1 and u^2 to be determined. The 2 by 3 matrices η_a and the 3 by 2 matrix $\partial f/\partial r$ take the form

$$\begin{aligned} \eta_a &= \frac{\partial \lambda_{i_a}^A}{\partial u^\alpha} = \begin{pmatrix} \partial\lambda_{i_a}^1/\partial u^1 & \partial\lambda_{i_a}^1/\partial u^2 & \partial\lambda_{i_a}^1/\partial a \\ \partial\lambda_{i_a}^2/\partial u^1 & \partial\lambda_{i_a}^2/\partial u^2 & \partial\lambda_{i_a}^2/\partial a \end{pmatrix}, \quad a = 1, 2 \\ \frac{\partial f}{\partial r} &= \begin{pmatrix} \partial u^1/\partial r^1 & \partial u^1/\partial r^2 \\ \partial u^2/\partial r^1 & \partial u^2/\partial r^2 \\ \partial a/\partial r^1 & \partial a/\partial r^2 \end{pmatrix}. \end{aligned} \tag{4.28}$$

Equations (4.25) lead to the following differential conditions

$$\begin{aligned} &\frac{\partial u^1}{\partial r^1} + \frac{\partial u^1}{\partial r^2} + (u^1 - kav^2)\frac{\partial u^2}{\partial r^1} + (u^1 - kaw^2)\frac{\partial u^2}{\partial r^2} + u^2\left(\frac{\partial a}{\partial r^1} + \frac{\partial a}{\partial r^2}\right) = 0, \\ &v^1\frac{\partial u^1}{\partial r^1} + w^1\frac{\partial u^1}{\partial r^2} + u^1\left(v^1\frac{\partial u^2}{\partial r^1} + w^1\frac{\partial u^2}{\partial r^2}\right) + (u^2v^1 + kav^2)\frac{\partial a}{\partial r^1} \\ &\qquad + (u^2w^1 + kaw^2)\frac{\partial a}{\partial r^2} = 0, \\ &k\left(v^2\frac{\partial u^1}{\partial r^1} + w^2\frac{\partial u^1}{\partial r^2}\right) - (a - kv^2u^1)\frac{\partial u^2}{\partial r^1} \\ &\qquad - (a - w^2ku^1)\frac{\partial u^2}{\partial r^2} + (av^1 + kv^2u^2)\frac{\partial a}{\partial r^1} + (aw^1 + kw^2u^2)\frac{\partial a}{\partial r^2} = 0. \end{aligned} \tag{4.29}$$

Assuming that we have found v^A and w^A as functions of u^1 and u^2, we have to solve (4.26) for the unknown functions u^1 and u^2 in terms of r^1 and r^2. The resulting expressions in the equations (4.26) are rather complicated, hence we omit them here. Various rank-2 solutions are determined by a specification of functions v^A and w^A in terms of u^1 and u^2. By way of illustration we show how to obtain a solution which depends on one arbitrary function of two variables.

Let us suppose that we are interested in the rank-2 solutions invariant under the vector fields

$$X_1 = \partial_t + u^1\partial_x, \quad X_2 = \partial_t + u^2\partial_y. \tag{4.30}$$

So, the functions $r^1 = x - u^1 t$ and $r^2 = y - u^2 t$ are the Riemann invariants of these vector fields. Under this assumption, equations (4.25) and (4.26) can be easily solved to obtain the Jacobian matrix

$$J = \frac{\partial(u^1, u^2)}{\partial(r^1, r^2)} \neq 0 \tag{4.31}$$

which has the characteristic polynomial with constant coefficients. This means that the trace and determinant of J are constant,

$$\begin{aligned} &(i) \quad u^1_{r^1} + u^2_{r^2} = 2C_1, \\ &(ii) \quad u^1_{r^1} u^2_{r^2} - u^1_{r^2} u^2_{r^1} = C_2. \end{aligned} \qquad C_1, C_2 \in \mathbb{R} \tag{4.32}$$

The trace condition (4.32(i)) implies that there exists a function h of r^1 and r^2 such that the conditions

$$u^1 = C_1 r^1 + h_{r^2}, \quad u^2 = C_1 r^2 - h_{r^1}, \tag{4.33}$$

hold. The determinant condition (4.32(ii)) requires that the function $h(r^1, r^2)$ satisfies the Monge-Ampère equation

$$h_{r^1 r^1} h_{r^2 r^2} - h^2_{r^1 r^2} = C, \quad C \in \mathbb{R}. \tag{4.34}$$

Hence, the general integral of the system (4.23) has the implicit form defined by the relations between the variables t, x, y, u^1 and u^2

$$\begin{aligned} u^1 &= C_1(x - u^1 t) + \frac{\partial h}{\partial r^2}(x - u^1 t, y - u^2 t), \\ u^2 &= C_1(y - u^2 t) + \frac{\partial h}{\partial r^1}(x - u^1 t, y - u^2 t), \\ a &= a_0 \left((1 + C_1 t)^2 + C t^2\right)^{-1/k}, \quad a_0 \in \mathbb{R} \end{aligned} \tag{4.35}$$

where the function h obeys (4.34).

Note that the Gaussian curvature K expressed in curvilinear coordinates $(t, r^1, r^2) \in \mathbb{R}^3$ of the surface $\mathcal{S} = \{t = h(r^1, r^2)\}$ is not constant and is given by

$$K(r^1, r^2) = \frac{C}{1 + h^2_{r^1} + h^2_{r^2}}. \tag{4.36}$$

For example, a particular nontrivial class of solution of (4.23) can be obtained if we assume that $C = 0$. In this case the general solution of (4.23) depends on three parameters, $a_0, C_1, m \in \mathbb{R}$ and takes the form

$$\begin{aligned} u^1 &= C_1(x - u^1 t) + (1 - m)\left(\frac{x - u^1 t}{y - u^2 t}\right)^m, \\ u^2 &= C_1(y - u^2 t) - m\left(\frac{y - u^2 t}{x - u^1 t}\right)^{1-m}, \\ a(t) &= \frac{a_0}{(1 + C_1 t)^{2/k}}. \end{aligned} \tag{4.37}$$

Note that if $C = 0$ and $C_1 = 0$ then the Jacobian matrix J is nilpotent and the divergence of the vector $\vec{u}$ is equal to zero. Then the expression

$$\begin{aligned} u^1 &= (1-m)\left(\frac{x-u^1t}{y-u^2t}\right)^m, \quad a = a_0, \\ u^2 &= -m\left(\frac{y-u^2t}{x-u^1t}\right)^{1-m}, \end{aligned} \tag{4.38}$$

defines a solution $\vec{u} = (u^1, u^2)$ to incompressible Euler equations

$$\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} = 0, \quad \operatorname{div}\vec{u} = 0, \quad a = a_0. \tag{4.39}$$

Example 5. Now let us consider a more general case of to the autonomous system (4.23) in $p = n+1$ independent $(t, x^i) \in X$ and $q = n+1$ dependent $(a, u^i) \in U$ variables. We look for the rank-k solutions, when $k = n$. The change of variables in the system (4.23) under the point transformation

$$\bar{t} = t, \quad \bar{x}^1 = x^1 - u^1 t, \ldots, \bar{x}^n = x^n - u^n t, \quad \bar{a} = a, \quad \bar{u} = u \tag{4.40}$$

leads to the following system

$$\begin{aligned} &\frac{D\bar{u}}{D\bar{t}} = 0, \\ &\frac{D\bar{a}}{D\bar{x}} = 0, \quad \frac{D\bar{a}}{D\bar{t}} + k^{-1}\bar{a}\operatorname{Tr}\left(B^{-1}\frac{D\bar{u}}{D\bar{x}}\right) = 0, \quad \bar{a} \neq 0 \end{aligned} \tag{4.41}$$

where the total derivatives are denoted by

$$\frac{D}{D\bar{t}} = \frac{\partial}{\partial t} + \bar{u}^i_{\bar{t}}\frac{\partial}{\partial \bar{u}^i}, \quad \frac{D}{D\bar{x}^j} = \frac{\partial}{\partial \bar{x}^j} + \bar{u}^i_{\bar{x}^j}\frac{\partial}{\partial \bar{u}^i}, \quad j = 1, \ldots, n \tag{4.42}$$

and the n by n nonsingular matrix B has the form

$$B = \mathbb{I} + t\frac{\partial \bar{u}}{\partial \bar{x}}. \tag{4.43}$$

The general solution of the first equation in (4.41) is

$$\bar{u} = f(\bar{x}), \quad \bar{x} = (\bar{x}^1, \ldots, \bar{x}^n) \tag{4.44}$$

for some function $f : \mathbb{R}^n \to \mathbb{R}^n$. The second equation in (4.41) can be written in an equivalent form

$$\frac{\partial}{\partial \bar{t}}\left(\ln|\bar{a}(t)|^k\right) + \operatorname{Tr}\left[(\mathbb{I} + \bar{t}Df(\bar{x}))^{-1}Df(\bar{x})\right] = 0, \tag{4.45}$$

where the Jacobian matrix is denoted by

$$Df(\bar{x}) = \frac{\partial f}{\partial \bar{x}}(\bar{x}). \tag{4.46}$$

Differentiation of equation (4.45) with respect to $\bar{x}$ yields

$$\frac{\partial^2}{\partial\bar{x}\partial\bar{t}}\left(\ln\det\left(\mathbb{I}+\bar{t}Df(\bar{x})\right)\right)=0 \tag{4.47}$$

with general solution

$$\det\left(\mathbb{I}+\bar{t}Df(\bar{x})\right)=\alpha(\bar{x})\beta(\bar{t}) \tag{4.48}$$

for some functions $\alpha:\mathbb{R}^n\to\mathbb{R}$ and $\beta:\mathbb{R}\to\mathbb{R}$. Evaluating (4.48) at the initial data $t=0$ implies $\alpha(\bar{x})=\beta(0)^{-1}$. Therefore

$$\det\left(\mathbb{I}+\bar{t}Df(\bar{x})\right)=\frac{\beta(\bar{t})}{\beta(0)}. \tag{4.49}$$

So, we have

$$\frac{\partial}{\partial x}\det\left(\mathbb{I}+\bar{t}Df(\bar{x})\right)=0. \tag{4.50}$$

Now, let us write the determinant in the form of the characteristic polynomial

$$\det\left(\mathbb{I}+\bar{t}Df(\bar{x})\right)=\bar{t}^nP_n(\varepsilon,\bar{x}),\quad \varepsilon=\frac{1}{\bar{t}} \tag{4.51}$$

where

$$\det\left(\varepsilon\mathbb{I}+Df(\bar{x})\right)=\varepsilon^n+p_{n-1}(\bar{x})\varepsilon^{n-1}+\ldots+p_1(\bar{x})\varepsilon+p_0(\bar{x}). \tag{4.52}$$

Equation (4.50) holds if and only if the coefficients of the characteristic polynomial $p_0,\ldots,p_{n-1}$ are constants. So, equation (4.45) implies that

$$\frac{\partial}{\partial\bar{t}}\ln|\bar{a}(\bar{t})|^k+\frac{\partial}{\partial\bar{t}}\ln|\det\left(\mathbb{I}+\bar{t}Df(\bar{x})\right)|=0.$$

Then we have,

$$\frac{\partial}{\partial\bar{t}}\Big(|\bar{a}(\bar{t})|^k\det\left(\mathbb{I}+\bar{t}Df(\bar{x})\right)\Big)=0. \tag{4.53}$$

Solving equation (4.53) we obtain

$$\bar{a}(\bar{t})=\gamma\left(\det\left(\mathbb{I}+\bar{t}Df(\bar{x})\right)\right)^{-1/k},\quad 0\neq\gamma\in\mathbb{R}. \tag{4.54}$$

Thus, the general solution of system (4.23) is

$$u=f(\bar{x}),\quad a(\bar{t})=\gamma\left[1+p_{n-1}\bar{t}+\ldots+p_0\bar{t}^n\right]^{-1/k}, \tag{4.55}$$

for any differentiable function $f:\mathbb{R}^n\to\mathbb{R}^n$ and takes the form of a constant characteristic polynomial on the Cauchy data $t=0$

$$P_n(\varepsilon)=\varepsilon^n+p_{n-1}\varepsilon^{n-1}+\ldots+p_1\varepsilon+p_0. \tag{4.56}$$

Note that the function a is constant if and only if $P_n(\varepsilon)=\varepsilon^n$. This fact holds if and only if the Jacobian matrix $Df(\bar{x})$ is nilpotent.

Note that in the particular case when $p=2$, the general explicit solution of (4.23) is given by

$$u(x,t)=(\beta+\alpha x)(1+\alpha t)^{-1}, \quad a(t)=\gamma(1+\alpha t)^{-1/k}, \quad \alpha,\beta,\gamma\in\mathbb{R}. \tag{4.57}$$

An extension of this solution to the (n+1)-dimensional space X is as follows

$$u(x,t)=(\mathbb{I}+t\alpha)^{-1}(\beta+\alpha x), \quad a(t)=\gamma(\det(\mathbb{I}+\alpha t))^{-1/k}, \tag{4.58}$$

where β is any constant n-component vector and α is any n by n constant matrix. In this case the Jacobian matrix is constant

$$Df(\bar{x})=\alpha \tag{4.59}$$

for any $\bar{x}\in\mathbb{R}^n$.

Finally, a similar computation can be performed for the case in which the vector function $\vec{u}=(u^1,u^2,u^3)$ satisfies the overdetermined system (4.39). In the above notation, an invariant solution under the vector fields

$$X_a=\partial_t+u^a\partial_{(a)}, \quad a=1,2,3 \tag{4.60}$$

is given by $\bar{u}=f(\bar{x})$ and is a divergence free solution

$$\operatorname{div}\vec{u}=0 \tag{4.61}$$

if and only if the trace condition

$$\operatorname{Tr}\left(B^{-1}\frac{\partial\bar{u}}{\partial\bar{x}}\right)=0, \quad B=\mathbb{I}+t\frac{D\bar{u}}{D\bar{x}}(\bar{x}) \tag{4.62}$$

holds. But

$$\frac{D\vec{u}}{D\bar{x}}=\frac{\partial B}{\partial t}. \tag{4.63}$$

Therefore $\operatorname{div}\vec{u}=0$ if and only if

$$\operatorname{Tr}\left(B^{-1}\frac{\partial B}{\partial t}\right)=0, \tag{4.64}$$

or equivalently, if and only if

$$\frac{\partial}{\partial t}(\det B)=0. \tag{4.65}$$

This means that the Jacobian matrix $Df(\bar{x})$ has to be a nilpotent one and takes the form

$$Df(\bar{x})=\begin{pmatrix} 0 & f^1_{\bar{x}^2} & f^1_{\bar{x}^3} \\ 0 & f^2_s & -f^2_s \\ 0 & f^2_s & -f^2_s \end{pmatrix}, \tag{4.66}$$

where f^1 is an arbitrary function of two variables $\bar{x}^2$ and $\bar{x}^3$ and f^2 is an arbitrary function of one variable $s = \bar{x}^2 - \bar{x}^3$. Note that if $f^1_{\bar{x}^2} \neq f^1_{\bar{x}^3}$ then the Jacobian matrix $Df(\bar{x})$ has rank 2 (otherwise f^1 is any function of s and $Df(\bar{x})$ has rank 1). As a consequence, the matrix B has the form

$$B = \begin{pmatrix} 1 & tf^1_{\bar{x}^2} & tf^1_{\bar{x}^3} \\ 0 & 1 + tf^2_s & -tf^2_s \\ 0 & tf^2_s & 1 - tf^2_s \end{pmatrix}, \quad \det B = 1. \tag{4.67}$$

So, the condition (4.65) is identically satisfied. Thus, the general solution of system (4.39) is implicitly defined by the equations

$$u^1 = f^1\big(x^2 - tf^2(x^2 - x^3), x^3 - tf^2(x^2 - x^3)\big), \quad u^2 = u^3 = f^2(x^2 - x^3), \quad a = a_0, \tag{4.68}$$

where the functions f^1 and f^2 are arbitrary functions of their arguments. Equations (4.68) define a rank-2 solution but, according to the formula for the corresponding principle [10], it is not a double Riemann wave.

Obviously, other choices of the wave vectors λ^A (and the related vector fields X_a) lead to different classes of solutions. The problem of the classification of these solutions remains still open but some results are known (see e.g. the functorial properties of systems of equations determining Riemann invariants [9]).

5 Conclusions

In this paper we have developed a new method which serves as a tool for constructing rank-k solutions of multi-dimensional hyperbolic systems including Riemann waves and their superpositions. The most significant characteristic of this approach is that it allows us to construct regular algorithms for finding solutions written in terms of Riemann invariants. Moreover, this approach does not refer to any additional considerations, but proceeds directly from the given system of PDEs.

Riemann waves and their superpositions described by multi-dimensional hyperbolic systems have been studied so far only in the context of the generalized method of characteristics (GMC) [1, 10, 18]. The essence of this method can be summarized as follows. It requires the supplementation of the original system of PDEs (1.1) with additional differential constraints for which all first derivatives are decomposable in the following form

$$\frac{\partial u^\alpha}{\partial x^i}(x) = \sum_{A=1}^{k} \xi^A(x)\gamma^\alpha_A(u)\lambda^A_i(u), \tag{5.1}$$

where

$$\begin{aligned} &\left(\Delta^{\mu i}_{\alpha}(u)\lambda^A_i\right)\gamma^\alpha_{(A)} = 0, \quad A = 1, \dots, k \\ &\operatorname{rank}\left(\Delta^{\mu i}_{\alpha}(u)\lambda^A_i\right) < l. \end{aligned} \tag{5.2}$$

Here, $\xi^A \neq 0$ are treated as arbitrary scalar functions of x and we assume that the vector fields $\{\gamma_1, \dots, \gamma_k\}$ are locally linearly independent. The necessary and sufficient conditions for the existence of k-wave solutoins (when $k > 1$) of the system (5.1) in terms of Riemann invariants impose some additional differential conditions on the wave vectors λ^A and the corresponding vector fields γ_A, namely [18]

$$\begin{aligned} &[\gamma_A, \gamma_B] \in \operatorname{span}\{\gamma_A, \gamma_B\}, \\ &\mathcal{L}_{\gamma_\beta} \lambda^A \in \operatorname{span}\{\lambda^A, \lambda^B\}, \quad \forall A \neq B = 1, \ldots, k, \end{aligned} \tag{5.3}$$

where $[\gamma_A, \gamma_B]$ denotes the commutator of the vector fields γ_A and γ_B, $\mathcal{L}_{\gamma_B}$ denotes the Lie derivatives along the vector fields γ_B.

Due to the homogeneity of the wave relation (5.2) we can choose, without loss of generality, a holonomic system for the fields $\{\gamma_1, \ldots, \gamma_k\}$ by requiring a proper length for each vector γ_A such that

$$[\gamma_A, \gamma_B] = 0, \quad \forall A \neq B = 1, \ldots, k. \tag{5.4}$$

It determines a k-dimensional submanifold $\mathcal{S} \subset U$ obtained by solving the system of PDEs

$$\frac{\partial f^\alpha}{\partial r^A} = \gamma^\alpha_A(f^1, \ldots, f^k) \tag{5.5}$$

with solution $\pi : F \to U$ defined by

$$\pi : (r^1, \ldots, r^k) \to \Big(f^1(r^1, \ldots, r^k), \ldots, f^q(r^1, \ldots, r^k)\Big). \tag{5.6}$$

The wave vectors λ^A are pulled back to the submanifold $\mathcal{S}$ and then λ^A become functions of the parameters $r^1, \ldots, r^k$. Consequently, the requirements (5.1) and (5.3) take the form

$$\frac{\partial r^A}{\partial x^i}(x) = \xi^A(x)\lambda^A_i(r^1, \ldots, r^k), \tag{5.7}$$

$$\frac{\partial \lambda^A}{\partial r^B} \in \operatorname{span}\{\lambda^A, \lambda^B\}, \quad \forall A \neq B = 1, \ldots, k \tag{5.8}$$

respectively. It has been shown [18] that the conditions (5.5) and (5.8) ensure that the set of solutions of system (1.1) subjected to (5.1), depends on k arbitrary functions of one variable. It has also been proved [17] that all solutions, i.e. the general integral of the system (5.7) under conditions (5.8) can be obtained by solving, with respect to the variables $r^1, \ldots, r^k$, the system in implicit form

$$\lambda^A_i(r^1, \ldots, r^k)x^i = \psi^A(r^1, \ldots, r^k), \tag{5.9}$$

where ψ^A are arbitrary functionnally independent differentiable functions of k variables $r^1, \ldots, r^k$. Note that solutions of (5.7) are constant on $(p-k)$-dimensional hyperplanes perpendicular to wave vectors λ^A satisfying conditions (5.8).

As one can observe, both methods discussed here exploit the invariance properties of the initial system of equations (1.1). In the GMC, they have the purely geometric character for which a form of solution is postulated by subjecting the original system (1.1) to the side conditions (5.1). In contrast, in the case of the approach proposed here we augment the system (1.1) by differential constraints (3.9).

There are, however, at least two basic differences between the GMC and our proposed approach. Riemann multiple waves defined from (5.1), (5.5) and (5.8) constitute a more limited class of solutions than the rank-k solutions postulated by our approach. This difference results from the fact that the scalar functions ξ^A appearing in expression (5.1) (which describe the profile of simple waves entering into a superposition) are substituted in our case (see expressions (3.3) or (3.4)) with a q by q or k by k matrix Φ^1 or Φ^2, respectively. This situation consequently allows for a much broader range of initial data. The second difference consists in fact that the restrictions (5.5) and (5.8) on the vector fields γ_A and λ^A, ensuring the solvability of the problem by GMC, are not necessary in our approach. This makes it possible for us to consider more general configurations of Riemann waves entering into an interaction.

A number of different attempts to generalize the Riemann invariants method and its various applications can be found in the recent literature on the subject (see e.g. [4, 6, 7, 8, 22]). For instance, the nonlinear k-waves superposition $u = f(r^1, \ldots, r^k)$ described in [16] can be regarded as dispersionless analogues to "n-gap solution" of (1.1) which require the resolution of a set of commuting diagonal systems for the Riemann invariants r^A, i.e.

$$r^A_{x^i} = \mu^A_{i(j)}(r) r^A_{x^j}, \quad A = 1, \ldots, k, \quad i \neq j = 1, \ldots, p. \tag{5.10}$$

That specific technique involves differential constraints on the functions μ^A_{ij} of the form [22]

$$\frac{\partial_j \mu^A_{i(j)}}{\mu^A_{i(j)} - \mu^A_{j(i)}} = \frac{\partial_j u^B_{i(j)}}{\mu^B_{i(j)} - \mu^B_{j(i)}}, \quad i \neq j, \quad A \neq B = 1, \ldots, k, \tag{5.11}$$

no summation. As in the case of Riemann k-waves if the system (5.11) is satisfied for the functions μ^A_{ij} then the general integral of the system (5.10) can be obtained by solving system (5.9) with respect to the variables $r^1, \ldots, r^k$.

In contrast, our proposed approach does not require the use of differential equations (5.10) and therefore does not impose constraints on the functions μ^A_{ij} when the 1-forms λ^A are linearly independent and $k < p$.

However, if one removes these assumptions and λ^A can be linearly dependent and $k \geq p$ then the approach presented in [7] is a valuable one and provides an effective tool for classification criterion of integrable systems.

In order to verify the efficiency of our approach we have used it for constructing rank-2 solutions of several examples of hydrodynamic type systems. The proposed approach proved to be a useful tool in the case of multi-dimensional hydrodynamic type systems (1.1), since it leads to new interesting solutions.

The examples illustrating our method clearly demonstrate its usefulness as it has produced several new and interesting results. Let us note that the outlined approach to rank-k solutions lends itself to numerous potential applications which arise in physics, chemistry and biology. It has to be stressed that for many physical systems, (e.g. nonlinear field equations, Einstein's equations of general relativity and the equations of continuous media, etc) there have been very few known examples of rank-k solutions. The approach proposed here offers a new and promising way to investigate and construct such type of solutions.

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