Two-parameter Alteration of B-splines

Yajuan Li
School of Science
Hangzhou Dianzi University
Hangzhou, China,310018

Mingzeng Yang
Mathematics department
Henan Institute of Education
Zhengzhou, China, 450014

Abstract—Altering one or more knots of B-spline curves and surfaces, points of curves and surfaces will move on well defined curves called paths. Letting the knot value tend to infinity, the limit of the paths will be computed in this paper. The effect of two-parameter alteration of B-splines is discussed.

I. INTRODUCTION

A B-spline curve is the composition of a number of curve segments, each of which is defined on a knot span[1]. As is well known, B-spline curves and surfaces(and their rational forms-NURBS curves and NURBS surfaces) are widely used in Computer Aided Geometric Design and Computer Graphics(CAGD&CG). In the designing process, one of the most important thing is modification of the shape of the model. Due to the popularity of the B-spline and NURBS models, there are many methods to modify the shape of the B-splines[2-12], among which modifying the knots is an important way.

Modifying knots of B-splines, the association between curve segments and knot spans will be changed, and hence the shape of the curve will be modified. For example, changing one or more knots of a B-spline curve, points of the curve will move on well defined curves called paths[6]. There are some publications studied the effect of the alteration of one or more knots on the shape of the curve[6-10]. By these theories, some authors worked on the constraint-based shape modification of B-spline curves and surfaces and obtained some interesting results on the restricted part of the path.

In this paper, we present some theorems describing the effect of alteration of knots of B-spline and NURBS surfaces based on the modification of two parameters.

This paper is organized as follows: In Section 2, the basic definitions and notations will be presented. Then in Section 3, we present the effect of the modification of two knots of NURBS curves by one parameter. Alteration of knots of tensor surfaces by two parameter will be given in Section 4 and 5.

II. MODIFY THE KNOTS OF B-SPLINES

A. B-SPLINES

A B-spline is a generalization of the Bézier curve. Let a vector known as the knot vector be defined \( U = \{u_l\}_{-\infty}^{+\infty} \), where \( U \) is a nondecreasing sequence. The basic definitions of basis functions are the followings.

Definition 1. Let \( U = \{u_l\}_{-\infty}^{+\infty} \) be a given knot sequence with \( u_l \leq u_{l+1} \), the normalized B-spline basis functions of order \( k \) (degree \( k - 1 \)) are defined recursively by the following equations:

\[
N_{l,k}(u) = \left\{ \begin{array}{ll}
1, & \text{if } u \in [u_l, u_{l+1}) \\
0, & \text{otherwise.}
\end{array} \right.
\]

\[
N_{l,k}(u) = \frac{u-u_l}{u_{l+k-1}-u_l} N_{l,k-1}(u) + \frac{u_{l+k}-u}{u_{l+k}-u_{l+1}} N_{l+k-1,k-1}(u)
\]

Here we define \( 0/0 = 0 \). Define control points \( p_l, \) \( l = 0, \) \( 1, \) \( \cdots, \) \( n \), then the curve defined by

\[
r(u) = \sum_{l=0}^{n} N_{l,k}(u)p_l, \quad u \in [u_{k-1}, u_n].
\]

is a B-spline curve. The ith arc can be written as

\[
r_i(u) = \sum_{l=i}^{n} N_{l,k}(u)p_l, \quad u \in [u_i, u_{i+k}],
\]

\( k-1 \leq i \leq n \).

Defining weights \( w_l \), we get the definition of Non Uniform Rational B-spline(NURBS) curve

\[
r(u) = \sum_{l=0}^{n} N_{l,k}(u)w_l p_l, \quad u \in [u_{k-1}, u_n].
\]

The definition of B-spline curve and NURBS curve can be generalized to the tensor product surfaces:

Definition 2. The surface \( s(u, v) \) defined by

\[
s(u, v) = \sum_{l=0}^{n} \sum_{g=0}^{r} N_{l,g,k}(u)N_{g,h}(v)p_{l,g},
\]

\( u \in [u_{k-1}, u_n], \quad v \in [v_{h-1}, v_m] \)

is called B-spline surface of order \( k \times h \), where \( \text{Nl}, k(u) \) and \( \text{Ngh}, h(v) \) are B-spline basis functions of order \( k \) and \( h \), which defined on given knot sequences \( U = \{u_l\}_{-\infty}^{+\infty} \) and \( V = \{v_g\}_{-\infty}^{+\infty} \) respectively, and \( p_{l,g} \) are control points. The \( i \times j \)th patch can be written as
s_r(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} N_{i,k}(u) N_{j,l}(v) p_{i,j},

Define weights w_{lg}, we have the definition of the NURBS surface

s(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} N_{i,k}(u) N_{j,l}(v) w_{lg} p_{i,j}

(1)

B. One-parameter alteration of B-spline curves

The knot vector is a nondecreasing sequence, so when a knot u_i is altered within the defined range [u_i-1, u_i+1], points of the B-spline curve will move on special rational curves s(u, v) called paths[6]. When we modify two knots u_i and u_j , i < j to the values u_i + t\lambda and u_j - t\lambda, i.e. (See Fig. 1)

\lim_{\lambda \to -\infty} r(u, t, \lambda) = p_j + (1-t) p_{j+2}, u \in [u_i, u_i+z), t \in [0, 1].

III. ONE-PARAMETER ALTERATION OF NURBS CURVES

By a parameter t \in [0, 1], we can modify two knots of a NURBS curve in the same way. The extended paths is obtained:

r(u, t, \lambda) = \sum_{i=0}^{n} N_{i,k}(u, t, \lambda) w_{i,j} p_{i,j}, u \in [u_{i-1}, u_{i+1}].

Denoting wpl by ql and letting \lambda \to \infty, by Lemma 1, we have

\lim_{\lambda \to -\infty} \sum_{i=0}^{n} N_{i,k}(u, t, \lambda) q_{i,j} = t q_j + (1-t) q_{j+2}, t \in [0, 1].

Thus we have the following proposition:

Theorem 1 Modifying the knots u_i = u_i + t\lambda and u_j = u_i + z-k \to u_i + t\lambda and u_j = u_i + z-k, i.e. (See Fig. 1)

\lim_{\lambda \to -\infty} r(u, t, \lambda) = t p_j + (1-t) p_{j+2}, u \in [u_i, u_i+z), t \in [0, 1].

IV. TWO-PARAMETER ALTERATION OF B-SPLINE SURFACES

Modifying four knots of a B-spline surface by two parameters t and f, 0 \leq t, f \leq 1, that is, changing the values of u_i, u_i+a, (a=1,2, ..., k) and v_j, v_j+b, (b=1,2, ..., h) to u_i + t\lambda and v_j + f\mu respectively, the extended paths of patches s(u, v, f, \lambda, \mu) can be expressed as:

s(u, v, t, f, \lambda, \mu) = \sum_{i=0}^{n} \sum_{j=0}^{m} N_{i,k}(u, t, \lambda) N_{j,l}(v, f, \mu) p_{i,j}

Here we give the following proposition:

Theorem 2. Altering four knots u_i, u_i+a, (a=1, 2, ..., k) and v_j, v_j+b, (b=1, 2, ..., h) of a B-spline surface by two parameters t, f, the extended paths of patches are

\lim_{\lambda \to -\infty} s(u, v, t, f, \lambda, \mu) = (t p_j + (1-t) p_{j+2}) + (1-f) q_{j+2} + (1-f) q_{j+3}.

(2)

Proof: From Section 2 of [15], we know that the following equations hold for u \in [u_i, u_i+a):

\lim_{\lambda \to -\infty} N_{i,k}(u, t, \lambda) = t, \lim_{\lambda \to -\infty} N_{i,k+1}(u, t, \lambda) = 1-t, a = 1, ..., k-1

\lim_{\lambda \to -\infty} N_{i,k+1}(u, t, \lambda) = 0, a = k.

It is the same for the v-direction:

\lim_{\lambda \to -\infty} N_{i,k}(v, f, \mu) = f, \lim_{\lambda \to -\infty} N_{i,k+1}(v, f, \mu) = 1-f, b = 1, ..., h-1.
\[ \lim_{\mu \to -\infty} N_{j,k}(v, f, \mu) = \lim_{\nu \to \infty} N_{j,h-k}(v, f, \mu) = 1, \quad b = h \]

The above equations yield (2) (See Fig. 2).

Therefore, we get the theorem for the two-parameter alteration of four knots of NURBS surfaces:

**Theorem 3.** Altering four knots \( u_i, u_{i+a}, (a=1, 2, \ldots, k) \) and \( v_j, v_{j+b}, (b=1, 2, \ldots, h) \) of a NURBS surface by two parameter \( t, f, 0 \leq t, f \leq 1 \), the extended paths of points of patches \( s(u, v, t, f, \lambda, \mu) \) converge to a point when \( \lambda \to -\infty \) and \( \mu \to -\infty \), that is:

\[
\begin{align*}
&\lim_{\lambda \to -\infty} s(u, v, t, f, \lambda, \mu) \\
&\quad = \frac{t(q_l + (1-f)b)w_{l,j+b} + (1-t)(q_{l+a} + (1-f)b)w_{l+a,j+b}}{t(fw_{l,1} + (1-f)b)w_{l,j+b} + (1-t)(fw_{l+a,1} + (1-f)b)w_{l+a,j+b}}
\end{align*}
\]

where \( q_l = w_{lg} q_{lg}, l = i, i+a-k, g = j, j+b-h \).

**ACKNOWLEDGMENT**

This work was Supported by the National Natural Science Foundation of China (No.60970079) and National Natural Science Foundation of China, Tian Yuan Special Foundation (No.11026107).

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