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Nonlocal Symmetries and the Complete Symmetry Group of $1 + 1$ Evolution Equations

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Abstract

The complete symmetry group of a $1 + 1$ linear evolution equation has been demonstrated to be represented by the six-dimensional Lie algebra of point symmetries $sl(2, R)\oplus_s W$, where W is the three-dimensional Heisenberg-Weyl algebra. The infinite number of solution symmetries does not play a role in the complete specification of the equation. In the absence of a sufficient number of point symmetries which are not solution symmetries one must look to generalized or nonlocal symmetries to remove the deficit. This is true whether the evolution equation be linear or not. We report two *Ansätze* which provide a route to the determination of the required nonlocal symmetry necessary to supplement the point symmetries for the complete specification of two nonlinear $1+1$ evolution equations which arise in the area of Financial Mathematics. The first of these, when reduced to its essential form, is the well-known Burgers' equation.

1 Introduction

The concept of a complete symmetry group as the group of the Lie symmetries required to specify completely a differential equation (equally a system of differential equations) was introduced some ten years ago by Krause [7, 8] in a study of the classical Kepler Problem. In general [2] a system of n second-order ordinary differential equations requires $2n + 1$ symmetries to specify it completely. The Newtonian equations for the Kepler Problem (section 3.4) possess just the five Lie point symmetries of the algebra $A_2 \oplus A_{3,9}$ representing invariance under time translation and rescaling on the one hand and the rotational invariance of $SO(3)$ on the other[1]. Krause had resort to the use of nonlocal symmetries to remedy the deficit[2] and devised an ingenious scheme for their determination.

[1]We use the classification scheme of Mubarakzyanov [16, 17, 18, 19].

[2]We must note that the use of nonlocal symmetries in the first application of this concept of a complete symmetry group should not be taken to imply that nonlocal symmetries are a necessary concommitant. That nonlocal symmetries have played an important role in the determination of the complete symmetry group in a number of instances [9, 21, 10, 11] should not obscure the reality that point symmetries have played an important role in the theoretical development as well as certain applications [1, 2, 3] of complete symmetry groups.

Unfortunately nonlocal symmetries of differential equations in general have a property in common with symmetries of first-order differential equations. Although they are more numerous than the grains of sand by the sea, there is no finite algorithm for their general determination[3].

Until recently the determination of complete symmetry groups has been confined to systems of ordinary differential equations. In a study of the complete symmetry group of the $1+1$ heat equation and some related equations which arise in Financial Mathematics we [20] showed that the number of Lie point symmetries required to specify the $1+1$ heat equation is six. The classical heat equation, as a linear partial differential equation, possesses an infinite number of Lie point symmetries. Specifically we write them as $5+1+\infty$ symmetries to indicate that there are three classes of symmetry in terms of provenance. The class containing the infinite number of Lie point symmetries comprises solutions of the equation. This is a feature of linear equations, be they ordinary or partial. Given that the order of an ordinary differential equation is usually not high this feature is perhaps of no great interest for them. However, in the case of partial differential equations the existence of an infinite number of solution symmetries is important, particularly if the partial differential equation under study happens to be nonlinear. The possession implies that there is a route to linearisation. The one-dimensional abelian subalgebra is a consequence of the homogeneity of the equation. The five remaining symmetries are critical for the successful group theoretical analysis of the equation. These nongeneric symmetries are determined by the particular structure of the equation and are the maximal number which this heat equation can possess. In the case of the heat equation in its standard form, *videlicet*

$$u_t = u_{xx} \tag{1.1}$$

in a usual notation, the nongeneric Lie point symmetries are

$$\begin{aligned}
\Gamma_1 &= \partial_x \\
\Gamma_2 &= t\partial_x - \tfrac{1}{2}xu\,\partial_u \\
\Gamma_3 &= \partial_t \\
\Gamma_4 &= t\partial_t + \tfrac{1}{2}x\,\partial_x - \tfrac{1}{4}u\,\partial_u \\
\Gamma_5 &= t^2\partial_t + tx\,\partial_x - \tfrac{1}{4}(x^2+2t)u\,\partial_u
\end{aligned} \tag{1.2}$$

and comprise two groups. The symmetries, Γ_3, Γ_4 and Γ_5, constitute the Lie algebra $sl(2,R)$ which is characteristic of ordinary differential equations of maximal symmetry and of Ermakov-Pinney systems. The two remaining symmetries, Γ_1 and Γ_2, correspond to the solution symmetries of the one-dimensional free particle. In general a scalar second-order ordinary differential equation derivable from a variational principle possesses at most five Noether point symmetries [15]. They are the counterparts of the five nongeneric Lie point symmetries of the heat equation. The connection is more easily seen through the corresponding Schrödinger equation [13] to which the heat equation is related by a simple point transformation.

The Lie algebra which characterises (1.1) comprises the five symmetries listed in (1.2) plus the homogeneity symmetry, $\Gamma_6 = u\partial_u$. The six Lie point symmetries split into two

[3]See also Sjoberg and Mahomed and the references cited therein [23] for a discussion of the theory and application of nonlocal symmetries.

three-dimensional subalgebras. One is the algebra $sl(2,R)$ mentioned above. The other is the three-dimensional Heisenberg-Weyl algebra of the three symmetries Γ_1, Γ_2 and Γ_6. The six-dimensional algebra has the structure $sl(2,R) \oplus_s W$. In the more systematic notation of the Mubarakzyanov classification scheme this is written as $A_{3,8} \oplus_s A_{3,3}$. It so happens that these six symmetries are also a representation of the complete symmetry group of (1.1). We should emphasize that the number of Lie point symmetries of a given differential equation and the number of symmetries required to specify it completely have no particular relation. In the case of the Kepler Problem not only is the number of Lie point symmetries insufficient to specify the system completely but certain of the point symmetries, those of the representation of the rotation group, play no role in the specification of the system. By way of contrast a scalar linear second-order ordinary differential equation has the eight Lie point symmetries of the algebra $sl(3,R)$, but requires only three symmetries to specify it completely. There are at least four combinations of the eight symmetries which perform the purpose [2].

In this paper we address the problem of identifying the symmetries which completely specify a given $1+1$ evolution equation when the number of Lie point symmetries is insufficient to the purpose. In particular we consider two nonlinear 1+1 evolution equations which arise in the Mathematics of Finance. The first of these equations possesses only three Lie point symmetries and is evidently not linearizable by means of a point transformation. For a specific relationship between the parameters in the equation the number of Lie point symmetries increases to five. In this case the equation, which then becomes the well-known Burgers' equation, can be converted by means of the equally well-known Cole-Hopf transformation $w = 2W_x/W$ to the standard heat equation and consequently solved [12]. The increase in the number of symmetries and the convertibility to the linear heat equation are still insufficient to specify the equation completely [20]. Here we consider the complete specification of the equation when there are five Lie point symmetries. The second equation possesses just four Lie point symmetries and no amount of playing with special values of the parameters increases that number. For both equations it is necessary to find additional symmetry to complete the specification and the additional symmetry must necessarily be nonlocal. The problem is the determination of the nonlocal symmetry. This is a nontrivial task for any equation, be it ordinary or partial, with even a modest pretence to complexity of structure[4]. We manage partially to obviate this difficulty by imposing an extra condition on the structure of the equation we seek to specify. This enables us to make progress using the Lie point symmetries at our disposal. Finally we must return to the *Ansatz* of the extra condition to determine the nonlocal symmetry behind the imposition of the specific structure.

We structure the paper as follows. In Section 2 we examine the equation with five Lie point symmetries. In Section 3 we extend the *Ansatz* which enabled us to determine the additional symmetry required for the complete specification of that equation to deal with the situation in which we have only four Lie point symmetries. In both instances our stratagem leads to the additional nonlocal symmetry required. We conclude the paper with some comments in Section 4.

[4]The calculation of nonlocal symmetries in the case of partial differential equations is even less obvious than that for ordinary differential equations for which more than a certain amount of ingenuity is often required [4]. By way of contrast the calculation of the Lie point symmetries for the equations which we consider in this paper is easily performed by one of the classic codes developed for the purpose [6, 22].

2 The First Nonlinear Equation

The equation

$$u_t + u_{xx} + (u + x)u_x - (Eu + Dx) = 0, \tag{2.1}$$

which arises in Financial Mathematics [12], possesses three Lie point symmetries for general values of the parameters, E and D. However, in the special case that $E = -1$ the equation

$$u_t + u_{xx} + (u + x)u_x + u - Dx = 0 \tag{2.2}$$

has the five Lie point symmetries

$$\begin{aligned} \Lambda_1 &= \partial_t \\ \Lambda_{2\pm} &= \exp[\pm Bt]\left\{\partial_x \pm (B \mp 1)\partial_u\right\} \\ \Lambda_{3\pm} &= \exp[\pm 2Bt]\left\{\partial_t \pm Bx\partial_x + \left(2B^2x \mp 2Bx \mp Bu\right)\partial_u\right\}, \end{aligned} \tag{2.3}$$

where $B^2 = D + 1$.

We can make the analysis of the equation in the form (2.2). However, for the purposes of this discussion we look to a simpler form. Under the transformation $w = u + x$ (2.2) becomes

$$w_t + w_{xx} + ww_x = B^2x. \tag{2.4}$$

The parameter B may be set at unity by rescaling. As the number of symmetries is unaffected, we take $B = 0$. The Lie point symmetries of

$$w_t + w_{xx} + ww_x = 0, \tag{2.5}$$

which is now the well-known Burgers' equation, are

$$\begin{aligned} \Delta_1 &= \partial_x \\ \Delta_2 &= t\partial_x + \partial_w \\ \Delta_3 &= \partial_t \\ \Delta_4 &= t\partial_t + \tfrac{1}{2}x\partial_x - \tfrac{1}{2}w\partial_w \\ \Delta_5 &= t^2\partial_t + tx\partial_x + (x - tw)\partial_w. \end{aligned} \tag{2.6}$$

The Lie Brackets of the symmetries in (2.6) are

$$\begin{array}{ll} [\Delta_1, \Delta_2]_{LB} = 0 & [\Delta_2, \Delta_3]_{LB} = -\Delta_1 \\ [\Delta_1, \Delta_3]_{LB} = 0 & [\Delta_2, \Delta_4]_{LB} = -\tfrac{1}{2}\Delta_2 \\ [\Delta_1, \Delta_4]_{LB} = \tfrac{1}{2}\Delta_1 & [\Delta_2, \Delta_5]_{LB} = 0 \\ [\Delta_1, \Delta_5]_{LB} = \Delta_2 & [\Delta_3, \Delta_4]_{LB} = \Delta_3 \\ [\Delta_4, \Delta_5]_{LB} = \Delta_5 & [\Delta_3, \Delta_5]_{LB} = 2\Delta_4. \end{array}$$

It is evident that the algebra is $sl(2, R) \oplus_s 2A_1$ with Δ_1 and Δ_2 constituting the two-dimensional abelian subalgebra.

To determine the complete symmetry group of (2.5) we commence with the general equation

$$w_{xx} = f(t, x, w, w_t, w_x), \tag{2.7}$$

where f is initially an arbitrary function of its arguments. We impose the symmetries in turn so that the functional form of f is established. These five Lie point symmetries are insufficient to specify completely equation (2.5). To determine the complete symmetry group we make use of an approach which we call the method of the 'implicit complete symmetry group'. This type of complete symmetry group is achieved by imposing an extra condition on the structure of the equation we are trying to specify. This condition then removes an argument from our arbitrary function and thereby makes the number of point symmetries required to specify the equation one fewer than that required for the complete symmetry group.

In principle we are saying that there is a nonlocal symmetry which allows us to impose this extra condition. Once we have imposed the desired condition, we then return to find the nonlocal symmetry. We illustrate this method with equation (2.5) for which we assume the general second-order evolution partial differential equation of the form (2.7).

The application of $\Delta_1 = \partial_t$ and $\Delta_3 = \partial_x$ gives

$$w_{xx} = f(w, w_x, w_t). \tag{2.8}$$

We now impose the condition that the function f is of the form

$$f(w, w_x, w_t) = h(w, w_x) - w_t \tag{2.9}$$

so that we have

$$w_{xx} + w_t = h(w, w_x). \tag{2.10}$$

The application of $\Delta_5 = t^2\partial_t + tx\,\partial_x + (x - tw)\partial_w$ on (2.10) gives

$$-3t\,w_{xx} - w - 3t\,w_t - x\,w_x = (x - tw)\frac{\partial h}{\partial w} + (1 - 2t)w_x\frac{\partial h}{\partial w_x}. \tag{2.11}$$

When we take (2.10) into account, (2.11) becomes

$$-3th - w - x\,w_x = (x - tw)\frac{\partial h}{\partial w} + (1 - 2t)w_x\frac{\partial h}{\partial w_x}.$$

Since x and t are not in h, we can extract coefficients to obtain

$$\begin{aligned}
\text{For } t \quad &: \quad 3h = w\frac{\partial h}{\partial w} + 2w_x\frac{\partial h}{\partial w_x} \\
\text{For } x \quad &: \quad -w_x = \frac{\partial h}{\partial w} \\
\text{For neither } x \text{ nor } t \quad &: \quad -w = w_x\frac{\partial h}{r\partial w_x}.
\end{aligned} \tag{2.12}$$

We substitute the second and third equations into the first equation so that

$$h = -w\, w_x.$$

Hence

$$w_{xx} + w_t + w\, w_x = 0$$

and we have recovered (2.5).

Now we need to find a nonlocal symmetry that allows us to write $f = h - w_t$. The procedure is as follows. We require that the characteristics for equation (2.8) which is invariant under Δ_1 and Δ_3 produced by the associated Lagrange's system be

$$w, w_x, w_t + f$$

from the associated Lagrange's system

$$\frac{\mathrm{d}w}{0} = \frac{\mathrm{d}w_x}{0} = \frac{\mathrm{d}w_t}{g(\cdot)} = \frac{\mathrm{d}f}{-g(\cdot)}, \tag{2.13}$$

where g is some arbitrary function of its arguments and arises from the fact that the associated Lagrange's system is always up to a common multiplier in the denominator. Assume that the nonlocal symmetry is of the form

$$\Delta_6 = \xi\partial_x + \tau\partial_t + \eta\partial_w \tag{2.14}$$

without specifying the nature of the dependence in ξ, τ and η.

For linear evolution equations the required terms of the second extension are given by

$$\Delta_6^{[2]} = \xi\partial_x + \tau\partial_t + \eta\partial_w + \eta_x\partial_{w_x} + \eta_t\partial_{w_t} + \eta_{xx}\partial_{w_{xx}}, \tag{2.15}$$

where

$$\begin{aligned}
\eta_x &= \frac{\partial\eta}{\partial x} + \left[\frac{\partial\eta}{\partial w} - \frac{\partial\xi}{\partial x}\right] w_x - \frac{\partial\tau}{\partial x} w_t \\
\eta_t &= \frac{\partial\eta}{\partial t} + \left[\frac{\partial\eta}{\partial w} - \frac{\partial\tau}{\partial t}\right] w_t - \frac{\partial\xi}{\partial t} w_x \\
\eta_{xx} &= \frac{\partial^2\eta}{\partial x^2} + \left[2\frac{\partial^2\eta}{\partial x\partial w} - \frac{\partial^2\xi}{\partial x^2}\right] w_x - \frac{\partial^2\tau}{\partial x^2} w_t + \left[\frac{\partial\eta}{\partial w} - 2\frac{\partial\xi}{\partial w}\right] w_{xx}.
\end{aligned}$$

We apply (2.15) to (2.8) and then demand that the coefficients in the equation give the associated Lagrange's system (2.13). We obtain that

$$\eta = 0, \quad \eta_x = 0, \quad \eta_{xx} + \eta_t = 0.$$

These result in the system of partial differential equations for the coefficient functions of the symmetry to be

$$\frac{\partial\xi}{\partial x} w_x + \frac{\partial\tau}{\partial x} w_t = 0$$

$$\left(\frac{\partial^2\tau}{\partial x^2} + \frac{\partial\tau}{\partial t}\right) w_t + \left(\frac{\partial^2\xi}{\partial x^2} + \frac{\partial\xi}{\partial t}\right) w_x + 2\frac{\partial\xi}{\partial x} w_{xx} = 0.$$

Substituting the upper equation into the lower we obtain, after dividing by w_t,

$$\frac{\partial^2\tau}{\partial x^2} + \frac{\partial\tau}{\partial t} - 2\frac{w_{xx}}{w_x}\frac{\partial\tau}{\partial x} + \left(\frac{\partial^2\xi}{\partial x^2} + \frac{\partial\xi}{\partial t}\right)\frac{w_x}{w_t} = 0.$$

For convenience[5] we choose $\tau = x$.

Then

$$\frac{\partial^2\xi}{\partial x^2} + \frac{\partial\xi}{\partial t} = 2\frac{w_{xx}w_t}{w_x^2}$$

for which, using Fourier transforms, Duhamel's principle or Green's function, one can derive the solution of a nonhomogeneous diffusion problem [14]. Here

$$\begin{aligned} \xi_t + \xi_{xx} &= g(x,t), \qquad x \in \mathcal{R}, \quad t > 0. \\ \xi(x,0) &= \xi_0(x), \qquad x \in \mathcal{R}, \end{aligned}$$

The solution is given by

$$\xi(x,t) = \int_{\mathcal{R}} K(x-y;t)\xi_0(y)\mathrm{d}y + \int_0^t \int_{\mathcal{R}} K(x-y,t-s)g(y,s)\mathrm{d}y\,\mathrm{d}s, \tag{2.16}$$

where $K(x,t)$ is the diffusion kernel given by

$$K(x,t) = \left(\frac{1}{4\pi t}\right)^{\frac{1}{2}} \exp\left(-x^2/4t\right)$$

and ξ_0 and $g(x,t)$ are continuous bounded functions. Hence

$$\Delta_6 = \xi(x,t)\partial_x + x\,\partial_t$$

with $\xi(x,t)$ given by the integral equation (2.16) for the function g given by

$$g(x,t) = 2\frac{w_{xx}w_t}{w_x^2}.$$

It must be noted that the solution to the above system of partial differential equations, (2.12), is not unique. A different choice of τ or ξ would produce a different solution. Hence there is not a unique nonlocal symmetry producing the same characteristics in (2.14).

3 Quasi-implicit complete symmetry groups

In the process of finding a complete symmetry group of a partial differential equation one sometimes has to specify more than one condition, which in turn reduces the number of independent variables in the general second-order evolution partial differential equation,

$$F(x,u,u_x,u_t,u_{xx}) = 0, \tag{3.1}$$

[5]In the case that one confines symmetries to be point symmetries the assumption of a specific expression for one of the coefficient functions is at least potentially a major restriction. In the case of generalized and/or nonlocal symmetries one is simply choosing one symmetry of an equivalence class rather like one chooses a gauge in Field Theory.

by more than the one seen in §2. This necessarily produces more than one nonlocal symmetry.

The most important step in this type of analysis for the symmetry group is to identify at what point in the analysis a nonlocal symmetry is required. The guideline is at a point where the arbitrary function found after the application of a particular point symmetry still depends on the variable that one is trying to remove. We illustrate this by an example drawn from the Mathematics of Finance.

The equation we consider is a nonlinear partial differential equation for volatility. The economic model [5] presents the necessary and sufficient conditions which permit the driving standard Brownian motion to be expressed as a scale change of the stock price process.

The economic model assumes frictionless markets, no arbitrage and that the underlying stock price process is a one-dimensional diffusion starting from a positive value. It also assumes a proportional risk-neutral drift of $r-q$, where $r \geq 0$ is the constant risk-free rate and $q \geq 0$ is the constant dividend yield. The absolute volatility rate is a positive $C^{2,1}$ function $u(x,t)$ of the stock price $x \in (0,\infty)$ and time $t \in (0,T)$, where T is some distant horizon exceeding the longest maturity of the option to be priced.

Carr, Tari and Zariphopoulou [5] derive the nonlinear partial differential equation

$$u^2 u_{xx} + (r-q)xu_x + u_t - (r-q)u = 0. \tag{3.2}$$

We rescale the variables to achieve an equation simpler in appearance, *videlicet*

$$u^2 u_{xx} + xu_x + u_t - u = 0, \tag{3.3}$$

and it is for this equation that we find the complete symmetry group.

The Lie point symmetries of (3.3) are

$$\begin{aligned} \Sigma_1 &= \partial_t \\ \Sigma_2 &= e^t\,\partial_x \\ \Sigma_3 &= \partial_t + x\,\partial_x + u\,\partial_u \\ \Sigma_4 &= t\,\partial_t + tx\,\partial_x + \left(t - \tfrac{1}{2}\right) u\,\partial_u . \end{aligned} \tag{3.4}$$

The Lie Brackets are

$$\begin{array}{lll} [\Sigma_1,\ \Sigma_2]_{LB} = \Sigma_2 & & \\ [\Sigma_1\ ,\Sigma_3]_{LB} = 0 & [\Sigma_2\ ,\Sigma_3]_{LB} = \Sigma_2 & \\ [\Sigma_1,\ \Sigma_4]_{LB} = \Sigma_3 & [\Sigma_2,\ \Sigma_4]_{LB} = 0 & [\Sigma_3,\ \Sigma_4]_{LB} = 0. \end{array}$$

The symmetries Σ_2, Σ_3 and Σ_4 form an $A_1 \oplus A_2$ subalgebra and the symmetries Σ_1, Σ_3 and Σ_4 form an $A_{3,1}$ (Weyl) subalgebra. This equation makes one very curious since it has an even number of symmetries and there is no trace of the $sl(2,R)$ subalgebra, which is a common phenomenon in many equations arising in finance. Equation (3.3) is not linearizable to the heat equation or one of its variations by means of a point transformation since the algebra (3.4) does not contain an infinite abelian subalgebra.

For a complete symmetry group we consider the general second-order evolution partial differential equation

$$F(x,t,u,u_x,u_t,u_{xx}) = 0.$$

By the Implicit Function Theorem the equation can be written in solved form of one of its essential arguments. We choose to write this as $u_t = f(x, t, u, u_x, u_{xx})$ since we wish to express the equation as an evolution equation with u_t as the subject[6].

Application of $\Sigma_1 = \partial_t$ gives

$$u_t = f(x, u, u_x, u_{xx}). \tag{3.5}$$

The second extension of $\Sigma_2 = e^t\partial_x$ is

$$\Sigma_2^{[2]} = e^t\partial_x + (0)\partial_{u_x} - e^t u_x\,\partial_{u_t} + (0)\partial_{u_{xx}}$$

and its application to (3.5) yields

$$-u_x = \frac{\partial f}{\partial x} \Rightarrow f = -x\,u_x + h(u_x, u_{xx}, u).$$

This is not good since h still depends explicitly on u_x. Before applying Σ_2 we become proactive and require that

$$u_t = f(u, xu_x, u_{xx}).$$

This is just imposing that the equation must have a Euler structure in x as far as u_x is concerned.

Then there is a nonlocal symmetry which allows the above operation. We find it as follows.

The characteristics would have been

$$u_t, u, u_{xx}, xu_x$$

which come from the associated Lagrange's system

$$\frac{du_x}{-u_x} = \frac{du}{0} = \frac{du_{xx}}{0} = \frac{du_t}{0} = \frac{dx}{x}.$$

This suggests that the second extension of the nonlocal symmetry, say $\Sigma_5 = \xi\partial_x + \tau\partial_t + \eta\partial_u$, is

$$\Sigma_5^{[2]} = \xi\,\partial_x + \tau\,\partial_t + \eta\,\partial_u + \zeta_x\,\partial_{u_x} + \zeta_t\,\partial_{u_t} + \zeta_{xx}\,\partial_{u_{xx}},$$

where

$$\xi = x \ \ , \ \ \eta = 0,$$

ζ_x, ζ_t and ζ_{xx} are the extensions of the operator Σ_5 relevant to the derivatives indicated. Specifically they are given by

$$\zeta_x = \frac{\partial\eta}{\partial x} + \left[\frac{\partial\eta}{\partial u} - \frac{\partial\epsilon}{\partial x}\right]u_x - \frac{\partial\tau}{\partial x}u_t \tag{3.6}$$

[6]Naturally the argument can be carried out using the conventional form of the general second-order evolution equation, but one must prepared to do hard labor!

$$\zeta_t = \frac{\partial\eta}{\partial t} + \left[\frac{\partial\eta}{\partial u} - \frac{\partial\tau}{\partial t}\right] u_t - \frac{\partial\xi}{\partial t} u_x \tag{3.7}$$

$$\zeta_{xx} = \frac{\partial^2\eta}{\partial x^2} + \left[2\frac{\partial^2\eta}{\partial x\partial u} - \frac{\partial^2\xi}{\partial x^2}\right] u_x - \frac{\partial^2\tau}{\partial x^2} u_t - 2\frac{\partial\xi}{\partial x} u_{xx}. \tag{3.8}$$

The symmetry generating function/system is

$$\zeta_{xx} = 0 \quad \zeta_t = 0 \quad \zeta_x = -u_x,$$

ie, when one makes use of the expressions of ζ_x, ζ_t and ζ_{xx}, above one obtains

$$\begin{aligned} &-\frac{\partial^2\xi}{\partial x^2} u_x - \frac{\partial^2\tau}{\partial x^2} u_t - 2\frac{\partial\xi}{\partial x} u_{xx} = 0 \\ &-\frac{\partial\tau}{\partial t} u_t - \frac{\partial\xi}{\partial t} u_x = u_t \Rightarrow \frac{\partial\tau}{\partial t} u_t + \frac{\partial\xi}{\partial t} u_x = 0 \\ &-\frac{\partial\xi}{\partial x} u_x - \frac{\partial\tau}{\partial x} u_t = -u_x \Rightarrow \frac{\partial\xi}{\partial x} u_x + \frac{\partial\tau}{\partial x} u_t = u_x. \end{aligned} \tag{3.9}$$

When we add the second and the third of (3.9), we have

$$\left(\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial x}\right) u_t + \left(\frac{\partial\xi}{\partial t} + \frac{\partial\xi}{\partial x}\right) u_x = u_x, \tag{3.10}$$

but we have assumed that

$$\xi = x.$$

Then

$$\left(\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial x}\right) u_t = 0$$

so that

$$\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial x} = 0. \tag{3.11}$$

From the first of (3.9) we obtain, after substituting $\xi = x$,

$$\begin{aligned} \frac{\partial^2\tau}{\partial x^2} u_t &= -2u_{xx} \\ \frac{\partial^2\tau}{\partial x^2} &= -2\frac{u_{xx}}{u_t}. \end{aligned}$$

The first integration gives

$$\frac{\partial\tau}{\partial x} = -2\int \frac{u_{xx}}{u_t} dx + c(t), \tag{3.12}$$

where c is some function of t. We substitute (3.12) into (3.11) so that

$$\frac{\partial\tau}{\partial t} = -c(t) + 2\int \frac{u_{xx}}{u_t} dx = 2\int \frac{u_{xx}}{u_t} dx - c(t)$$

and hence

$$\tau(x,t) = 2\int\int \frac{u_{xx}}{u_t} dx\, dt - \int c(t)dt + \beta, \tag{3.13}$$

where β is a constant of integration.

The nonlocal symmetry is

$$\Sigma_5 = x\,\partial_x + \tau\,\partial_t,$$

where $\tau(x,t)$ is given by (3.13). Note that in this case there is only one nontrivial solution of the system of partial differential equations producing the nonlocal symmetry. So the nonlocal symmetry producing the above characteristics is unique.

Hence we have the desired result that

$$u_t = f(u, x\,u_x, u_{xx}).$$

We further proceed with the application of the remaining Lie point symmetries. The application of the second extension of $\Sigma_2 = e^t\partial_x$ gives

$$-u_x = \frac{\partial f}{\partial(x\,u_x)} \cdot u_x,$$

ie,

$$\frac{\partial f}{\partial(x\,u_x)} = -1 \Rightarrow f = -x\,u_x + h(u, u_{xx})\ .$$

$$u_t + x\,u_x = h(u, u_{xx}). \tag{3.14}$$

The second extension of Σ_4 is

$$\begin{aligned} \Sigma_4^{[2]} &= t\,\partial_t + tx\,\partial_x + \left(t - \tfrac{1}{2}\right) u\,\partial_u - \tfrac{1}{2}u_x\,\partial_{u_x} \\ &\quad + \left[u + \left(t - \tfrac{3}{2}\right) u_t - x\,u_x\right] \partial_{u_t} - \left(t + \tfrac{1}{2}\right) u_{xx}\,\partial u_{xx} \end{aligned} \tag{3.15}$$

and the action of (3.14) on (3.15) gives $\Sigma_4^{[2]}(u_t + x\,u_x - h) = 0$ which implies that

$$u + \left(t - \tfrac{3}{2}\right) u_t - x\,u_x = -tx\,u_x + \left(t - \tfrac{1}{2}\right) u\,\frac{\partial h}{\partial u} + \tfrac{1}{2}x\,u_x - \left(t + \tfrac{1}{2}\right) u_{xx}\frac{\partial h}{\partial u_{xx}}. \tag{3.16}$$

Extracting coefficients of t which is not present in (3.16) we have

$$\text{For } \ t: \quad u_t = -x\,u_x + u\frac{\partial h}{\partial u} - u_{xx}\frac{\partial h}{\partial u_{xx}}$$

$$\Rightarrow u_t + x\,u_x = u\frac{\partial h}{\partial u} - u_{xx}\frac{\partial h}{\partial u_{xx}},$$

ie,

$$h = u\frac{\partial h}{\partial u} - u_{xx}\frac{\partial h}{\partial u_{xx}}. \tag{3.17}$$

The remaining terms of (3.16) give

$$u - \tfrac{3}{2}u_t - x\,u_x = -\tfrac{1}{2}u\frac{\partial h}{\partial u} + \tfrac{1}{2}x\,u_x - \tfrac{1}{2}u_{xx}\frac{\partial h}{\partial u_{xx}},$$

ie

$$u - \tfrac{3}{2}h = -\tfrac{1}{2}u\frac{\partial h}{\partial u} - \tfrac{1}{2}u_{xx}\frac{\partial h}{\partial u_{xx}}. \tag{3.18}$$

When one substitutes (3.17) into (3.18), the result is

$$u = u\frac{\partial h}{\partial u} - 2u_{xx}\frac{\partial h}{\partial u_{xx}}.$$

The associated Lagrange's system is

$$\frac{dh}{u} = \frac{du}{u} = \frac{du_{xx}}{-2u_{xx}}.$$

The characteristics are

$$h - u; \;\; u^2u_{xx}$$

so that $h = u + g(u^2u_{xx})$.

When we resubstitute h into equation (3.17), we get

$$\begin{aligned} u + g &= u(1 + 2uu_{xx}g') - u_{xx}u^2g' \\ g &= 2u^2u_{xx}g' - u^2u_{xx}g' \\ g &= u^2u_{xx}g', \end{aligned}$$

ie,

$$\frac{g'}{g} = \frac{1}{u^2u_{xx}}.$$

When this is integrated and exponentiated, we obtain

$$g = \gamma u^2u_{xx},$$

This gives the characteristic $g/(u^2u_{xx})$ so that

$$u_t + xu_x = u + \gamma u^2u_{xx},$$

where γ is an arbitrary constant. We require that $\gamma = -1$. This can easily done by rescaling or by the use of this not very nice nonlocal symmetry[7]

$$\Sigma_6 = \tau\,\partial_t,$$

with τ given by

$$\tau(x,t) = -2\int\int\frac{u_{xx}}{u_tu_x}dx\,dt - \int c_1(t)dt + \beta,$$

where $c_1(t)$ and β are function and constant of integration respectively.

[7]There are other nonlocal symmetries which provide a similar result.

4 Conclusion

The implicit and quasi-implicit complete symmetry group approach not only provides us with the sufficient number of symmetries to form a complete symmetry group but also provides a more direct way to find nonlocal symmetries. The nonlocal symmetries found are known to have specific functions in the development of a partial differential equation. Further the nonlocal symmetries producing the desired result may be unique as seen in Section 3 or not. Furthermore work needs be done to determine whether are they unique. Also the nonlocal symmetry, Δ_6, is an *extracting* symmetry since it removes a variable from the arbitrary function while the nonlocal symmetry, Σ_5, is a *combining* symmetry as it combines the variables inside the arbitrary function. It turns out that, when an arbitrary function we are trying to specify contains more than three arguments, *ie* contains either the space or the time variable in addition to the u and its derivatives, the extracting nonlocal symmetry simply becomes a Lie point symmetry.

An interesting point to note is that the nonlocal symmetry, Δ_6, is not only the symmetry for equation (2.3) but it turns out that all evolution equations which can be written in the form,

$$w_{xx} + w_t = h(w, w_x), \tag{4.1}$$

have Δ_6 as the nonlocal symmetry. Similarly the nonlocal symmetry, Σ_5, is a symmetry for all equations of the form

$$u_t = f(u, xu_x, u_{xx}).$$

These types of symmetries are said to be *generic* to these structures of equations. One can proceed in a similar way to find other *generic* symmetries for other structures of equations. The implicit and quasi-implicit ideas presented in this paper were inspired from a consideration of nonlinear partial differential equations. They can also be applied quite easily to linear partial differential equations. Consider the 1 + 1 linear evolution equation

$$u_t + u_{xx} + \frac{u}{x^2} = 0$$

from our previous paper [20]. This equation has the Lie point symmetries

$$\begin{aligned}
G_1 &= \partial_t \\
G_2 &= t\partial_t + \tfrac{1}{2}x\,\partial_x - \tfrac{1}{4}u\,\partial_u \\
G_3 &= t^2\partial_t + tx\,\partial_x + \tfrac{1}{4}(x^2 - 2t)u\,\partial_u \\
G_4 &= u\,\partial_u \\
G_5 &= g(t, x)\partial_u,
\end{aligned}$$

The application of G_1 to the general second-order evolution equation leads to

$$u_{xx} = f(x, u, u_x, u_t)\ .$$

A nonlocal symmetry of the form

$$\Gamma = x\partial_x + \tau(x, t)\partial_t + 2u\partial u\ , \tag{4.2}$$

where

$$\tau(x,t) = 2\int \frac{1}{u_x}\left(\int \frac{u_{xx}}{u_t}\,dx\right)dt + \int \frac{\beta(t)}{u_x}\,dt + \gamma,$$

$\beta(t)$ is some arbitrary function of t and γ is a constant of integration, guarantees that

$$f(x,u,u_x,u_t) = h\left(\frac{u}{x^2}, u_x, u_t\right) \tag{4.3}$$

and (4.2) is generic to (4.3).

Note that (4.3) can also be written in the form (4.1) using a nonlocal symmetry of the form Σ_5 to obtain

$$u_{xx} + u_t = g\left(\frac{u}{x^2}, u_x\right). \tag{4.4}$$

The function g is arbitrary as is seen by replacing the first argument in (4.1) with those of g in (4.4). From this point onwards one can easily use the remaining Lie point symmetries $G_2 - G_5$ to specify completely the equation.

Another point worth mentioning is that the level of complexity increases when one has an insufficient number of point symmetries to constitute a complete symmetry group. This was more evident in [20] when dealing with an $1+1$ evolution equation with $\infty+1+3$ Lie point symmetries. However, once all necessary nonlocal symmetries have been determined, the complete symmetry group follows immediately.

The open question in the implicit/quasi-implicit complete symmetry group is the determination of the exact point at which a nonlocal symmetry is required. So far we have been successful in following our intuition and we hope to provide a concise guideline/algorithm to avoid the frustration of attempting the exercise when experience, intuition and luck are absent. The beauty of this algorithm is that we can then proceed (*iff*) to find most of these generic nonlocal symmetries of evolution equations.

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