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Isomorphisms of Maximal Subsemigroups of D-classes of Finite Full Transformation Semigroup

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***Abstract*—In this paper, by using *Green*'s equivalences, we obtaine isomorphisms of the maximal subsemigroups of *Green*-classes(*R*-classes, *L*-classes and *D*-classes).**

Keywords—green-classes; maximal subsemigroups; isomorphism

I. Introduction and Main Result

Let T_n be the finite full transformation semigroup on the set $X_n = \{1, 2, \cdots, n\}(n \geq 3)$, from the Gomes and Howie[1], the Green equicvalences in T_n can be characterized as:

$$\alpha L\beta \Leftrightarrow im\alpha = im\beta;$$

$$\alpha R\beta \Leftrightarrow \ker\alpha = \ker\beta;$$

$$\alpha H\beta \Leftrightarrow im\alpha = im\beta, \ker\alpha = \ker\beta,$$

where $\alpha, \beta \in D_r$, it is well known that T_n has exactly n *D*-classes. The *D*-classes of T_n have form $D_r = \{\alpha \in T_n : |im\alpha| = r\}$, for $1 \leq r \leq n$. Especially, when $r = n$, D_r denotes as a symmetric group on X_n. That is to say, D_r consists of all bijections on X_n.

If ϕ is a mapping from a semigroup S into a semigroup T, we say that ϕ is a homomorphism if $\forall x, y \in S, \phi(x)\phi(y) = \phi(xy)$ .And if ϕ is both one-one and onto,we shall call it an isomorphism.

In recent years, many scholars have done lots of research deeply on full transformation semigroups and their subsemigroups[2-7]. Especially, In 1998 chein.B. M. and Teclezghi. B. in [2] have studied endomorphisms of finite full transformation semigroups. Tang and Yang in [3,4] have described homomorphisms of two finite full transformation semigroups. But isomorphism of transfor-mation semigroups has been studied rarely. We have obtained the structure of the maximal subsemigroups of *Green*-classes(*R*-classes, *L*-classes and *D*-classes) of T_n, and showed the maximal subsemigroups of *R*-classes and *L*-classes are also the maximal subsemigroups of *D*-classes(see[5,6]). In this paper, we consider the isomorphism of maximal subsemigroups of *D*-classes.

The main result can be described as follows:

Theorem 1.1. Let S, T be arbitrary maximal subsemigroups of *R*-classes in $D_r (2 \leq r \leq n-1)$. If $|E(S)| = |E(T)|$ (denote $E(S)$ the set of all idempotents of S), then $S \cong T$.

Theorem 1.2. Let S, T be arbitrary maximal subsemigroups of *L*-classes in $D_r (2 \leq r \leq n-1)$. If $|E(S)| = |E(T)|$, then $S \cong T$.

Theorem 1.3. Let S, T be arbitrary maximal subsemigroups in the same *D*-class. For each $\alpha \in S, \beta \in T$, we have

$$S \cong T \Leftrightarrow |E(R_\alpha \cap S)| = |E(R_\beta \cap T)|, |E(L_\alpha \cap S)| = |E(L_\beta \cap T)|.$$

Denote R_α the *R*-class containing the element α.

This paper is organized as follow: in Section 2, we give some lemmas which will be used to prove out main result, and the proof is given in Section 3.

II. Some Preliminary Results

In the section, we give some lemmas in preparation for the proof of our main result. Definitions and terms that are not defined in this article see[1].

Lemma 2.1. Let a be an element of a regular *D*-class D in a semigroup S.

(1)If $a' \in V(a)$ (the set of inverse of a), then $a' \in D$ and the two *H*-classes $R_a \cap L_{a'}, L_a \cap R_{a'}$ contain, respectively, the idempotent aa' and $a'a$.

(2)If b in D is such that $R_a \cap L_b$ and $L_a \cap R_b$ contain idempotents e, f, respectively, then H_b contains an inverse a^* of a such that $aa^* = e, a^*a = f$.

(3)No *H*-class contains more than one inverse of a.

Proof. See [1].

The following result usually known as *Green*'s Lemma:

Lemma 2.2.[1] Let a,b be *R*-equivalent elements in a semigroups, and let s,s' in $S^1\left(S\bigcup\{1\}\right)$ be such that

$$as=b, bs'=a.$$

Then the right translations $\rho_s\left|L_a, \rho_{s'}\right|L_b$ are mutually inverse *R*-class preserving bijections from L_a onto L_b and L_b onto L_a, respectively.

Lemma 2.3.[1] Let a,b be *L*-equivalent elements in a semigroups, and let t,t' in S^1 be such that

$$at=b, bt'=a.$$

Then the left translations $\lambda_t\left|R_a, \lambda_{t'}\right|R_b$ are mutually inverse *L*-class preserving bijections from R_a onto R_b and R_b onto R_a, respectively.

By Lemma 2.2 and 2.3, let $H=H_e$ and $K=H_f$ are two *H*-classes in the same *R*-class, where $e\in E(H)$, $f\in E(K)$, then ρ_f is an isomorphism from H_e onto H_f, with inverse ρ_e from H_f onto H_e. Similarly, if $H=H_g$ and $K=H_h$ are two *H*-classes in the same *L*-class, where $g\in E(H)$, $h\in E(K)$, then λ_g is an isomorphism from H_h onto H_g, with inverse λ_h from H_g, onto H_h.

Lemma 2.4. Let S,T be arbitrary finite semigroups. If $S\cong T$, then $\left|E(S)\right|=\left|E(T)\right|$.

Proof. If φ be isomorphism from S onto T, then $\varphi(E(S))\subseteq E(T)$, since S,T be finite semigroups, we have $\left|E(S)\right|=\left|\varphi(E(S))\right|\le\left|E(T)\right|$. Similarly, we have $\left|E(T)\right|\le\left|E(S)\right|$, further $\left|E(T)\right|=\left|E(S)\right|$.

Lemma 2.5. Let S,T be arbitrary semigroups, a map $\varphi: S\to T$ be semigroup morphism. Then we have

$$aRb\Leftrightarrow\varphi(a)R\varphi(b), \forall a,b\in S; \quad (1)$$

$$aLb\Leftrightarrow\varphi(a)L\varphi(b), \forall a,b\in S; \quad (2)$$

$$e\in E(S)\Rightarrow\varphi(e)\in E(S). \quad (3)$$

Proof. (1)Suppose aRb, there exist $u,v\in S$ such that $a=bu, b=av$, then

$$\varphi(a)=\varphi(bu)=\varphi(b)\varphi(u),\ \varphi(b)=\varphi(av)=\varphi(a)\varphi(v).$$

Therefore $\varphi(a)R\varphi(b)$.

(2)Similar to (1);

(3)Let $e\in E(S)$, then $\varphi(e)=\varphi(ee)=\varphi(e)\varphi(e)$, thus $\varphi(e)\in E(S)$.

Therefore, one of the necessary conditions about isomorphism of maximal subsemigroups of D_r is that the number of idempotents is equal.

III. PROOF OF THEOREM

In this section, we give the proof of Theorem 1.1,1.2 and 1.3, and study the isomorphism of maximal subsemigroups of *R*-classes, *L*-classes and *D*-classes.

Proof of Theorem 1.1. Let

$$S=\bigcup_{i=1}^{r}H_{e_i}, T=\bigcup_{i=1}^{r}H_{f_i}\ (2\le r\le n-1),$$

where $e_i\in E(S), f_i\in E(T)$. Let $\varphi_1: H_{e_1}\to H_{f_1}$, $x\to a'xa$. By Lemma 3.1, there exist $a\in R_{e_1}\bigcap L_{f_1}$, $a'\in L_{e_1}\bigcap R_{f_1}$ such that $aa'=e_1, a'a=f_1$. Then we have that φ_1 is an isomorphism from H_{e_1} onto H_{f_1}.

Let

$$\rho_{e_i}: H_{e_1}\to H_{e_i}, x\to xe_i,\ \rho_{f_i}: H_{f_1}\to H_{f_i}, x\to xf_i.$$

Then ρ_{e_i},ρ_{f_i} are isomorphic from H_{e_1} onto H_{e_i} and H_{f_1} onto H_{f_i}, respectively.

Thus the map $\varphi_i=\rho_{e_i}^{-1}\varphi_1\rho_{f_i}\ (i\ne1)$ is an isomorphism from H_{e_i} onto H_{f_i}.

Give a function from S to T

$$g(x)=\begin{cases}\varphi_1(x) & x\in H_{e_1}\\ \varphi_i(x) & x\in H_{e_i}(i\ne1)\end{cases}.$$

It's easy to see that g is a bijection.

For arbitrary $x_1\in H_{e_1}, x_2\in H_{e_i}(i\ne1)$, we have

$$g(x_1x_2)=\varphi_i(x_1x_2)=(x_1x_2)\rho_{e_i}^{-1}\varphi_1\rho_{f_i}=a'x_1x_2e_1af_i=a'x_1x_2af_i,$$

$$g(x_1)g(x_2)=\varphi_1(x_1)\varphi_i(x_2)=a'x_1aa'x_2e_1af_i$$
$$=a'x_1e_1x_2af_i=a'x_1x_2af_i.$$

Thus $g(x_1x_2)=g(x_1)g(x_2)$, similarly, we have

$$g(x_2x_1)=g(x_2)g(x_1).$$

For arbitrary $x_1\in H_{e_i}, x_2\in H_{e_j}\ (i\neq j, i, j\neq 1)$,

we have

$$g(x_1x_2)=\varphi_j(x_1x_2)=(x_1x_2)\rho_{e_j}^{-1}\varphi_1\rho_{f_j}$$
$$=a'x_1x_2e_1af_j=a'x_1x_2af_j,$$

$$g(x_1)g(x_2)=\left(x_1\rho_{e_i}^{-1}\varphi_1\rho_{f_i}\right)\left(x_2\rho_{e_j}^{-1}\varphi_1\rho_{f_j}\right)$$
$$=a'x_1e_1af_ia'x_2e_1af_j=a'x_1af_ia'x_2af_j$$
$$=a'x_1aa'x_2af_j=a'x_1x_2af_j.$$

Thus $g(x_1x_2)=g(x_1)g(x_2)$.

Therefore, g is an isomorphism from S onto T.

The proof of Theorem 1.2 is similarly to Theorem 1.1.

Proof of Theorem 1.3. Firstly, we shall prove the sufficient condition.

Since the structural of S,T and for arbitrary $e\in E(S)$, $f\in E(T)$, we have

$$\left|E\left(L_e\cap S\right)\right|=\left|E\left(L_f\cap T\right)\right|, \left|E\left(R_e\cap S\right)\right|=\left|E\left(R_f\cap T\right)\right|.$$

Then the arrangement of idempotents of S and T can be seen as two matrix with equal rank, we denote them by A_{kl} and B_{kl}, where e_{ij}, f_{ij} standing the i row and j column in A_{kl} and B_{kl} respectively, and

$$e_{ij}\in E\left(A_{kl}\right), f_{ij}\in E\left(B_{kl}\right).$$

Similar to Theorem 1.1, define an isomorphism

$$\varphi_{ij}:H_{e_{ij}}\to H_{f_{ij}}.$$

If $i=1, j=1$,

$$\varphi_{11}(x)=a'xa,$$

where

$$a\in R_{e_{11}}\cap L_{f_{11}}, a'\in R_{f_{11}}\cap L_{e_{11}},\ aa'=e_{11},\ a'a=f_{11},$$

it's easy to see by Lemma 2.1.

If $i=1, j\neq 1$,

$$\varphi_{1j}=\rho_{e_{1j}}^{-1}\varphi_{11}\rho_{f_{1j}},$$

where $\rho_{e_{1j}}^{-1}=\rho_{e_{11}}$.

If $i\neq 1, j=1$,

$$\varphi_{i1}=\lambda_{e_{i1}}^{-1}\varphi_{11}\lambda_{f_{i1}},$$

where $\lambda_{e_{i1}}^{-1}=\lambda_{e_{11}}$.

If $i\neq 1, j\neq 1$,

$$\varphi_{ij}=g_1^{-1}\varphi_{11}g_2,$$

where

$$g_1:H_{e_{11}}\to H_{e_{ij}}, x\to x\rho_{e_{1j}}\lambda_{e_{ij}},\ g_2:H_{f_{11}}\to H_{f_{ij}}, x\to x\rho_{f_{1j}}\lambda_{f_{ij}}.$$

Thus

$$\varphi_{ij}=\rho_{e_{i1}}\lambda_{e_{11}}\varphi_{11}\lambda_{f_{i1}}\rho_{f_{ij}}.$$

Further, we can give a map $g:S\to T$:

$$g(x)=\begin{cases}a'xa, & x\in H_{e_{11}},\\ \varphi_{ij}(x), & x\in H_{e_{ij}}.\end{cases}$$

It is easy to show that g is a bijection from S onto T. Next, we distinguish 6 cases:

Case 1. For arbitrary

$$x_1\in H_{e_{11}}, x_2\in H_{e_{1j}}\ (j\neq 1),$$

then

$$g\left(x_1x_2\right)=g\left(x_1\right)g\left(x_2\right), g\left(x_2x_1\right)=g\left(x_2\right)g\left(x_1\right)$$

by Theorem 1.1.

For arbitrary

$$x_1 \in H_{e_{11}}, x_2 \in H_{e_{i1}} (i \neq 1),$$

similarly, we have

$$g(x_1x_2) = g(x_1)g(x_2), g(x_2x_1) = g(x_2)g(x_1).$$

Case 2. For arbitrary

$$x_1 \in H_{e_{11}}, x_2 \in H_{e_{ij}} (i, j \neq 1),$$

then

$$g(x_1x_2) = \varphi_{1j}(x_1x_2) = a'x_1x_2e_{11}af_{1j} = a'x_1x_2af_{1j},$$
$$g(x_1)g(x_2) = a'x_1af_{i1}a'e_{11}x_2e_{i1}af_{ij}$$
$$= a'x_1aa'e_{11}x_2e_{ij}e_{11}af_{i1}f_{1j}$$
$$= a'x_1(x_2e_{ij})(e_{11}af_{i1})f_{1j}$$
$$= a'x_1x_2af_{1j}.$$

Thus, $g(x_1x_2) = g(x_1)g(x_2)$. Similarly,

$$g(x_2x_1) = g(x_2)g(x_1).$$

Case 3. For arbitrary

$$x_1 \in H_{e_{1j}}, x_2 \in H_{e_{i1}} (i, j \neq 1),$$

then

$$g(x_1x_2) = \varphi_{11}(x_1x_2) = a'x_1x_2a,$$
$$g(x_1)g(x_2) = a'x_1e_{11}af_{1j}f_{i1}a'e_{11}x_2a$$
$$= a'x_1af_{11}a'e_{11}x_2a$$
$$= a'x_1e_{11}x_2a$$
$$= a'(x_1e_{1j})(e_{i1}x_2)a$$
$$= a'x_1x_2a.$$

Thus, $g(x_1x_2) = g(x_1)g(x_2)$. Similarly,

$$g(x_2x_1) = g(x_2)g(x_1).$$

Case 4. For arbitrary

$$x_1 \in H_{e_{1m}}, x_2 \in H_{e_{ij}} (m, i, j \neq 1),$$

then

$$g(x_1x_2) = \varphi_{1j}(x_1x_2) = a'x_1x_2e_{11}af_{1j} = a'x_1x_2af_{1j},$$
$$g(x_1)g(x_2) = a'x_1e_{11}af_{1m}f_{i1}a'e_{11}x_2e_{i1}af_{ij}$$
$$= a'x_1af_{11}a'x_2e_{ij}e_{11}af_{i1}f_{1j}$$
$$= a'x_1e_{11}x_2e_{ij}(e_{11}a)f_{1j}$$
$$= a'(x_1e_{1m})(e_{i1}x_2e_{ij})af_{1j}$$
$$= a'x_1x_2af_{1j}.$$

Thus, $g(x_1x_2) = g(x_1)g(x_2)$. Similarly,

$$g(x_2x_1) = g(x_2)g(x_1).$$

Case 5. For arbitrary

$$x_1 \in H_{e_{n1}}, x_2 \in H_{e_{ij}} (n, i, j \neq 1),$$

then

$$g(x_1x_2) = \varphi_{nj}(x_1x_2) = f_{n1}a'e_{11}x_1x_2e_{n1}af_{nj} = f_{n1}a'x_1x_2e_{nj}e_{11}af_{nj}$$
$$= f_{n1}a'x_1x_2af_{n1}f_{1j} = f_{n1}a'x_1x_2af_{1j},$$
$$g(x_1)g(x_2) = f_{n1}a'e_{11}x_1af_{i1}a'e_{11}x_2e_{i1}af_{ij}$$
$$= f_{n1}a'x_1aa'(x_2e_{ij})(e_{11}af_{n1})f_{1j} = f_{n1}a'x_1x_2f_{1j}.$$

Thus, $g(x_1x_2) = g(x_1)g(x_2)$. Similarly,

$$g(x_2x_1) = g(x_2)g(x_1).$$

Case 6. For arbitrary

$$x_1 \in H_{e_{ij}}, x_2 \in H_{e_{mn}} (i, j, m, n \neq 1),$$

then

$$g(x_1x_2) = \varphi_{in}(x_1x_2) = f_{i1}a'e_{11}x_1x_2e_{i1}af_{in}$$
$$= f_{i1}a'(x_1x_2)e_{in}(e_{11}af_{i1})f_{1n} = f_{i1}a'x_1x_2af_{1n},$$
$$g(x_1)g(x_2) = f_{i1}a'e_{11}x_1e_{i1}af_{ij}f_{m1}a'e_{11}x_2e_{m1}af_{mn}$$
$$= f_{i1}a'(x_1e_{ij})(e_{11}a)f_{ij}f_{m1}a'(x_2e_{mn})(e_{11}a)f_{mn}$$
$$= f_{i1}a'x_1(af_{i1})a'x_2(af_{m1})f_{1n} = f_{i1}a'x_1e_{11}x_2af_{1n}$$
$$= f_{i1}a'(x_1e_{1j})(e_{m1}x_2)af_{1n} = f_{i1}a'x_1x_2af_{1n}.$$

Thus,

$$g(x_1x_2) = g(x_1)g(x_2).$$

From the discussion above, we have shown that g is an isomorphism from S onto T, thus we have shown the sufficient condition.

Next, we shall show the necessary condition. Let φ be a semigroup isomorphism from S onto T. Then for arbitrary $e \in E(S)$, by Lemma 2.5 we have

$$|E(R_e \cap S)| \leq |E(R_{\varphi(e)} \cap T)|, \ |E(L_e \cap S)| \leq |E(L_{\varphi(e)} \cap T)|.$$

For arbitrary $\alpha \in S, \beta \in T$, by Lemma 2.4 we deduce

$$|E(R_e \cap S)| = |E(R_\alpha \cap S)|, \ |E(L_e \cap S)| = |E(L_\alpha \cap S)|,$$

$$\left|E\left(R_{\varphi(e)} \cap T\right)\right| = \left|E\left(R_{\beta} \cap T\right)\right|,\ \left|E\left(L_{\varphi(e)} \cap T\right)\right| = \left|E\left(L_{\beta} \cap T\right)\right|.$$

Further,

$$\left|E\left(R_{\alpha} \cap S\right)\right| \le \left|E\left(R_{\beta} \cap T\right)\right|,\ \left|E\left(L_{\alpha} \cap S\right)\right| \le \left|E\left(L_{\beta} \cap T\right)\right|.$$

Notice that φ^{-1} is an isomorphism from T onto S, thus in the same way we have

$$\left|E\left(R_{\alpha} \cap S\right)\right| \ge \left|E\left(R_{\beta} \cap T\right)\right|,\ \left|E\left(L_{\alpha} \cap S\right)\right| \ge \left|E\left(L_{\beta} \cap T\right)\right|,$$

for arbitrary $\alpha \in S, \beta \in T$.

Hence,

$$\left|E\left(R_{\alpha} \cap S\right)\right| = \left|E\left(R_{\beta} \cap T\right)\right|,\ \left|E\left(L_{\alpha} \cap S\right)\right| = \left|E\left(L_{\beta} \cap T\right)\right|.$$

We have shown the necessary condition.

From the discussion above, the proof of Proposition 1.3 is complete.

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