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Feynman-Jackson integrals

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Abstract

We introduce perturbative Feynman integrals in the context of q-calculus generalizing the Gaussian q-integrals introduced by Díaz and Teruel. We provide analytic as well as combinatorial interpretations for the Feynman-Jackson integrals.

1 Introduction

Feynman integrals are a main tool in high energy physics since they provide a universal integral representation for the correlation functions of any Lagrangian quantum field theory whose associated quadratic form is non-degenerated. In some cases the degenerated situation may be approached as well by including odd variables as is usually done in the BRST-BV procedure. Despite its power Feynman integrals still await a proper definition from a rigorous mathematical point of view. The main difficulties in understanding Feynman integrals are the following

1. The output of a perturbative Feynman integral is a formal power series in infinitely many variables, i.e., an element of $\mathbb{C}[[g_1, ..., g_n, ..]]$. This fact goes against our strongly held believe that the output of an integral should be a number.

2. There is no guarantee that the formal series mentioned above will be convergent, not even in an asymptotic sense. General statements in this matter are missing.

3. Feynman integrals of greatest interest are performed over spaces of infinite dimension. In this situation the coefficients of the series in variables $\mathbb{C}[[g_1, ..., g_n, ..]]$ referred above are given by finite dimensional integrals which might be divergent. In this case additional care must be taken in order to renormalize the values of these integrals. The renormalization procedure, when applies, is done in two steps one of analytic nature called regularization, and a further step of algebraic nature which may be regarded as a fairly general form of the inclusion-exclusion principle of combinatorics.

4. In process 1 to 3 above a number of choices must be made. No general statements showing the unicity of the result are known.

Finite dimensional Feynman integrals are also of interest for example in Matrix theory. They still present difficulties 1 and 2 above but issues 3 and 4 become null. The goal of this paper is to construct a q-analogue of Feynman integrals which we call Feynman-Jackson integrals. We consider only the simplest case of 1-dimensional integrals. Our approach is to use the q,k-generalized gamma function and the q,k-generalized Pochhamer symbol introduced in [6] and [5].

The computation of a 1-dimension Feynman integrals, for example an integral of the form $\int e^{h(x)}dx$ where $h(x) = \dfrac{-x^2}{2} + \displaystyle\sum_{j=1}^{\infty} h_j \frac{x^j}{j!}$ is done in four steps

1. The integral is obtain perturbatively, meaning that the integrand $h(x)$ should be replaced by a formal power series $\dfrac{-x^2}{2} + \displaystyle\sum_{j=1}^{\infty} g_j h_j \frac{x^j}{j!} \in \mathbb{C}[[g_1, ..., g_n, ..]]$ where $\{g_j\}_{j=1}^{\infty}$ is a countable set of independent variables.

2. One uses the key identity $e^{\frac{-x^2}{2}+\sum_{j=1}^{\infty} g_j h_j \frac{x^j}{j!}} = e^{-\frac{x^2}{2}} \; e^{\sum_{j=1}^{\infty} g_j h_j \frac{x^j}{j!}}$.

3. $e^{\sum_{j=1}^{\infty} g_j h_j \frac{x^j}{j!}}$ is expanded as a formal power series in $\mathbb{C}[[g_1, ..., g_n, ..]]$ using the series expansion of e^x. This step reduces the computation of the Feynman integrals to computing a countable number of Gaussian integrals.

4. Compute the Gaussian integrals obtained in step 3 which yield as output an element in $\mathbb{C}[[g_1, ..., g_n, ..]]$.

Steps 1,3 and 4 can be carried out in q-calculus without much difficulty . The subtle issue to be tackled is the unpleasant fact that the identity $e^{x+y} = e^x e^y$ does not hold in q-calculus.

This paper is organized as follows: in Section 2 after a quick review of q-calculus we introduce Gauss-Jackson integrals based on the definition of the function $\Gamma_{q,2}$ introduced in [5]. In Section 3 we introduce the combinatorial tools that shall be needed to formulate our main theorem. In Section 4 we introduce the algebraic properties of the q-exponential that will allow us to overcome the fact that $E_q^{x+y} \neq E_q^x E_q^y$. In Section 5 we shall enunciate and prove our main result Theorem 16

$$\frac{1}{\Gamma_{q,2}(1)} \int_{-\nu}^{\nu} E_{q,2}^{\frac{-q^2x^2}{[2]_q}+\sum_{j=1}^{\infty} g_j h_j \frac{x^j}{[j]_q!}} d_q x = \sum_{\Lambda \in \mathrm{Ob}(\mathbf{Graph}_q)/\sim} h_q(\Lambda) \frac{\omega_q(\Lambda)}{aut_q(\Lambda)}$$

which gives a q-analogue of 1-dimensional Feynman integrals.

2 Gauss-Jackson integrals

In this section we review several definitions in q-calculus. The reader may find more information in [5], [3], [12] and [4] . We focus upon the q,k-generalizations of the Pochhammer symbol, the gamma function and its integral representations [5].

Let us fix $0 < q < 1$ and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be any map. The q-derivative $\partial_q(f)$ of f is given by $\partial_q(f) = \dfrac{d_q f}{d_q x} = \dfrac{I_q(f) - f}{(q-1)x}$, where $I_q : \mathbb{R} \longrightarrow \mathbb{R}$ is given by $I_q(f)(x) = f(qx)$ for all $x \in \mathbb{R}$, and $d_q(f) = I_q(f) - f$.

The definite Jackson integral (see [10] and [11]) of a map $f : [0, b] \longrightarrow \mathbb{R}$ is given by

$$\int_0^b f(x) d_q x = (1-q) b \sum_{n=0}^{\infty} q^n f(q^n b).$$

The improper Jackson integral of a map $f : [0, \infty) \longrightarrow \mathbb{R}$ is given by

$$\int_0^{\infty/a} f(x) d_q x = (1-q) \sum_{n \in \mathbb{Z}} \frac{q^n}{a} f\left(\frac{q^n}{a}\right).$$

For all $t \in \mathbb{Z}^+$ the q-factorial is given by $[t]_q! = [t]_q [t-1]_q \cdots [1]_q$, where $[t]_q = \dfrac{(1-q^t)}{(1-q)}$ is the q-analogue of a real number t. The q-factorial is an instance of the q, k-generalized Pochhammer symbol which is given by

$$[t]_{n,k} = [t]_q [t+k]_q [t+2k]_q \ldots [t+(n-1)k]_q = \prod_{j=0}^{n-1} [t+jk]_q, \quad \text{for all} \quad t \in \mathbb{R}.$$

In this paper we shall mainly use the $q, 2$- generalized Pochhamer symbol evaluated at $t = 1$, namely

$$[1]_{n,2} = [1]_q [3]_q [5]_q \ldots [2n-1]_q = \prod_{j=0}^{n-1} [1+2j]_q. \tag{2.1}$$

We remark that $[1]_{n+1,2} = [2n+1]_q [1]_{n,2}$. We shall use the following notation. Let $x, y, t \in \mathbb{R}$ and $n \in \mathbb{Z}^+$ we set

$$(x+y)^n_{q,2} := \prod_{j=0}^{n-1} (x + q^{2j} y) \quad \text{and} \quad (1+x)^t_{q,2} := \frac{(1+x)^\infty_{q,2}}{(1+q^{2t}x)^\infty_{q,2}},$$

$$\text{where } (1+x)^\infty_{q,2} := \prod_{j=0}^{\infty} (1 + q^{2j} x)$$

Recall that one can define two q-analogues of the exponential function given as follows

$$\begin{aligned} E^x_{q,2} &= \sum_{n=0}^{\infty} \frac{q^{n(n-1)} x^n}{[n]_{q^2}!} = (1 + (1-q^2)x)^\infty_{q,2} \\ e^x_{q,2} &= \sum_{n=0}^{\infty} \frac{x^n}{[n]_{q^2}!} = \frac{1}{(1-(1-q^2)x)^\infty_{q,2}}. \end{aligned}$$

The $q, 2$-gamma function $\Gamma_{q,2}(t)$ is given by the explicit formula $\Gamma_{q,2}(t) = \dfrac{(1-q^2)^{\frac{t}{2}-1}_{q,2}}{(1-q)^{\frac{t}{2}-1}}$ for a real number $t > 0$, and has a representation in terms of $E^x_{q,2}$ given by the following

Jackson integral

$$\Gamma_{q,2}(t) = \int_0^{\left(\frac{[2]_q}{(1-q^2)}\right)^{\frac{1}{2}}} x^{t-1} E_{q,2}^{-\frac{q^2x^2}{[2]_q}} d_qx, \quad t > 0. \tag{2.2}$$

Similarly one can define $\gamma_{q,2}^{(a)}(t)$ for $a > 0$ using $e_{q,2}^x$ by the following Jackson integral

$$\gamma_{q,2}^{(a)}(t) = \int_0^{\infty/a(1-q^2)^{\frac{1}{2}}} x^{t-1} e_{q,2}^{-\frac{x^2}{[2]_q}} d_qx, \quad t > 0. \tag{2.3}$$

Both integral representations are related by $\Gamma_{q,2}(t) = c(a,t)\gamma_{q,2}^{(a)}(t)$, where the function $c(a,t)$ is given by

$$c(a,t) = \frac{a^t[2]_q^{\frac{t}{2}}}{1+[2]_qa^2}\left(1+\frac{1}{[2]_qa^2}\right)_{q,2}^{\frac{t}{2}}\left(1+[2]_qa^2\right)_{q,2}^{1-\frac{t}{2}}, \quad \text{for} \quad a > 0 \quad \text{and} \quad t \in \mathbb{R}.$$

We proceed to introduce two different q-analogues of the Gaussian integral and give a Jackson integral representation for each one. The Gaussian integrals are related to each other by the function $c(a,t)$ in a similar way as the integral representations of the $q,2$-generalized gamma functions are related to each other.

Definition 1. *Let $\nu = \left(\frac{[2]_q}{(1-q^2)}\right)^{\frac{1}{2}}$ and $\varepsilon^{(a)} = \infty/a(1-q^2)^{\frac{1}{2}}$, the Gaussian-Jackson integrals are given by*

$$G(t) := \frac{1}{2}\int_{-\nu}^{\nu} x^{t-1} E_{q,2}^{-\frac{q^2x^2}{[2]_q}} d_qx = \frac{1}{2}\int_0^{\nu} x^{t-1} E_{q,2}^{-\frac{q^2x^2}{[2]_q}} d_qx + \frac{1}{2}\int_{-\nu}^{0} x^{t-1} E_{q,2}^{-\frac{q^2x^2}{[2]_q}} d_qx, \quad t > 0.$$

$$G^{(a)}(t) := \frac{1}{2}\int_{-\varepsilon^{(a)}}^{\varepsilon^{(a)}} x^{t-1} e_{q,2}^{-\frac{x^2}{[2]_q}} d_qx = \frac{1}{2}\int_0^{\varepsilon^{(a)}} x^{t-1} e_{q,2}^{-\frac{x^2}{[2]_q}} d_qx + \frac{1}{2}\int_{-\varepsilon^{(a)}}^{0} x^{t-1} e_{q,2}^{-\frac{x^2}{[2]_q}} d_qx, \quad t > 0.$$

Notice that if $t-1$ is an odd integer both integrals in Definition 1 are zero because then x^{t-1} is an odd function while $E_{q,2}^{-\frac{q^2x^2}{[2]_q}}$ and $e_{q,2}^{-\frac{x^2}{[2]_q}}$ are even functions.

3 Combinatorial interpretation of $[1]_{n,2}$

In this section we introduce the combinatorial tools that will be needed in order to describe q-analogue of 1-dimensional Feynman integrals. The interested reader may consult [1], [2], [8] for further information.

Definition 2. *A partition of $a \in \mathbb{Z}^+$ is a finite sequence of positive integers $a_1, a_2, \ldots, a_r$ such that $\sum_{i=1}^{r} a_i = a$. For $a, d \in \mathbb{Z}^+$, $p_d(a)$ denotes the number of partitions of a into less than d parts.*

Definition 3. *Let $n \in \mathbb{Z}^+$ and $a_1, a_2, \ldots, a_n$ be a partition of a. The q-multinomial coefficient is given by*

$$\left[\begin{array}{c} a_1 + a_2 + \cdots + a_n \\ a_1, a_2, \ldots, a_n \end{array}\right]_q = \frac{[a_1 + a_2 + \ldots a_n]_q!}{[a_1]_q![a_2]_q! \ldots [a_n]_q!}.$$

Denote by $[[n]]$ the set $\{1, \ldots, n\}$ ordered in the natural way. $|X|$ denotes the cardinality of set X and S_X denotes the group of permutations on X.

Definition 4. *Let $a_1, \ldots, a_n$ be a partition of a. We denote by $S(a_1, \ldots, a_n)$ the set of all maps $f : [[a]] \longrightarrow [[n]]$ such that $|f^{-1}(i)| = a_i$ for all $i \in [[n]]$. We set $\mathrm{inv}(f) := |\{(i,j) \in [[a]] \times [[a]] : i < j$ and $f(i) > f(j)\}|$.*

The following result is proved using induction.

Theorem 5.

$$\left[\begin{array}{c} a_1 + a_2 + \cdots + a_n \\ a_1, a_2, \ldots, a_n \end{array}\right]_q = \sum_{f \in S(a_1, \ldots, a_n)} q^{\mathrm{inv}(f)}.$$

Notice that this result implies that

$$[n]_q! = \left[\begin{array}{c} 1 + 1 + \cdots + 1 \\ 1, 1, \ldots, 1 \end{array}\right]_q = \sum_{f \in S(1, \ldots, 1)} q^{\mathrm{inv}(f)} = \sum_{f \in S_{[[n]]}} q^{\mathrm{inv}(f)}.$$

Definition 6. *A paring α on a totally ordered set R of cardinality $2n$ is a sequence $\alpha = \{ \ (a_i, b_i) \ \}_{i=1}^n \in (R^2)^n$ such that*

1. $a_1 < a_2 < \cdots < a_n$.
2. $a_i < b_i, \quad i = 1, \ldots, n.$
3. $R = \bigsqcup_{i=1}^{n} \{a_i, b_i\}.$

We denote by $P(R)$ the set of pairings on R.

Definition 7. *For $\alpha \in P([[2n]])$ we set*

1. $((a_i, b_i)) = \{j \in [[2n]] : a_i < j < b_i\}$ *for all* $(a_i, b_i) \in \alpha$.
2. $P_i(\alpha) = \{b_j : 1 \leq j < i\}$.
3. $w(\alpha) = \prod_{i=1}^{n} q^{|((a_i,b_i)) \backslash P_i(\alpha)|} = q^{\sum_{i=1}^n |((a_i,b_i)) \backslash P_i(\alpha)|}$. *We call $w(\alpha)$ the weight of α.*

Example 8. *Let α be the pairing on $[[12]]$ shown in Figure 1. The weight $w(\alpha)$ can be computed as follows*

$$q^{|((a_1,b_1)) \backslash P_1(\alpha)|} = q^8, \qquad q^{|((a_2,b_2)) \backslash P_2(\alpha)|} = q^5, \qquad q^{|((a_3,b_3)) \backslash P_3(\alpha)|} = q^0,$$

$$q^{|((a_4,b_4)) \backslash P_4(\alpha)|} = q^{6-2}, \quad q^{|((a_5,b_5)) \backslash P_5(\alpha)|} = q^{4-2}, \quad q^{|((a_6,b_6)) \backslash P_6(\alpha)|} = q^{1-1}.$$

Hence $w(\alpha) = q^{19}$.

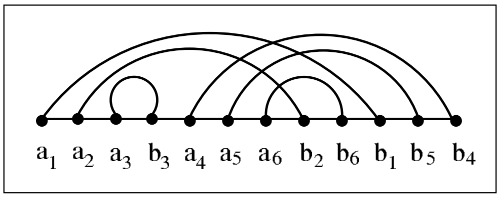

Figure 1. Example of a pairing α on $[[12]]$

Theorem 9. *Given $n \in \mathbb{N}$ the following identity holds*

$$[1]_{n,2} = \sum_{\alpha \in P([[2n]])} w(\alpha). \tag{3.1}$$

Proof. We use induction on n. For $n = 1$, we have $[1]_{1,2} = 1$. Suppose identity (3.1) holds for n, we prove it for $n+1$ as follows

$$\begin{aligned}
\sum_{\alpha \in P([[2n+2]])} w(\alpha) &= \sum_{\alpha \in P([[2n+2]])} w(\alpha - \{(a_1, b_1)\}) q^{|((a_1,b_1))|} \\
&= \sum_{2 \leq b_1 \leq 2n+2} q^{b_1 - 2} \sum_{\beta \in P([[2n+2]] \setminus \{(a_1,b_1)\})} w(\beta) \\
&= \sum_{2 \leq b_1 \leq 2n+2} q^{b_1 - 2} \sum_{\beta \in P([[2n]])} w(\beta) \\
&= \sum_{2 \leq b_1 \leq 2n+2} q^{b_1 - 2} \quad [1]_{n,2} \\
&= [2n+1]_q [1]_{n,2} = [1]_{n+1,2}.
\end{aligned}$$

■

Notices that as $q \longrightarrow 1$ we recover the well known identity

$$(2n-1)(2n-3)\dots 1 = |\{\text{pairings on } [[2n]]\}|.$$

Example 10. *By definition $[1]_{2,2} = [1]_q[3]_q = [3]_q$. Consider the pairings of a four elements ordered set. Figure 2 shows that there are 3 such pairings and that the sum of their weights is $1 + q + q^2$ as it should.*

4 Algebraic properties of the q-exponentials

The q-exponential maps e_q^x and E_q^x are good q-analogues of the exponential map e^x since they satisfy $\partial_q e_q^x = e_q^x$, e_q^0=1 and $\lim_{q \longrightarrow 1} e_q^x = e^x$, and $\partial_q E_q^x = E_q^{qx}$, $E_q^0 = 1$ and $\lim_{q \longrightarrow 1} E_q^x = e^x$. From a differential point of view e_q^x is the right q-analogue of e^x. However both e_q^x and E_q^x lack the fundamental algebraic property of the exponential, namely that $e^x : (\mathbb{R}, +) \longrightarrow$

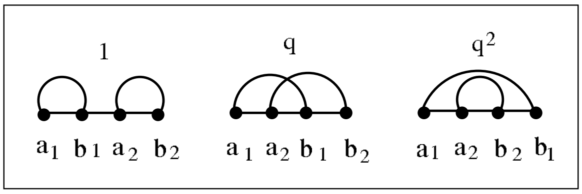

Figure 2. Combinatorial meaning of $[1]_{2,2}$

$(\mathbb{R},\cdot)$ is a group homomorphism. Indeed one checks that $e_q^{x+y} \neq e_q^x e_q^y$ and also that $E_q^{x+y} \neq E_q^x E_q^y$. Nevertheless we still have the remarkable identity $(e_q^x)^{-1} = E_q^{-x}$.

A possible algebraic solution to this problem is to assume that $yx = qxy$. Using this relation one verifies that $e_q^{x+y} = e_q^x e_q^y$ and $E_q^{x+y} = E_q^x E_q^y$. However we still have to deal with the fact that $e_q^{x+y} \neq e_q^x e_q^y$ and $E_q^{x+y} \neq E_q^x E_q^y$ for commuting variables $x, y \in \mathbb{R}$. Theorems 11 and 12 below provide tools that allow us to overcome this obstacle in the process of computing Feynman integrals, as discussed in the Introduction.

Theorem 11. $E_{q,2}^{x+y} = E_{q,2}^x \left(\sum_{c,d\geq 0} \lambda_{c,d} x^c y^d \right)$, *where* $\lambda_{c,d} = \sum_{k=0}^{c} \frac{(-1)^{c-k}\binom{d+k}{k} q^{(d+k)(d+k-1)}}{[d+k]_{q^2}![c-k]_{q^2}!}$.

Proof.

$$\begin{aligned} E_{q,2}^{x+y} e_{q,2}^{-x} &= \left(\sum_{n=0}^{\infty} \frac{q^{n(n-1)}(x+y)^n}{[n]_{q^2}!} \right) \left(\sum_{m=0}^{\infty} \frac{(-1)^m x^m}{[m]_{q^2}!} \right) \\ &= \sum_{n,m,k\leq n} \frac{(-1)^m \binom{n}{k} q^{n(n-1)}}{[n]_{q^2}![m]_{q^2}!} x^{m+k} y^{n-k} \end{aligned}$$

Making the change $c = m + k$ and $d = n - k$, we get

$$E_{q,2}^{x+y} e_{q,2}^{-x} = \sum_{c,d\geq 0} \left(\sum_{k=0}^{c} \frac{(-1)^{c-k}\binom{d+k}{k} q^{(d+k)(d+k-1)}}{[d+k]_{q^2}![c-k]_{q^2}!} \right) x^c y^d. \tag{4.1}$$

■

Theorem 12. $e_{q,2}^{x+y} = e_{q,2}^x \left(\sum_{c,d\geq 0} \kappa_{c,d} x^c y^d \right)$, *where* $\kappa_{c,d} = \sum_{k=0}^{c} \frac{(-1)^{c-k}\binom{d+k}{k} q^{(c-k)(c-k-1)}}{[d+k]_{q^2}![c-k]_{q^2}!}$.

Proof.

$$\begin{aligned} e_{q,2}^{x+y} E_{q,2}^{-x} &= \left(\sum_{n=0}^{\infty} \frac{(x+y)^n}{[n]_{q^2}!} \right) \left(\sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m-1)} x^m}{[m]_{q,2}!} \right) \\ &= \sum_{n,m,k\leq n} \frac{(-1)^m \binom{n}{k} q^{m(m-1)}}{[n]_{q^2}![m]_{q^2}!} x^{m+k} y^{n-k} \end{aligned}$$

Fixing $c = m + k$ and $d = n - k$, we have

$$e_{q,2}^{x+y} E_{q,2}^{-x} = \sum_{c,d \geq 0} \left(\sum_{k=0}^{c} \frac{(-1)^{c-k} \binom{d+k}{k} q^{(c-k)(c-k-1)}}{[d+k]_{q^2}! [c-k]_{q^2}!} x^c y^d \right). \tag{4.2}$$

■

Lemma 13. *For $c, d \in \mathbb{N}$, $\lim_{q \to 1} \lambda_{c,d} = \frac{1}{d!} \delta_{c,0}$.*

Proof.

$$\lim_{q \to 1} \lambda_{c,d} = \sum_{k=0}^{c} (-1)^{c-k} \frac{\binom{d+k}{k}}{(d+k)!(c-k)!} = \frac{1}{d!c!} \sum_{k=0}^{c} (-1)^{c-k} \binom{c}{k} = \frac{1}{d!} \delta_{c,0}.$$

■

5 Feynman-Jackson integrals

We denote by **Graph** the category whose objects Ob(**Graph**) are graphs. Recall that a graph Λ is triple (V, E, b) where V and E are finite sets, called the set of vertices and the set of edges respectively, and b is a map that assigns to each edge $e \in E$ a subset of V a cardinality one or two. To each graph we associate a map $\mathrm{val} : V \longrightarrow \mathbb{N}$ defined by $\mathrm{val}(s) = |\{e : s \in b(e)\}|$. All graphs considered in this paper are such that $\mathrm{val}(s) \geq 1$ for all $s \in V$. Morphisms in **Graph** from Λ_1 to Λ_2 are pairs (φ_V, φ_E) such that

1. $\varphi_V : V(\Lambda_1) \longrightarrow V(\Lambda_2)$.
2. $\varphi_E : E(\Lambda_1) \longrightarrow E(\Lambda_2)$.
3. $b(\Lambda_2)(\varphi_E(e)) = \varphi_V(b(\Lambda_1)(e))$, for all $e \in E(\Lambda_1)$.

The essence of 1-dimensional Feynman integrals, see [7], may be summarized in the following identity

$$\frac{1}{\sqrt{2\pi}} \int e^{\frac{-x^2}{2} + \sum_{j=1}^{\infty} g_j h_j \frac{x^j}{j!}} = \sum_{\Lambda \in \mathrm{Ob}(\mathbf{Graph})/\sim} h(\Lambda) \frac{\omega(\Lambda)}{aut(\Lambda)}. \tag{5.1}$$

In identity (5.1) the following notation is used

1. Ob(**Graph**)$/\sim$ denotes the set of isomorphisms classes of graphs.
2. $h(\Lambda) = \prod_{s \in V} h_{\mathrm{val}(s)}$.
3. $\omega(\Lambda) = \prod_{s \in V} g_{\mathrm{val}(s)}$.
4. $aut(\Lambda) = |Aut(\Lambda)|$ where $Aut(\Lambda)$ denotes the set of isomorphisms from graph Λ into itself, for all $\Lambda \in \mathrm{Ob}(\mathbf{Graph})$.

Theorem 16 below provides a q-analogue of identity (5.1). We first prove the following

Theorem 14.

$$\frac{1}{\Gamma_{q,2}(1)}\int_{-\nu}^{\nu} E_{q,2}^{\frac{-q^2x^2}{[2]_q}+\sum_{j=1}^{\infty} g_j h_j \frac{x^j}{[j]_q!}} d_qx = \sum_{m=0}^{\infty} \chi_m q^m$$

where

$$\chi_m = \sum_{\alpha,c,d,j,k,l,f} \frac{(-1)^k g_l h_l}{[2]_q^c[2j]_q![d+k]_{q^2}![c-k]_{q^2}!}\binom{d+k}{k}.$$

The sum above runs over all $c,d,j,k \in \mathbb{Z}^+$, such that $k \leq c$, $l \in p_d(2j)$, $f \in S(l_1,\dots,l_d)$, $\alpha \in P([[2c+2j]])$ and $\sum_{i=1}^{c+j} |((a_i,b_i)) \setminus P_i(\alpha)| + \mathrm{inv}(f) + (d+k)(d+k-1) + 2c = m$.

Proof. Making the changes $x \longrightarrow \dfrac{-q^2x^2}{[2]_q}$ and $y \longrightarrow \displaystyle\sum_{j=1}^{\infty} \frac{g_j h_j x^j}{[j]_q!}$ in Theorem 11, we get

$$\begin{aligned}
E_{q,2}^{\frac{-q^2x^2}{[2]_q}+\sum_{j=1}^{\infty} g_j h_j \frac{x^j}{[j]_q!}} &= E_{q,2}^{\frac{-q^2x^2}{[2]_q}} \sum_{c,d=0}^{\infty}\left(\frac{\lambda_{c,d}(-1)^c q^{2c}x^{2c}}{[2]_q^c}\left(\sum_{j=1}^{\infty} g_j h_j \frac{x^j}{[j]_q!}\right)^d\right)\\
&= E_{q,2}^{\frac{-q^2x^2}{[2]_q}} \sum_{c,d=0}^{\infty}\left(\frac{\lambda_{c,d}(-1)^c q^{2c}x^{2c}}{[2]_q^c}\left(\sum_{j=1}^{\infty}\left(\sum_{l\in p_d(j)} \frac{[j]_q! g_{l_1}\dots g_{l_d} h_{l_1}\dots h_{l_d}}{[l_1]_q![l_2]_q!\dots[l_d]_q!}\right)\frac{x^j}{[j]_q!}\right)\right)\\
&= E_{q,2}^{\frac{-q^2x^2}{[2]_q}} \sum_{c,d,j,l} \frac{\lambda_{c,d}(-1)^c q^{2c} g_{l_1}\dots g_{l_d} h_{l_1}\dots h_{l_d}}{[2]_q^c}\begin{bmatrix} j \\ l_1,\dots,l_d\end{bmatrix}_q x^{2c+j}.
\end{aligned}$$

Using the expression given in Theorem 11 for $\lambda_{c,d}$ and using the convention that $g_l = g_{l_1}\dots g_{l_d}$ and $h_l = h_{l_1}\dots h_{l_d}$ for $l \in p_d(j)$ we get

$$E_{q,2}^{\frac{-q^2x^2}{[2]_q}+\sum_{j=1}^{\infty} g_j h_j \frac{x^j}{[j]_q!}} = E_{q,2}^{\frac{-q^2x^2}{[2]_q}} \sum_{c,d,j,k,l} \frac{(-1)^{2c-k} g_l h_l q^{(d+k)(d+k-1)+2c}}{[2]_q^c[j]_q![d+k]_{q^2}![c-k]_{q^2}!}\binom{d+k}{k}\begin{bmatrix} j \\ l_1,\dots,l_d\end{bmatrix}_q x^{2c+j}. \tag{5.2}$$

Multiply by $\frac{1}{\Gamma_{q,2}(1)}$ and integrate both sides of the equation (5.2) from $-\nu$ to ν (which cancels out all terms with j odd), one gets for $l \in p_d(2j)$

$$\frac{1}{\Gamma_{q,2}(1)}\int_{-\nu}^{\nu} E_{q,2}^{\frac{-q^2x^2}{[2]_q}+\sum_{j=1}^{\infty} g_j h_j \frac{x^j}{[j]_q!}} d_qx \tag{5.3}$$

$$=\sum_{c,d,j,k,l} \frac{(-1)^{2c-k} g_l h_l q^{(d+k)(d+k-1)+2c}\binom{d+k}{k}}{[2]_q^c[2j]_q![d+k]_{q^2}![c-k]_{q^2}!}\begin{bmatrix} 2j \\ l_1,\dots,l_d\end{bmatrix}_q \frac{1}{\Gamma_{q,2}(1)}\int_{-\nu}^{\nu} E_{q,2}^{\frac{-q^2x^2}{[2]_q}} x^{2c+2j} d_qx \tag{5.4}$$

$$=\sum_{c,d,j,k,l} \frac{(-1)^{k} g_l h_l q^{(d+k)(d+k-1)+2c}\binom{d+k}{k}}{[2]_q^c[2j]_q![d+k]_{q^2}![c-k]_{q^2}!}\begin{bmatrix} 2j \\ l_1,\dots,l_d\end{bmatrix}_q [1]_{c+j,2}. \tag{5.5}$$

Notice that equation (5.5) is obtained from equation (5.4) using Definition 1. Using Theorem 9 and (4) in the right-hand side of (5.5) one obtains

$$\sum_{\alpha,c,d,j,k,l,f} \frac{(-1)^{2c-k} g_l h_l \binom{d+k}{k}}{[2]_q^c [2j]_q! [d+k]_{q^2}! [c-k]_{q^2}!} q^{\sum_{i=1}^{c+j} |((a_i,b_i))\setminus P_i(\alpha)| + \mathrm{inv}(f) + (d+k)(d+k-1)+2c}. \quad (5.6)$$

Which yields the desired result. ■

Using Lemma 13 one notices that the limit as q goes to 1 of (5.6) is

$$\sum \frac{g_l h_l}{(2j)! d!} \binom{2j}{l_1, \ldots, l_d} |\text{pairings on } [[2j]]|$$

which is well known to be equivalent to formula (5.1)

Definition 15. *We denote by* $\boldsymbol{Graph}_q$ *the category whose objects* $\mathrm{Ob}(\boldsymbol{Graph}_q)$ *are planar* q*-graphs* (V, E, b, f) *such that*

1. $V = \{\bullet\} \sqcup V^1 \sqcup V^2$ *where* $V^1 = \{\otimes_1, \ldots, \otimes_{|V_1|}\}$ *and* $V^2 = \{\circ_1, \ldots, \circ_{|V_2|}\}$.
2. $E = E^1 \sqcup E^2 \sqcup E^3$.
3. b *is a map that assigns to each edge* $e \in E$ *a subset of* V *a cardinality two.*
4. *Set* $F_\circ = \{(\circ_i, e) : i \in [[|V^2|]]$ *and* $\bullet \notin b(e)\}$. *We require that* $|F_\circ|$ *be even.* $f : F_\circ \longrightarrow [[|V^2|]]$ *is any map.*
5. $|b^{-1}(\{\otimes_i, \bullet\})| \in \{0, 1\}$ *for all* $i \in [[|V^1|]]$ *and* $|b^{-1}(\circ_i, \bullet)| = 1$ *for all* $i \in [[|V^2|]]$. *If* $|b^{-1}(\{\otimes_i, \bullet\})| = 1$ *then* $|b^{-1}(\{\otimes_i, \bullet\})| = 1$ *for all* $i \geq j$; *and* $\bullet \in b(e)$ *for any* $e \in E^3$.
6. $\mathrm{val}(\otimes_i) \in \{2, 3\}$. *If* $\mathrm{val}(\otimes_i) = 3$ *then* $\mathrm{val}(\otimes_j) = 3$ *for all* $i \geq j$, *and* $|E^2| \leq |V^1|$.

Morphisms in $\mathbf{Graph}_q$ are defined in the obvious way. Figure 3 shows an example of a planar q-graph with $n = 4$ and $m = 5$. Edges in E^1 (E^2, E^3) are depicted by dark (dotted, regular) lines, respectively. The map f can be read off the numbering of half-edges in E^3 attached to vertices $\{\circ_1, \ldots, \circ_m\}$.
Notice that associated to any graph $\Lambda \in \mathrm{Ob}(\mathbf{Graph}_q)$ there exists a pairing α on the naturally ordered set $\{(v, e) : v \in V^1 \sqcup V^2$ and $\bullet \notin b(e)\}$. Similarly, associated to any graph there is a map $\widehat{f} : [[|F_\circ|]] \longrightarrow [[|V^2|]]$ which is constructed from f and the natural ordering on $F_\circ$. For $\Lambda \in \mathrm{Ob}(\mathbf{Graph}_q)$ we set

1. $h_q = \displaystyle\prod_{i=1}^{|V^2|} h_{\mathrm{val}(\circ_i)-1}$.
2. $\omega_q = (-1)^{|E^2|} q^{2|V^1| + \binom{|V^2|+|E^2|}{2}} \omega(\alpha)\, \mathrm{inv}(\widehat{f}) \displaystyle\prod_{i=1}^{|V^2|} g_{\mathrm{val}(\circ_i)-1}$.
3. $aut_q(\Lambda) = [2]_q^n\ [|F_\circ|]_q!\ [|V^2| + |E^2|]_{q^2}!\ [|V^1| - |E^2|]_{q^2}!$

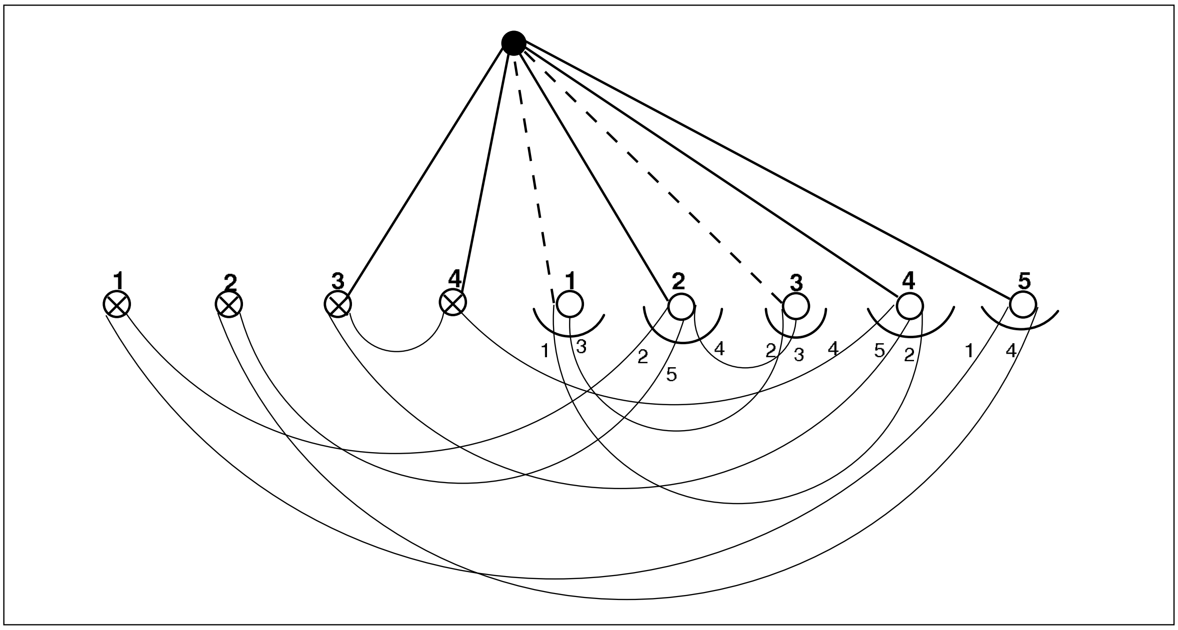

Figure 3. Feynman q-diagram

Using the notion of planar q-graphs introduced above Theorem 14 may be rewritten as follows

Theorem 16.

$$\frac{1}{\Gamma_{q,2}(1)}\int_{-\nu}^{\nu} E_{q,2}^{\frac{-q^2x^2}{[2]_q}+\sum_{j=1}^{\infty} g_j h_j \frac{x^j}{[j]_q!}} d_q x = \sum_{\Lambda\in \mathrm{Ob}(\boldsymbol{Graph}_q)/\sim} h_q(\Lambda)\frac{\omega_q(\Lambda)}{aut_q(\Lambda)}$$

Setting $h_j = 1$ in Theorem 16 one gets

Corolary 17.

$$\frac{1}{\Gamma_{q,2}(1)}\int_{-\nu}^{\nu} E_{q,2}^{\frac{-q^2x^2}{[2]_q}+\sum_{j=1}^{\infty} g_j \frac{x^j}{[j]_q!}} d_q x = \sum \frac{(-1)^{|E^2|}q^{2|V^1|+\binom{|V^2|+|E^2|}{2}}\omega(\alpha)\,\mathrm{inv}(\widehat{f})\prod_{i=1}^{|V^2|} g_{\mathrm{val}(\circ_i)-1}}{[2]_q^n\ [|F_\circ|]_q!\ [|V^2|+|E^2|]_{q^2}!\ [|V^1|-|E^2|]_{q^2}!}$$

where the sum runs over all $\Lambda \in \mathrm{Ob}(\boldsymbol{Graph}_q)/\sim$.

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