Theoretical Investigation on the Resonance Effect in the Acoustics

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Abstract—**In this paper, the resonance interaction at the acoustic vibration caused by the inharmonic vibration is discussed. Based on the perturbation theory, the influences of the resonance interaction between the two and three vibrant levels on the level energy, wave-function, and the spectral line intensity are investigated in detail respectively. The analytical expressions of the vibration level energy, wave-function, and spectral line intensity with the consideration of the resonance interaction are also obtained under the third order inharmonic approximation.**

Keyword-Resonance; Vibration level; Wave-function; Spectral line intensity

I. INTRODUCTION

The analysis and application of acoustic resonance phenomenon had been researched. For example ,Sun Guang-rong in the university of Nanking carried on a research responding to Steady State Frequency Response in rooms—Acoustics of Room and Small Room to indoor steady state frequency [1] . Vera Markovi Etc has carried on theory analysis to the acoustic resonance [2-4]. The intensity and structural of vibrant acoustic spectrum take place to change according to the resonance interaction between the difference vibrant levels. The resonance interaction of the different vibrant levels causes spectral structure complicated and bring for difficulty for analysis of acoustic vibration structure. Such as figure 1 show:
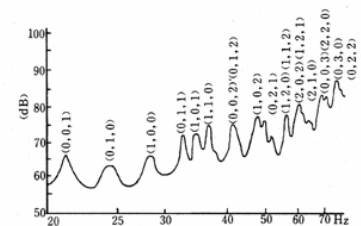

Fig.1. The actually measured resonance effect

Consequently, the theory study of the effect on acoustic spectrum structure with the consideration of the resonance interaction between the vibration levels have important instructional function for the correct analysis of the test of acoustic resonance spectrum .In this paper ,the basic model of the resonance interaction between the three vibration levels on the level energy, wave-function and the spectral line intensity are investigated in theory respectively and are also obtained the express of the theory investigated in detail .

II. THE BASIC THEORY OF RESONANCE INTERACTION

It is the most roughly approximation regarding vibration as harmonic vibration .It belongs to zero classes approximation .Actually, vibration is inharmonic . Therefore in motive Hamiltonian operator, higher order term of normal coordinate Q_i should be considered for potential energy operator. In order to simplify the theory calculation , suppose in these harmonic vibration model Q_1 、 Q_2 and their overtones vibration have homological symmetry and vibrant frequency satisfy $\upsilon_1 \approx 2\upsilon_2$. The potential energy operator is

evolved to three cubed of normal coordinate under the three level inharmonic approximation. Its expression:

$$V = u_e + \frac{1}{2}\sum_k \lambda_k Q_k^2 + \sum_{i,j,k}\alpha_{i,j,k}Q_iQ_jQ_k \qquad (1)$$

Then at this time vibration Hamiltonian can write for:

$$\hat{H} = \hat{H}_0 + \hat{H}' \qquad (2)$$

Among which

$$\hat{H}_0 = -\frac{\eta^2}{2}\sum\frac{\partial^2}{\partial Q_k^2} + \mu_e + \frac{1}{2}\sum_k \lambda_k Q_k^2 \qquad (3)$$

$$\hat{H}' = \sum_{ijk}\alpha_{ijk}Q_iQ_jQ_k$$

$$= \alpha_{1,1,1}Q_1^3 + \alpha_{1,1,2}Q_1^2Q_2 + \alpha_{1,2,2}Q_1Q_2^2 + \alpha_{2,2,2}Q_2^3 \quad (4)$$

α_{ijk} is three ranks inharmonic coefficient, the higher rank number is, the more small inharmonic coefficient is. The H' is a resonance function item, can be regarded as a perturbation item, be adopted as perturbation theory processing, We adopt three ranks inharmonic approximation in concrete calculation. Considering the resonance interaction, state function is the linear addition of the zero function with interaction of vibration state and keep same alike symmetry with the zero function, but make interaction of level energy take place to change, for $\hat{H}_0$ there have

$$\hat{H}_0\psi_n^{(0)} = E_n^{(0)}\psi_n^{(0)} \qquad (5)$$

Then, considering resonance interaction there have

$$\left(\hat{H}_0 + \hat{H}'\right)\psi = E\psi \qquad (6)$$

For solving the equation (6), will launch ψ to the liner combining of zero class state function, then

$$\psi = \sum_n a_n\psi_n^{(0)} \qquad (7)$$

got by the (5, 6, 7)

$$\sum_n\left(E - E_n^{(0)}\right)a_n\psi_n^{(0)} - \hat{H}'\sum_n a_n\psi_n^{(0)} = 0 \quad (8)$$

The equation (8) both sides is left to multiply $\psi_m^{(0)*}$ and integral calculus, can get

$$\sum_n[H'_{mn} - \left(E - E_m^{(0)}\right)\delta_{mn}]a_n = 0 \qquad (9)$$

$$\det\left|H'_{mn} - \left(E - E_m^{(0)}\right)\delta_{mn}\right| = 0 \qquad (10)$$

Among which $\hat{H}'_{mn}$ is the resonance interaction matrix element, from(9) and(10) can get the energy of vibration levels $E_1, E_2, E_3 \Lambda$ and state function $\psi_1, \psi_2, \psi_3, \ldots$.

III.   RESONANCE INTERACTIONS BETWEEN TWO VIBRATION STATE

A   energy level and state function after the interaction

Suppose the state function of vibration energy level (V_1, V_2, V_3) is ψ_1, the state function of vibration energy level $(V_1 + 1, V_2 - 2, V_3)$ is ψ_2, then these two vibrations energy levels are almost degenerate, by equation (10) can get:

$$\begin{vmatrix} E_1^0 + H'_{11} - E & H'_{12} \\ H'_{12} & E_2^0 + H'_{22} - E \end{vmatrix} = 0 \qquad (11)$$

Under three ranks inharmonic approximation there have:

$$\hat{H}'_{11} = H'_{22} = 0 \qquad (12)$$

$$H'_{12} = \langle V_1V_2V_3|H'|V_1 + 1, V_2 - 2, V_3\rangle$$

$$= \alpha_{122}\langle V_1V_2V_3|Q_1Q_2^2|V_1 + 1, V_2 - 2, V_3\rangle$$

$$= \alpha_{122}\left(\frac{1}{2a}\right)^{3/2}\sqrt{(V_1 + 1)(V_2 - 1)V_2} \qquad (13)$$

In equation $a = \frac{4\pi^2 v_e\mu}{\eta}$, v_e is vibrant frequency, μ is reduced mass, $\alpha_{122} = \frac{1}{6}\left(\frac{\partial^3 U}{\partial Q_1\partial Q_2^2}\right)$ is three ranks inharmonic coefficient, from equation (11-13) can get:

$$E_\pm = \frac{1}{2}\left(E_1^0 + E_2^0\right) \pm \frac{1}{2}\sqrt{\left(E_1^0 + E_2^0\right) + 4|H'_{12}|^2} \quad (14)$$

Thus get the abruption value of energy level:

$$W_\chi = \frac{1}{2}\left(\sqrt{4|H'_{12}|^2 + \Delta_0^2} - \Delta_0\right), \ \ \chi = 1,2 \quad (15)$$

Among which Δ_0 Is difference between E_1^o and E_2^o without considering perturbation, From equation (15) we can see the relation between energy level's splitting with resonance matrix element. The change of level energy is shown as below diagram considering interaction, among them Δ is difference between E_+ and $\boldsymbol{E}_-$ for considering perturbation.
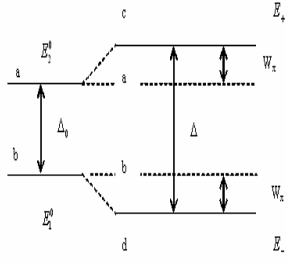
Fig.2. The influence of the resonance interaction on the energy of the two vibration levels

The state function of interaction can empress as linear combination of originally state function .

$$\psi_k = a_k \psi_1^o + b_k \psi_2^0 \text{, (} k=1,2 \text{,)} \quad (16)$$

Substitute the value of equation (14) into equation (9) can get the combination coefficient of state function respectively after

considering interaction .

$$a_k = \left(\frac{H_{12}'^2}{\left(E_k - E_1^0\right)^2 + H_{12}'^2} \right)^{1/2} \quad b_k = \left(\frac{\left(E_k - E_1^0\right)^2}{\left(E_k - E_1^0\right)^2 + H_{12}'^2} \right)^{1/2} \quad \text{(k=1,2)}$$

$$\quad (17)$$

Then there have

$$\frac{a_k}{b_k} = \left| \frac{H_{12}'}{E_k - E_1^0} \right| \quad (18)$$

Through reasonable approximation there have

$$\frac{a_k}{b_k} = \left| \frac{(E_1^0 + E_2^0)H_{12}'}{E_k^0(E_2^0 + E_1^0) + H_{12}'^2} \right| \quad (19)$$

Under the general situation has

$$H_{12}'^2 < E_k^0(E_2^0 + E_1^0) \quad \text{(k=1、2)} \quad (20)$$

Can prove

$$\frac{d}{dH_{12}'}\left(\frac{a_k}{b_k}\right) > 0 \quad (21)$$

Therefore, the more H_{12}' is, the proportion of $\psi_1^{(0)}$ in ψ_1 and ψ_2 increase .

B spectral line intensity considering interaction

Spectral line intensity taking no account of interaction is:

$$I_k^o \propto \left| \int \varphi_0 \varphi_k^0 d\tau \right|^2 \quad (22)$$

Among them φ_0 is lower state for transition, φ_k^0 is upper state for transition, $k = 1,2$, considering interaction spectral line intensity is:

$$I_k \propto \left| \int \varphi_0 \varphi_k d\tau \right|^2$$

$$= \left| \int \varphi_0 \left(a_k \varphi_1^0 + b_k \varphi_2^0\right) d\tau \right|^2$$

$$= a_k^2 I_1^0 + b_k^2 I_2^0 + 2a_k b_k \sqrt{I_1^0 I_2^0}$$

$$= \left(\frac{H_{12}'^2}{\left(E_k - E_1^0\right)^2 + H_{12}'^2} \right) I_1^0 + \left(\frac{\left(E_k - E_1^0\right)^2}{\left(E_k - E_1^0\right)^2 + H_{12}'^2} \right) I_2^0 + \frac{2H_{12}'\left(E_k - E_1^0\right)}{\left(E_k - E_1^0\right)^2 + H_{12}'^2} \sqrt{I_1^0 I_2^0}$$

$$(23)$$

IV. THE INTERACTION OF THREE RESONANCE VIBRATION STATE

A Energy levels and state function after considering interaction

State function of vibration energy level $(V_1 - 1, V_2 + 2, V_3)$ is ψ_1 , state function of vibration energy level (V_1, V_2, V_3) is ψ_3 , these three vibration energy levels are approximately combined and belong to identical homology of the symmetry ,from equation (10) there have

$$\left\| \begin{matrix} E_1^0 + H_{11}' - E & H_{12}' & H_{13}' \\ H_{12}' & E_2^0 + H_{22}' - E & H_{23}' \\ H_{13}' & H_{23}' & E_3^0 + H_{33}' - E \end{matrix} \right\| = 0 \quad (24)$$

Under three ranks inharmonic approximation there have

$$H_{11}' = H_{22}' = H_{33}' = 0 \quad (25)$$

$$H_{12}' = \langle V_1 - 1, V_2 + 2, V_3 | H' | V_1 V_2 V_3 \rangle$$

$$= \alpha_{122} \langle V_1 - 1, V_2 + 2, V_3 | Q_1 Q_2^2 | V_1 V_2 V_3 \rangle$$

$$= \alpha_{122} \left(\frac{1}{2a} \right)^{3/2} \sqrt{V_1(V_2 + 2)V_2} \quad (26)$$

$$H_{23}' = \langle V_1, V_2, V_3 | H' | V_1 + 1, V_2 - 2, V_3 \rangle$$

$$= \alpha_{122} \langle V_1 V_2 V_3 | Q_1 Q_2^2 | V_1 + 1, V_2 - 2, V_3 \rangle$$

$$= \alpha_{122} \left(\frac{1}{2a} \right)^{3/2} \sqrt{(V_1 + 1)(V_2 - 1)V_2} \quad (27)$$

$$H_{13}' = 0 \quad (28)$$

After considering interaction from equation (24-28), we canget energy of the three approximately combined vibration

energy level respectively:

$$E_1 = \sqrt{\frac{-P}{3}}\left(-\cos\frac{\phi}{3} + \sqrt{3}\sin\phi\right) \; ;$$

$$E_2 = 2\sqrt{\frac{-P}{3}}\left(-\cos\frac{\phi}{3}\right) ;$$

$$E_3 = \sqrt{\frac{-P}{3}}\left(-\cos\frac{\phi}{3} - \sqrt{3}\sin\phi\right) \tag{29}$$

Among which

$$\phi = \cos^{-1}\left(\frac{-q}{2\sqrt{-p^3/27}}\right)$$

$$p = \frac{E_1^0 E_2^0 + E_2^0 E_3^0 + E_1^0 E_3^0 - E_1^{2\,0} - E_2^{2\,0} - E_3^{2\,0} - 3H_{12}'^2 - 3H_{23}'^2}{3}$$

$$\tag{30}$$

$$q = \frac{1}{27}\left[3E_1^{2\,0}E_2^0 + 3E_1^{2\,0}E_3^0 + 3E_2^{2\,0}E_3^0 + 3E_1^0 E_2^{2\,0} + 3E_3^{2\,0}E_1^0 + 3E_3^{2\,0}E_2^0 - 12E_1^0 E_2^0 E_3^0\right.$$
$$\left. - 9(H_{12}'^2 + H_{23}'^2)(E_1^0 + E_2^0 + E_3^0) + 27E_1^0 H_{23}'^2 + 27E_3^0 H_{12}'^2 - 2E_1^{3\,0} - 2E_2^{3\,0} - 2E_3^{3\,0}\right]$$

$$\tag{31}$$

The state function of interaction can empress as linercombination of originally state function $\varphi_k = a_k \varphi_1^0 + b_k \varphi_2^0 + c_k \varphi_3^0$. Here, $k = 1,2,3$, Substitute the value of equation (29) into equation (9) can get the combination coefficient of state function respectively after considering interaction .

$$a_k = \left[\frac{(E_k - E_3^0)^2 H_{12}'^2}{(E_k - E_3^0)^2 H_{12}'^2 + (E_1^0 - E_k)^2(E_3^0 - E_k)^2 + (E_1^0 - E_k)^2 H_{23}'^2}\right]^{1/2}$$

$$b_k = \left[\frac{(E_1^0 - E_k)^2(E_3^0 - E_k)^2}{(E_k - E_3^0)^2 H_{12}'^2 + (E_1^0 - E_k)^2(E_3^0 - E_k)^2 + (E_1^0 - E_k)^2 H_{23}'^2}\right]^{1/2}$$

$$\text{(k=1,2,3)} \tag{32}$$

$$c_k = \left[\frac{(E_1^0 - E_k)^2 H_{23}'^2}{(E_k - E_3^0)^2 H_{12}'^2 + (E_1^0 - E_k)^2(E_3^0 - E_k)^2 + (E_1^0 - E_k)^2 H_{23}'^2}\right]^{1/2}$$

A series of formula of above can simply write:

$$a_k = \left[\frac{m_k^2 H_{12}'^2}{m_k^2 H_{12}'^2 + n_k^2 m_k^2 + n_k^2 H_{23}'^2}\right]^{1/2} ;$$

$$b_k = \left[\frac{m_k^2 n_k^2}{m_k^2 H_{12}'^2 + n_k^2 m_k^2 + n_k^2 H_{23}'^2}\right]^{1/2} ; \tag{33}$$

$$c_k = \left[\frac{n_k^2 H_{23}'^2}{m_k^2 H_{12}'^2 + n_k^2 m_k^2 + n_k^2 H_{23}'^2}\right]^{1/2}$$

Among which $m_k = E_k - E_3^0$; $n_k = E_1^0 - E_k$;
$$k = 1,2,3$$

B   spectral line intensity considering interaction

Spectral line intensity taking no account of interaction is:

$$I_k^0 \propto \left|\int \varphi_0 \varphi_k^0 d\tau\right|^2$$

Among them φ_0 is lower state for transition, φ_k^0 is upper state for transition , $k = 1,2,3$, considering interaction spectral line intensity is:

$$I_k \propto \left|\int \varphi_0 \varphi_k d\tau\right|^2$$

$$= \left|\int \varphi_0 \left(a_k \varphi_1^0 + b_k \varphi_2^0 + c_k \varphi_3^0\right)d\tau\right|^2$$

$$= a_k^2 I_1^0 + b_k^2 I_2^0 + c_k^2 I_3^0 + 2a_k b_k\sqrt{I_1^0 I_2^0} + 2a_k c_k\sqrt{I_1^0 I_3^0} + 2b_k c_k\sqrt{I_2^0 I_3^0} =$$

$$\left[\frac{m_k^2 H_{12}'^2}{m_k^2 H_{12}'^2 + n_k^2 m_k^2 + n_k^2 H_{23}'^2}\right]I_1^0 + \left[\frac{m_k^2 n_k^2}{m_k^2 H_{12}'^2 + n_k^2 m_k^2 + n_k^2 H_{23}'^2}\right]I_2^0$$

$$+ \left[\frac{n_k^2 H_{23}'^2}{m_k^2 H_{12}'^2 + n_k^2 m_k^2 + n_k^2 H_{23}'^2}\right]I_3^0 + \frac{2m_k^4 n_k^2 H_{12}'^2}{m_k^2 H_{12}'^2 + m_k^2 n_k^2 + n_k^2 H_{23}'^2}\sqrt{I_1^0 I_2^0}$$

$$+ \frac{2m_k^2 n_k^2 H_{12}'^2 H_{23}'^2}{m_k^2 H_{12}'^2 + m_k^2 n_k^2 + n_k^2 H_{23}'^2}\sqrt{I_1^0 I_3^0} + \frac{2m_k^2 n_k^4 H_{23}'^2}{m_k^2 H_{12}'^2 + m_k^2 n_k^2 + n_k^2 H_{23}'^2}\sqrt{I_2^0 I_3^0}$$

$$\tag{34}$$

Among which $m_k = E_k - E_3^0$, $n_k = E_1^0 - E_k$
（ $k = 1,2,3$ ）

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