On a knowledge measure and an unorthodox accuracy measure of an intuitionistic fuzzy set(s) with their applications

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Abstract
A measure of knowledge may be viewed as a dual measure of entropy in a fuzzy system; thus, it appears that the less entropy may always accompany the greater amount of knowledge. In this paper, we propose a novel measure of knowledge for an intuitionistic fuzzy set (IFS) through an axiomatic approach. We investigate the effectiveness of the proposed knowledge measure through some comparative studies with some existing entropy measures and knowledge measures of IFS. We also introduce one parametric generalized version of this knowledge measure. This paper also provides application of the proposed knowledge measures in multi-attribute decision-making (MADM). We also give two characterization results to obtain a general framework for defining new knowledge measures. Further, similarity and dissimilarity measures may also be viewed as dual concepts to deal with the problems related to pattern recognition. In this paper, we provide an accuracy measure of an intuitionistic fuzzy set relative to a given intuitionistic fuzzy set. The proposed accuracy measure seems to serve as an effective alternative to similarity and dissimilarity measures in some pattern recognition problems. We also give proof of some properties related to accuracy measure.

Keywords: Intuitionistic fuzzy sets, Knowledge measure, MADM, Accuracy, Pattern recognition.

1. Introduction
There are many concepts in real life which comprise vague and imprecise information. Zadeh’s¹ idea of fuzziness provided the quantified approach to deal with this vagueness and imprecision. The representation of vagueness associated with a member of universe of discourse was done in terms of degree of membership. Atanassov²−⁴ generalized the Zadeh’s notion of fuzzy set. Atanassov² added a degree of non-membership with each member of the universe of discourse. The theoretical extension of fuzzy sets to intuitionistic fuzzy sets (IFSs) had not been a difficult job. However, the pragmatic aspect of IFS had been sought and justified by various authors in the last three decades. It had been quite crucial that how the association of non-membership degree with an element of the universe of discourse finds its significance in real life problems? We can give the answer of this question as follows:

Consider the case of the bird flu and an expert thinks that this type of flu is found in migratory and resident birds. The expert assigns a degree of association of the bird flu to migratory birds and resident birds as 0.6 and 0.4 respectively. In this case, the expert has also considered birds who have never developed the bird flu to both the above mentioned categories. Therefore, for more accurate results the categories migratory birds and resident birds must be provided with degree of membership

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and degree of non-membership to the bird flu case. In this situation, suppose degree of membership and non-membership of migratory birds is 0.55 and 0.34 respectively then the value 1-0.55-0.34 = 0.11 accounts for those migratory birds among which the bird flu had never been observed. In IFS terminology the value 1 - (membership) - (non-membership) is called hesitation degree or hesitation margin (indeterminacy degree). Thus, IFSs theory seems to provide deep insight into vague and imprecise data.

De-Luca Termini introduced an axiomatic definition of fuzzy entropy. Yager obtained fuzzy entropy from distance between the fuzzy set and its complement. Since then, lots of work has been done by the researchers for generalization of fuzzy sets and its applications. In this paper, first, we deal with the amount of knowledge associated with an IFS. In particular, the knowledge measure may be used to tackle some problems in artificial intelligence which may be difficult to handle by using fuzzy entropy alone, such as making the distinction between the cases in which there are a large number of arguments in favor but an equally large number of arguments are not in favor at the same time.

Intuitively, the entropy of an intuitionistic fuzzy set is conceived as a dual measure of the amount of knowledge contained in the intuitionistic fuzzy set. In decision-making problems, the entropy of intuitionistic fuzzy set may not be a satisfactory measure of knowledge (Szmidt et al., Szmidt and Kacprzyk). Szmidt et al. pointed out that fuzziness does not consider the peculiarities of how the fuzziness is distributed. Consider the two situations: first, when membership function and non-membership function are both equal to 0.5 and the second situation, when the membership function and non-membership function are both equal to 0. In both situations, intuitionistic fuzzy entropy assumes maximum value (equal to 1). However, from the pragmatic point of view, these two situations are clearly different. The existing entropy measures of IFS fail to capture this unique feature of IFS. To handle these situations, Szmidt et al. proposed a measure of knowledge for an intuitionistic fuzzy set which may be considered as a dual measure of intuitionistic fuzzy entropy. This measure claimed to capture some additional features which might be useful in decision-making problems. In the context of IFSs, Szmidt et al., Guo and Song, Guo developed knowledge measures to capture some interesting features of IFSs. They also showed the effectiveness and applications of their knowledge measure in and . The linguistic hedges are the essential features of fuzzy and intuitionistic fuzzy theory. The linguistic hedge argument explains the effectiveness of an entropy or a knowledge measure (more detail in section 4). Depending on the hesitation degree, different knowledge measures are suitable for different situations. Guo showed the effectiveness of his knowledge measure from the aspect of structured linguistic variables (linguistic hedges) in the IFS having high hesitation degree. However, it is not so effective for IFS having small hesitation degree. The entropy measure and the knowledge measure are conceptually dual to each other. But in intuitionistic fuzzy routine, a knowledge measure is not the hard complement of entropy measure and vice-versa. Therefore, a problem dealt with entropy measure alone apprehend to miss out some intriguing features of IFS. Thus, a problem with uncertain dataset(s) handled by both, knowledge measure and entropy measure arguably seems to augment the knowledge base of expert system in vague environment. Consequently, better solutions to optimization problems may be obtained. These facts motivated us to obtain a general framework for defining the knowledge measure of IFS for different situations. In particular, we propose a novel knowledge measure for IFS and study its application in MADM problems. Furthermore, we obtain a one parametric generalization of the proposed knowledge measure. The generalized knowledge measure provides the flexibility of application and the parameter α may be considered as a sensitivity parameter to detect the adaptive changes.

Similarity and dissimilarity are important concepts associated with two data sets. These are very helpful in problems related to pattern recognition. Since fuzzy methods are adaptive and provide soft solutions to real life problems, therefore the notions of similarity and dissimilarity have been extended to fuzzy sets and intuitionistic fuzzy sets by many researchers and lots of similarity and dissimilarity have been put forward (refer and the references therein). Wu et al. investigated the similarity measure models and algorithms for hierarchical case based reasoning (CBR). They developed a similarity evaluation model for hierarchical case (HC) trees by aggregating conceptual similarity and the value sim-
ilarity of two HC-trees. The effectiveness of the hierarchical case similarity evaluation model had also been demonstrated with the help of illustrative examples. Wu et al. 38 developed a recommender system to recommend a suitable e-learning activity to a learner according to his profile and requirements. In their study, they proposed a fuzzy category similarity measure to evaluate the semantic similarity between learning activities/learner profile. In this study, when a subtree is matched to a target tree then an asymmetric similarity measure is desirable. Our proposed accuracy measure between two IFSs may be qualified to be an asymmetric similarity measure. Zhang et al. 39 presented a hybrid similarity measure method for analysis of patent portfolios. In this model, categorical similarity of international patent classifications (IPCs) and the semantic similarity of textual elements have been fused to obtain a hybrid similarity measure. Zhang et al. 39 also presented a case study of firms in China’s medical device industry using the proposed hybrid similarity measure. Due to ever widening multiple perspectives in Science, Technology and Innovations (ST&I), the Technological Roadmapping (TRM) is an essential process for research and development, planning and strategic management. The analysis of ST&I data plays an instrumental role in augmenting the capabilities of domain experts while dealing with real world problems. Zhang et al. 40 utilized similarity measures for topical analysis of Science, Technology and Innovations. Boran and Akay 7 analysed some existing similarity measures for an intuitionistic fuzzy set along with some counter-intuitive cases. Xiao et al. 41 presented detail analysis of existing distance measures of intuitionistic fuzzy sets along with their counter-intuitive cases. Xiao et al. 41 also introduced an intuitive distance measure for intuitionistic fuzzy sets and discussed its application in pattern recognition. Since similarity and dissimilarity are dual concepts. Therefore, both concepts have been equally applied to pattern recognition problems by many researchers. We have observed the following research gaps in the previous studies.

- In all the earlier studies there is no asymmetric similarity measure which is desirable for certain problems (e.g. Wu et al. 37).
- Due to counter-intuitive situations one model can’t solve all problems related to a particular class (e.g. pattern recognition).

These facts motivated us to develop asymmetric similarity measure between two IFSs. Pertaining to this, we propose a measure of the accuracy of an intuitionistic fuzzy set relative to a given intuitionistic fuzzy set. We consider this accuracy measure as a generalization of knowledge measure. The novelties of this paper are summarized as follows:

- We introduce a new knowledge measure of an IFS and demonstrate its superiority in certain situations.
- The application of knowledge measure in MADM problem is presented.
- We propose the notion of accuracy in an IFS B relative to a given IFS A. The effectiveness of accuracy measure is tested in pattern recognition problems. The result shows that some similarity/dissimilarity measures are unable to differentiate some IFSs but accuracy measure can do so.
- The accuracy measure captures some essential features of IFSs. So, we prove some properties of accuracy measure.

The remainder of the paper is organized as follows: Section 2 presents some preliminaries related to IFSs. In Section 3, we propose a new knowledge measure and prove some of its properties. To explore the linguistic aspect of knowledge measure and effectiveness of knowledge measure in an MADM problems, some comparative empirical studies are presented in Section 4. In Section 5, the application of knowledge measure is discussed in an MADM problem with completely unknown/incomplete criteria weights information. Section 6 introduces a generalized version of knowledge measure presented in Section 3 and discusses its effectiveness in certain situation. In Section 7, we prove some characterization theorems for the knowledge measure. Section 8, provides an axiomatic framework to define an accuracy measure of an IFS B relative to a given IFS A and proof of some results related to accuracy measure. In Section 9, we discuss applicability and efficiency of accuracy measure in problems related to pattern recognition. Finally, Section 10 concludes and discusses scope for future research work.

2. Preliminaries

Zadeh 1 introduced the notion of fuzzy sets as follows.
Definition 1 let $X = \{x_1, x_2, ..., x_n\}$ be a universal set, then a fuzzy subset of universal set $X$ is defined as

$$A = \{(x, \xi_A(x)) | x \in X\},$$

where $\xi_A(x) : X \to [0, 1]$ represents a membership function. The value $\xi_A(x)$ describes the extent of presence of $x \in X$ in $A$.

Atanassov ², ⁴ generalized the notion of FS as follows.

Definition 2 Let $X$ be the universe of discourse. Then an intuitionistic fuzzy set $A$ is represented as follow $X$ is given by

$$A = \{(x, \xi_A(x), \eta_A(x)) | x \in X\},$$

where $\xi_A(x) : X \to [0, 1]$ and $\eta_A(x) : X \to [0, 1]$ are membership and non-membership functions subject to the condition

$$0 \leq \xi_A(x) + \eta_A(x) \leq 1,$$

$\forall x \in X$. The values $\xi_A(x)$ and $\eta_A(x)$ denote the extent of belongingness and the degree of non-belongingness of $x$ to $A$, respectively. In addition for each IFS $A$ in $X$ if

$$\pi_A(x) = 1 - \xi_A(x) - \eta_A(x),$$

then $\pi_A(x)$ is called hesitancy margin/degree of $x$ to $A$.

Definition 3 Let the family of all IFSs over a universe of discourse $X$ be $IFS(X)$. Let $A, B \in IFS(X)$ are

$$A = \{(x, \xi_A(x), \eta_A(x)) | x \in X\},$$

$$B = \{(x, \xi_B(x), \eta_B(x)) | x \in X\},$$

and the operations defined on $IFS(X)$ are given, for every $x \in X$, as:

- $A \subseteq B$ if and only if $\xi_A(x_i) \leq \xi_B(x_i)$ and $\eta_A(x_i) \geq \eta_B(x_i)$;
- $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$;
- $A^c = \{(x, \eta_A(x), \xi_A(x)) | x \in X\}$;
- $A \cup B = \{(x, \xi_A(x) \vee \xi_B(x), \eta_A(x) \wedge \eta_B(x))\}$;
- $A \cap B = \{(x, \xi_A(x) \wedge \xi_B(x), \eta_A(x) \vee \eta_B(x))\}$;

Szmidt and Kacprzyk ¹⁹ proposed the axiomatic definition of entropy of IFS as follows.

Definition 4 An entropy on $IFS(X)$ is a real-valued function $E : IFS(X) \to [0, 1]$, satisfying the following four axioms.

(P1) $E(A) = 0$ iff $A$ is a crisp set; that is, $\xi_A(x_i) = 0$, $\eta_A(x_i) = 1$ or $\xi_A(x_i) = 1$, $\eta_A(x_i) = 0$ for all $x_i \in X$.

(P2) $E(A) = 1$ iff $\xi_A(x_i) = \eta_A(x_i)$ for all $x_i \in X$.

(P3) $E(A) \leq E(B)$ iff $A \subseteq B$, that is, if $\xi_A(x_i) \leq \xi_B(x_i)$ and $\eta_A(x_i) \geq \eta_B(x_i)$, for $\xi_B(x_i) \leq \eta_B(x_i)$, or if $\xi_A(x_i) \geq \xi_B(x_i)$ and $\eta_A(x_i) \leq \eta_B(x_i)$, for $\xi_B(x_i) \geq \eta_B(x_i)$ for any $x_i \in X$.

(P4) $E(A) = E(A^c)$.

Montes et al. ⁴² proposed the following definition of divergence measure between two IFSs.

Definition 5 Let $X$ be a finite universe, and let $IFS(X)$ denote the set of all intuitionistic fuzzy sets on $X$. A map $D_IF : IFS(X) \times IFS(X) \to \mathbb{R}$ is a divergence measure for IFS if for every $A, B \in IFS(X)$ it fulfills the following properties:

(D1) $D_IF(A, B) = D_IF(B, A)$.

(D2) $D_IF(A, A) = 0$.

(D3) $D_IF(A \cap C, B \cap C) \leq D_IF(A, B)$.

(D4) $D_IF(A \cup C, B \cup C) \leq D_IF(A, B)$.

Dengfeng and Cheng ¹³ introduced the following definition of intuitionistic fuzzy similarity measure.

Definition 6 A function $S : IFS(X) \times IFS(X) \to [0, 1]$ is called a similarity measure, if $S$ has the following properties:

(SM1) $S(A, B) = S(B, A), \forall A, B \in IFS(X)$.

(SM2) $S(A, B) = 1$, if $A = B$.

(SM3) $0 \leq S(A, B) \leq 1 \forall A, B \in IFS(X)$.

(SM4) $\forall A, B, C \in IFS(X)$, if $A \subseteq B \subseteq C$, then $S(A, B) \geq S(A, C)$ and $S(B, C) \geq S(A, C)$.

Gupta ²⁸ proposed the following axiomatic definition of IFS’ knowledge measure.

Definition 7 A mapping $K : IFS(X) \to [0, 1]$ is called a knowledge measure on $AIFS(X)$, if $K$ has the following properties:

(KP$_{AIFS}$) $K(A) = 1$ iff $A$ is a crisp set;

(KP$_{AIFS}$) $K(A) = 0$ iff $\pi_A(x_i) = 1 \forall x_i \in X$;

(KP$_{AIFS}$) $K(A) \geq K(B)$ if $A$ is less fuzzy than $B$, i.e., $A \subseteq B$ for $\xi_B(x_i) \leq \eta_B(x_i)$ $\forall x_i \in X$ or $A \supseteq B$ for $\xi_B(x_i) \geq \eta_B(x_i)$ $\forall x_i \in X$;

(KP$_{AIFS}$) $K(A^c) = K(A)$. 
For example, the following two knowledge measures satisfy the axiomatic requirements of a knowledge measure.

\[
K(x) = \frac{1}{n} \sum_{i=1}^{n} (1 - 0.5[E(x_i) + \pi(x_i)]). \tag{1}
\]

(Szmidt et al.\textsuperscript{27})

\[
K_G(A) = 1 - 2n \sum_{i=1}^{n} (1 - |\xi_A(x_i) - \eta_A(x_i)|)
\]

\[
(1 + \pi_A(x_i)). \tag{2}
\]

(Guo\textsuperscript{28})

In the next section, we propose a new knowledge measure of an intuitionistic fuzzy set A and study some of its properties.

3. Novel knowledge measure of IFS

Let \( A \in IFS(X) \), then we propose the following measure of knowledge in the IFS \( A \)

\[
K(A) = \frac{1}{n} \sum_{i=1}^{n} (\xi_A^2(x_i) + \eta_A^2(x_i)). \tag{3}
\]

First, we establish that (3) is a valid knowledge measure and then we prove its valuation property.

**Theorem 1** \( K(A) \) is a valid knowledge measure.

**Proof** We prove the axiomatic requirements (\( KP_{AIFS1} \) - \( KP_{AIFS4} \)).

(\( KP_{AIFS1} \)) Let \( A \) be a crisp set. This implies \( \xi_A(x_i) = 1 \) or \( \eta_A(x_i) = 1 \) \( \forall x_i \in X \); thus \( K(A) = 1 \). On the other hand, let \( K(A) = 1 \) then

\[
K(A) = \frac{1}{n} \sum_{i=1}^{n} [\xi_A^2(x_i) + \eta_A^2(x_i)]
\]

\[
= 1 = \frac{1}{n} \sum_{i=1}^{n} [\xi_A^2(x_i) + \eta_A^2(x_i)].
\]

which means \( \xi_A(x_i) = 1 \) or \( \eta_A(x_i) = 1 \) \( \forall x_i \in X \) and thus \( A \) is a crisp set.

(\( KP_{AIFS2} \)) Let \( \pi_A(x_i) = 1 \) \( \forall x_i \in X \). This implies \( \xi_A(x_i) = \eta_A(x_i) = 0 \) \( \forall x_i \in X \); thus \( K(A) = 0 \). On the other hand, let \( K(A) = 0 \) further algebraic manipulation lead to

\[
\frac{1}{n} \sum_{i=1}^{n} [\xi_A^2(x_i) + \eta_A^2(x_i)] = 0
\]

Now, \( \xi_A(x_i) \geq 0, \eta_A(x_i) \geq 0 \). Therefore, Eq.(4) yields \( \xi_A^2(x_i) + \eta_A^2(x_i) = 0 \) \( \forall x_i \in X \). But then \( \pi_A(x_i) = 1 \).

(\( KP_{AIFS3} \)) We empirically test the axiom (\( KP_{AIFS3} \)) for \( K(A) \), by generation of IFSs satisfying the conditions \( \xi_A(x_i) \leq \xi_B(x_i) \leq \eta_B(x_i) \leq \eta_A(x_i) \) and \( \xi_A(x_i) \geq \xi_B(x_i) \geq \eta_B(x_i) \geq \eta_A(x_i) \). In both the situations we observe that \( K(A) \geq K(B) \).

(\( KP_{AIFS4} \)) Follows from the definition of \( A^c \).

**Theorem 2** Let \( K(A) \) and \( K(B) \) be knowledge measures of IFSs \( A \) and \( B \) respectively. Then

\[
K(A \cup B) + K(A \cap B) = K(A) + K(B).
\]

**Proof** We consider two cases.

Case 1 First we consider the case when \( \xi_A(x_i) \geq \xi_B(x_i) \geq \eta_B(x_i) \geq \eta_A(x_i) \), \( 1 \leq i \leq n \). We have

\[
K(A \cup B) + K(A \cap B) = \frac{1}{n} \sum_{i=1}^{n} [\xi_A^2(x_i) + \eta_A^2(x_i)] + \frac{1}{n} \sum_{i=1}^{n} [\xi_B^2(x_i) + \eta_B^2(x_i)] \]

\[
= \frac{1}{n} \sum_{i=1}^{n} (\xi_A(x_i) + \xi_B(x_i))^2 + \frac{1}{n} \sum_{i=1}^{n} (\eta_A(x_i) + \eta_B(x_i))^2 \]

\[
= \frac{1}{n} \sum_{i=1}^{n} (\xi_A(x_i) + \eta_B(x_i))^2 + \frac{1}{n} \sum_{i=1}^{n} (\eta_A(x_i) + \xi_B(x_i))^2 \]

\[
= \frac{1}{n} \sum_{i=1}^{n} (\xi_A(x_i))^2 + \frac{1}{n} \sum_{i=1}^{n} (\eta_A(x_i))^2 \]

\[
= K(A) + K(B).
\]

Case 2 Now, we consider the case when \( \xi_A(x_i) \leq \xi_B(x_i) \leq \eta_B(x_i) \leq \eta_A(x_i) \), \( 1 \leq i \leq n \). By following the similar steps as in Case 1, we get \( K(A \cup B) + K(A \cap B) = K(A) + K(B) \).

Hence, the proof follows.

4. Comparative studies

In this section, we present some comparative studies of knowledge measure (3). In subsection 4.1, we investigate
the performance of (3) from the point of view of structural linguistic variables. We also compare the results with some entropies of an intuitionistic fuzzy set and two existing knowledge measures. Then in subsection 4.2, we investigate the effectiveness of $K(x)$ in a MADM problem and compare the results with Mao's intuitionistic fuzzy entropy method.

4.1. Effectiveness of new knowledge measure in structured linguistic framework

First, we examine the performance of our developed measure with the help of example given below (adapted from Hung and Yang [11] and Guo [28]).

**Example 1** Let $A = \{x, \xi_A(x), \eta_A(x)\} | x \in X$ be an IFS in $X$. De et al. [44] defined an IFS $A^m$, where $m$ is any real number which is given by

$$A^m = \{x, (\xi_A(x))^m, 1 - (1 - \eta_A(x))^m | x \in X\}.$$

1) Let $A$ be an IFS in $X = \{6, 7, 8, 9, 10\}$, which is given by

$$A = \{6, 0.1, 0.8\}, \{7, 0.3, 0.5\}, \{8, 0.6, 0.2\}, \{9, 0.9, 0.0\}, \{10, 1.0, 0.0\}\}.$$

Now, by using the above operation, we generate IFSs pertaining to $A$ which are given below:

$$A^{0.5} = \{6, 0.316, 0.553\}, \{7, 0.548, 0.293\}, \{8, 0.775, 0.106\}, \{9, 0.949, 0.0\}, \{10, 1.0, 0.0\}\}$$

$$A^{2} = \{6, 0.010, 0.960\}, \{7, 0.090, 0.750\}, \{8, 0.360, 0.360\}, \{9, 0.810, 0.0\}, \{10, 1.0, 0.0\}\}$$

$$A^{3} = \{6, 0.001, 0.992\}, \{7, 0.027, 0.875\}, \{8, 0.216, 0.488\}, \{9, 0.729, 0.0\}, \{10, 1.0, 0.0\}\}$$

and

$$A^{4} = \{6, 0.000, 0.998\}, \{7, 0.008, 0.938\}, \{8, 0.130, 0.590\}, \{9, 0.656, 0.0\}, \{10, 1.0, 0.0\}\}.$$

De et al. [44] regarded $A$ as LARGE in $X$ by considering the characterization of linguistic variables. In the same way, linguistic equivalent of $A^{0.5}$ is More or less LARGE, $A^{2}$ is Very LARGE, $A^{3}$ is Quite very LARGE, and $A^{4}$ is Very very LARGE.

Now by taking into account the mathematical operations, the entropy of these IFS should have the following order:

$$Entropy(A^{0.5}) > Entropy(A) > Entropy(A^{2}) >$$

$$Entropy(A^{3}) > Entropy(A^{4}).$$

(5)

Now, from structured linguistic point of view knowledge measure of IFS should follow the order:

$$K(A^{0.5}) < K(A) < K(A^{2}) < K(A^{3}) < K(A^{4}).$$

(6)

In this section, we consider three entropies $E_{sk}, E_{ld}, E_{CI}$, the knowledge measures $K_{sk}, K_{IFS}$ and the proposed knowledge measure $K(x)$. Each knowledge measure is suitable in some situation. We investigate the suitability of knowledge measures based on the hesitation margin. We examine the performance of knowledge measure (3) with these IFSs and interpret the results in the sense of the amount of knowledge associated with them. We also compare our knowledge-based results with the entropy of these IFSs, with the aim of showing the effectiveness of considered measure. The comparative results are shown in Table 1.

From Table 1, we observe that requirement (5) is satisfied by all entropies and requirement (6) is satisfied by Guo’s knowledge measure. Now, on reducing the hesitation margin in set $A$, we obtain set $B$ i.e.

$$B = \{6, 0.1, 0.8\}, \{7, 0.3, 0.5\}, \{8, 0.5, 0.4\}, \{9, 0.9, 0.0\}, \{10, 1.0, 0.0\}\}.$$

and construct Table 2.

From Table 2, we observe that requirement (5) is satisfied by all entropies and requirement (6) is not satisfied by any of the knowledge measure.

If we further reduce the hesitation margin, we obtain set $C$ i.e.

$$C = \{6, 0.1, 0.8\}, \{7, 0.3, 0.5\}, \{8, 0.5, 0.5\}, \{9, 0.9, 0.0\}, \{10, 1.0, 0.0\}\}.$$
Table 1. Comparative results by different models for IFSs pertaining to A

<table>
<thead>
<tr>
<th>IFSs</th>
<th>Entropy $E_{sk}$</th>
<th>Entropy $E_{ldl}$</th>
<th>Entropy $E_{zj}$</th>
<th>Knowledge $K_{skb}$</th>
<th>Knowledge $K_{AIFS}$</th>
<th>Knowledge $K(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A$^{0.5}$</td>
<td>0.319</td>
<td>0.471</td>
<td>0.249</td>
<td>0.794</td>
<td>0.785</td>
<td>0.668056</td>
</tr>
<tr>
<td>A</td>
<td>0.307</td>
<td>0.466</td>
<td>0.212</td>
<td>0.786</td>
<td>0.788</td>
<td>0.64</td>
</tr>
<tr>
<td>A$^{2}$</td>
<td>0.301</td>
<td>0.390</td>
<td>0.266</td>
<td>0.783</td>
<td>0.805</td>
<td>0.68152</td>
</tr>
<tr>
<td>A$^{3}$</td>
<td>0.212</td>
<td>0.317</td>
<td>0.095</td>
<td>0.827</td>
<td>0.854</td>
<td>0.713332</td>
</tr>
<tr>
<td>A$^{4}$</td>
<td>0.176</td>
<td>0.278</td>
<td>0.046</td>
<td>0.844</td>
<td>0.877</td>
<td>0.732496</td>
</tr>
</tbody>
</table>

Table 2. Comparative results by different models for IFSS pertaining to B

<table>
<thead>
<tr>
<th>IFSs</th>
<th>Entropy $E_{sk}$</th>
<th>Entropy $E_{ldl}$</th>
<th>Entropy $E_{zj}$</th>
<th>Knowledge $K_{skb}$</th>
<th>Knowledge $K_{AIFS}$</th>
<th>Knowledge $K(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>B$^{0.5}$</td>
<td>0.345</td>
<td>0.508</td>
<td>0.285</td>
<td>0.787</td>
<td>0.767</td>
<td>0.6485786</td>
</tr>
<tr>
<td>B</td>
<td>0.374</td>
<td>0.502</td>
<td>0.305</td>
<td>0.763</td>
<td>0.761</td>
<td>0.642</td>
</tr>
<tr>
<td>B$^{2}$</td>
<td>0.197</td>
<td>0.345</td>
<td>0.104</td>
<td>0.852</td>
<td>0.865</td>
<td>0.7241</td>
</tr>
<tr>
<td>B$^{3}$</td>
<td>0.131</td>
<td>0.352</td>
<td>0.038</td>
<td>0.888</td>
<td>0.911</td>
<td>0.7824282</td>
</tr>
<tr>
<td>B$^{4}$</td>
<td>0.109</td>
<td>0.200</td>
<td>0.016</td>
<td>0.899</td>
<td>0.926</td>
<td>0.8133984</td>
</tr>
</tbody>
</table>

Table 3. Comparative results by different models for IFSS pertaining to C

<table>
<thead>
<tr>
<th>IFSs</th>
<th>Entropy $E_{sk}$</th>
<th>Entropy $E_{ldl}$</th>
<th>Entropy $E_{zj}$</th>
<th>Knowledge $K_{skb}$</th>
<th>Knowledge $K_{AIFS}$</th>
<th>Knowledge $K(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>C$^{0.5}$</td>
<td>0.352</td>
<td>0.519</td>
<td>0.304</td>
<td>0.790</td>
<td>0.763</td>
<td>0.6556234</td>
</tr>
<tr>
<td>C</td>
<td>0.407</td>
<td>0.512</td>
<td>0.345</td>
<td>0.756</td>
<td>0.760</td>
<td>0.66</td>
</tr>
<tr>
<td>C$^{2}$</td>
<td>0.168</td>
<td>0.328</td>
<td>0.093</td>
<td>0.878</td>
<td>0.883</td>
<td>0.75468</td>
</tr>
<tr>
<td>C$^{3}$</td>
<td>0.110</td>
<td>0.229</td>
<td>0.035</td>
<td>0.907</td>
<td>0.923</td>
<td>0.812622</td>
</tr>
<tr>
<td>C$^{4}$</td>
<td>0.095</td>
<td>0.179</td>
<td>0.015</td>
<td>0.913</td>
<td>0.934</td>
<td>0.8379872</td>
</tr>
</tbody>
</table>

Table 3.

From Table 3, we observe that the preference order (6) is satisfied by the newly proposed knowledge measure. The comparative results are shown in Table 3. Similarly, there is still the greater entropy for the IFS B in Table 3, from which it is clear that the measures $E_{sk}$, $E_{zj}$, $K_{skb}$, and our developed model $K(A)$ are doing well this time. In summary, remarkable among the entropy above is the measure $E_{sk}$ that clearly out performs the others ones throughout the process. As far as the knowledge measurement models are concerned, our developed knowledge measure $K(A)$ is doing better throughout the process. Therefore, the model $K(A)$ may be considered to employ in some situations may be theoretical or practical.

4.2. Effectiveness of new knowledge measure in multiple-attribute decision-making

We examine the performance of the new knowledge measure with the help of example adapted from section 5.2 of Mao et al. 43.

Example 2 An investment company plans to invest some money in the best fund. Five possible funds $(x_1, x_2, x_3, x_4$ and $x_5)$ satisfies requirements under four attributes $(a_1, a_2, a_3$ and $a_4)$, in order to choose the best fund, the company makes some evaluations for these funds. The results have been given using intuitionistic fuzzy sets in Table 5 of Mao et al. 43. Under the condition that attributes are benefit type. The weight associated
The score function for each alternative is given by
\[ E(a_j) = \frac{1 - E(a_j)}{\sum_{k=1}^{4} (1 - E(a_k))}, \quad (1 \leq j \leq 4); \quad (7) \]
where \( E(a_j) = \sum_{i=1}^{5} E(x_i, a_j) \), and \( E(x_i, a_j) \) = Intuitionistic fuzzy entropy of every object under each attribute. The score function for each alternative is given by
\[ S(x_i) = \sum_{j=1}^{4} (\xi(x_i, a_j) - \eta(x_i, a_j)) \times W_i, \quad (1 \leq i \leq 5); \quad (8) \]
where \( \xi(x_i, a_j), \eta(x_i, a_j) \) represent the membership and non-membership degree of object \( x_i \) under attribute \( a_j \).

Since, the knowledge measure is considered as a dual of entropy of intuitionistic fuzzy set. Therefore, in the context of study of knowledge measure we can modify Eq. (7) as follows
\[ W_j = \frac{K(a_j)}{\sum_{k=1}^{4} K(a_k)}, j = 1, 2, 3, 4. \quad (9) \]

Using the data of Table 5 of Mao et al. \(^43\) and by using (11) the weights of alternatives \( x_1, x_2, x_3, x_4 \) and \( x_5 \) are as follows
\[ W = \{0.3186682521, 0.2068965517, 0.2366230678, 0.2378121284\}. \]

Consequently, using (8) we obtain scores of alternatives \( x_1, x_2, x_3, x_4, x_5 \) as follows:
\[ S(x_1) = 0.4141, S(x_2) = 0.3316, S(x_3) = 0.2031, S(x_4) = 0.4039 \text{ and } S(x_5) = 0.3377. \]

Here, we have \( S(x_1) > S(x_4) > S(x_3) > S(x_2) > S(x_5) \).

The preference order of five alternative is completely in agreement with the results obtained in Mao et al. \(^43\). Therefore, the measure (3) is effective in the problems of multi-attribute decision-making. In the next section, we apply \( K(A) \) in a MADM problem, with complete and partial information of weights.

5. A knowledge measure based approach for MADM problem

Problem Formulation: Let \( A = \{a_1, a_2, \ldots, a_n\} \) be the set of n-attributes, \( X = \{x_1, x_2, \ldots, x_m\} \) be the set of m-alternatives and \( W = \{w_1, w_2, \ldots, w_m\} \), where \( w_j \in [0, 1] \) be the weight vector of attributes which is not predefined.

Suppose IFS \( a_{ij} = (\xi_{ij}, \eta_{ij}) \) denote the expert’s assessment corresponds to the \( j \)th attribute \( a_j \) \( (j = 1, 2, \ldots, n) \) to evaluate \( i \)th alternative \( x_i \) \( (i = 1, 2, \ldots, m) \).

The decision matrix corresponding to above mentioned situations is as follows:
\[ M = \begin{bmatrix}
 x_1 & a_1 & a_2 & \cdots & a_n \\
 x_2 & a_{11} & a_{12} & \cdots & a_{1n} \\
 x_3 & a_{21} & a_{22} & \cdots & a_{2n} \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 x_m & a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix} \]

We discuss the problem of MADM in two situations. One, when weights of attributes are completely unknown and secondly, when we have partial information about weights of attributes.

5.1. A MADM model based on knowledge measure when there is no information about weights

The main steps of the model are as follows:

Step 1 Let
\[ M = [a_{ij}]_{n \times m} \quad (10) \]
be the intuitionistic fuzzy decision matrix (as described in the problem statement). Consequently, our proposed knowledge measure takes the form
\[ K(a_j) = \frac{1}{m} \sum_{i=1}^{m} (\xi_{ij}^2 + \eta_{ij}^2); \quad j = 1, 2, \ldots, n. \quad (11) \]

Step 2 The weights associated with attribute \( a_j \) can be obtained using equation (9).

Step 3 After evaluating the weights of attributes, the weighted aggregated values for each alternative are obtained by the weighted intuitionistic fuzzy arithmetic mean operator \(^48\) as follows:
\[ S_i = S(x_i) = \left(1 - \prod_{j=1}^{n} (1 - \xi_{a_{ij}})^{w_j}\right) \prod_{j=1}^{n} \eta_{a_{ij}}^{w_j}) \]
\[ = (\xi, \eta_i) \text{ (say).} \quad (12) \]
Step 4 Finally, to rank the alternative we compute the score of each alternative as follows 45:

\[ \text{Score}(S_i) = \xi_i - \eta_i; \ i = 1, 2, \ldots, m. \]  
(14)

Example 3 43 Consider the case of five alternatives with four attributes. Let the decision matrix be \( M = \)

\[
\begin{pmatrix}
    a_1 & a_2 & a_3 & a_4 \\
    s_1 & (0.7, 0.2) & (0.5, 0.3) & (0.6, 0.1) & (0.6, 0.2) \\
    s_2 & (0.7, 0.3) & (0.5, 0.2) & (0.7, 0.2) & (0.4, 0.5) \\
    s_3 & (0.6, 0.4) & (0.5, 0.4) & (0.5, 0.3) & (0.6, 0.3) \\
    s_4 & (0.8, 0.1) & (0.6, 0.3) & (0.3, 0.4) & (0.6, 0.2) \\
    s_5 & (0.6, 0.2) & (0.4, 0.3) & (0.7, 0.1) & (0.5, 0.3) \\
\end{pmatrix}
\]

Step 1 We obtain the knowledge associated with each attribute by solving the following programming model.

\[
K(a_1) = 0.536, K(a_2) = 0.348, K(a_3) = 0.398, K(a_4) = 0.4.
\]

Step 2 We calculate the weights of attributes using (9).

\[
w_1 = 0.3186682521, w_2 = 0.2068965517, w_3 = 0.2366230678, w_4 = 0.2378121284
\]

Step 3 We compute weighted aggregated value of each alternative using (12). We obtain the values

\[
S_1 = (0.382205, 0.1846), \\
S_2 = (0.393196, 0.282996), \\
S_3 = (0.441613, 0.348967) \; \text{and} \\
S_4 = (0.366134, 0.205478).
\]

Step 4 We obtain score of alternatives by using (13).

\[
\text{Score}(S_1) = 0.197605, \\
\text{Score}(S_2) = 0.110201, \\
\text{Score}(S_3) = 0.92645, \\
\text{Score}(S_4) = 0.160656 \\
\text{and} \ \text{Score}(S_5) = 0.225238
\]

The preference order in this case is

\[ S_5 > S_1 > S_4 > S_2 > S_3. \]

Hence \( x_5 \) has maximum score. Therefore, it is the best alternative.

5.2. Knowledge measure based MADM model when partial information regarding attribute weights is available.

In the problems related to MADM, the determination of attribute weight is very important. In real life situations,
Step 3 By using (12), we obtain the weighted aggregated score function.

Step 4 By using (14), we obtain scores.

Example 4 Consider the intuitionistic fuzzy decision matrix given in example 3.

Step 1 We obtain the knowledge based matrix as follows:

\[
M_k = \begin{pmatrix}
    x_1 & a_1 & a_2 & a_3 & a_4 \\
    x_2 & 0.53 & 0.34 & 0.37 & 0.4 \\
    x_3 & 0.58 & 0.29 & 0.29 & 0.41 \\
    x_4 & 0.52 & 0.4 & 0.34 & 0.45 \\
    x_5 & 0.65 & 0.45 & 0.25 & 0.4 \\
    x_6 & 0.4 & 0.25 & 0.5 & 0.34 \\
\end{pmatrix}
\]

Step 2 Let the partial information regarding attributes weight is given by the set H as follow:

\[ H = \{0.25 \leq w_1 \leq 0.75, 0.35 \leq w_2 \leq 0.60, 0.30 \leq w_3 \leq 0.35, 0.40 \leq w_4 \leq 0.45, w_1 + w_2 + w_3 + w_4 = 1\}. \]

Now, by using the model we have

\[ \text{Max.} K_W = 2.68w_1 + 1.73w_2 + 1.75w_3 + 2w_4 \]
subject to \( W \in H \).

Solving this model using MATLAB, we get

\[ w_1 = 0.25, \quad w_2 = 0.35, \quad w_3 = 0.30 \quad \text{and} \quad w_4 = 0.45. \]

Step 3 We obtain the weighted aggregated scores as follows:

\[ S_1 = (0.292055, 0.106589), \]
\[ S_2 = (0.321532, 0.190321), \]
\[ S_3 = (0.335562, 0.233925), \]
\[ S_4 = (0.288692, 0.135854), \]
\[ S_5 = (0.339266, 0.127925). \]

Step 4 Finally, to rank the alternatives, we determine the score of each alternative using (16) we have

\[ \text{Score}(S_1) = 0.185466, \quad \text{Score}(S_2) = 0.131211, \]
\[ \text{Score}(S_3) = 0.101637, \quad \text{Score}(S_4) = 0.152838 \quad \text{and} \quad \text{Score}(S_5) = 0.211331. \]

Therefore, the preference order is

\[ S_3 > S_2 > S_4 > S_5 > S_1. \]

Hence \( x_3 \) is best alternative.

However, the multi-attribute decision-making problem discussed in this section can also be solved by entropy methods under intuitionistic fuzzy environment. The knowledge measure based approach in MADM problems must be preferred due to following reasons.

- Less computational complexity of knowledge measure.
- In structured linguistic variables, the proposed knowledge measure is more effective in an intuitionistic fuzzy set with small hesitation margin (Section 4).

It is quite natural that in some problems the score of two or more alternatives may be equal. To handle such problems, for obtaining a strict preference order among alternatives we introduce a one parametric generalization of (3) in the next section and study its effectiveness in certain situations.

6. Generalized knowledge measure

We propose the following generalized knowledge measure

\[ K^\alpha(A) = \frac{1}{n(\alpha - 1)} \sum_{i=1}^{n} (\xi^\alpha(x_i) + \eta^\alpha(x_i)), \quad \alpha > 1. \]

(17)

For \( \alpha = 2 \), we recover \( K(A) \).

Theorem 3 \( K^\alpha(A) \) is a valid knowledge measure.

Proof Following the same steps as in proof of theorem 1 we can prove that, \( K^\alpha(A) \) is a valid knowledge measure.

Example 5 Consider the case of five alternatives \( x_i, i = 1, 2, 3, 4, 5 \) and four attributes \( a_j, j = 1, 2, 3, 4 \) with the following decision table (Table4).

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( a_3 )</th>
<th>( a_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>(0.7,0.2,0.1)</td>
<td>(0.5,0.3,0.2)</td>
<td>(0.6,0.1,0.3)</td>
<td>(0.6,0.2,0.2)</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>(0.7,0.3,0.0)</td>
<td>(0.5,0.2,0.2)</td>
<td>(0.7,0.2,0.1)</td>
<td>(0.4,0.5,0.1)</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>(0.6,0.4,0.0)</td>
<td>(0.5,0.4,0.1)</td>
<td>(0.5,0.3,0.1)</td>
<td>(0.6,0.3,0.1)</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>(0.8,0.1,0.1)</td>
<td>(0.6,0.3,0.1)</td>
<td>(0.3,0.4,0.3)</td>
<td>(0.6,0.2,0.2)</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>(0.6,0.2,0.2)</td>
<td>(0.7,0.4,0.3)</td>
<td>(0.6,0.1,0.2)</td>
<td>(0.2,0.3,0.2)</td>
</tr>
</tbody>
</table>

We construct generalized score function as follows:

\[ S^\alpha(x_i) = \sum_{j=1}^{4} \left( \frac{\xi^\alpha(x_i, a_j)}{\left( \sqrt{\xi^\alpha(x_i, a_j)} + \sqrt{\eta^\alpha(x_i, a_j)} \right) \times W_j^\alpha} \right), \]

\[ i = 1, 2, ..., 5. \]

(18)
where \( \xi(x_i,a_j) \) and \( \eta(x_i,a_j) \) are membership and non-membership degree of object \( x_i \) under attribute \( a_j \). Let,

\[
W^\alpha_j = \frac{K^\alpha(a_j)}{\sum_{i=1}^{n} K^\alpha(a_i)},
\]

(19)

denotes the generalized weight of attributes \( a_j \). Now we calculate weights for different values of \( \alpha \) by using (17) and (19). The corresponding weight vectors are obtained in Table 5.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( w^\alpha_1 )</th>
<th>( w^\alpha_2 )</th>
<th>( w^\alpha_3 )</th>
<th>( w^\alpha_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.2614</td>
<td>0.5256</td>
<td>0.2196</td>
<td>0.2113</td>
</tr>
<tr>
<td>1.1</td>
<td>0.2790</td>
<td>0.5261</td>
<td>0.2264</td>
<td>0.2313</td>
</tr>
<tr>
<td>1.3</td>
<td>0.2667</td>
<td>0.5218</td>
<td>0.2245</td>
<td>0.2269</td>
</tr>
<tr>
<td>1.5</td>
<td>0.2994</td>
<td>0.5298</td>
<td>0.2228</td>
<td>0.2224</td>
</tr>
<tr>
<td>1.7</td>
<td>0.3033</td>
<td>0.5272</td>
<td>0.2215</td>
<td>0.2179</td>
</tr>
<tr>
<td>1.9</td>
<td>0.3119</td>
<td>0.5242</td>
<td>0.2202</td>
<td>0.2135</td>
</tr>
</tbody>
</table>

Finally, we compute the score of each alternative corresponding to various values of \( \alpha \) and obtain Table 6. We observe that for \( \alpha = 2 \) score of alternative \( x_2 \) and \( x_5 \) is equal and hence we are not able to obtain a strict preference. But, when we change the values of \( \alpha \), we obtain a strict preference order among the alternatives. Moreover, it is interesting to note that the best alternative in all the cases is same i.e. \( x_1 \). Therefore, generalized knowledge measure is resolving a tie when a strict preference among alternative is desirable.

Now, in the next section, we prove some characterization results to obtain a general framework for defining new knowledge measures.

7. Characterization of knowledge measure

Theorem 4 Let \( G : [0,1]^2 \rightarrow [0,\infty] \) be a mapping. Then the following function \( K_{IF} : IFS(X) \rightarrow [0,1] \) defined by

\[
K_{IF}(A) = \frac{1}{n} \sum_{i=1}^{n} G(\xi_i, \eta_i).
\]

(\ast)

satisfies axioms \((K_{IFS1}) - (K_{IFS4})\) if \( G \) satisfies the following conditions:

(1) \( G(x,y)=0 \) with \( x+y \leq 1 \) if and only if \( x=y=0 \),

(2) \( G(x,y)=1 \) with \( x+y \leq 1 \) if and only if either \( x=0 \), \( y=1 \) or \( x=1 \), \( y=0 \),

(3) \( G(x,y)=G(y,x) \),

\( (4) G(x,y) \geq G(z,w) \) if \( x \leq z \leq w \leq y \) or \( x \geq z \geq w \geq y \).

Proof Consider \( K_{IF}(A) \) as given in equation(\ast) satisfies axioms \((K_{IFS1}) - (K_{IFS4})\). We show that \( G \) satisfies (1) if \( G(x,y)=0 \) if \( x,y \in [0,1] \) then we can take \( \xi_i = x \) and \( \eta_i = y \) for every \( i = 1,2,\ldots,n \). Then, we have

\[
K_{IF}(A) = \frac{1}{n} \sum_{i=1}^{n} G(\xi_i, \eta_i) = 0.
\]

From the axiom \((K_{IFS2})\), it can happen iff \( x = y = 0 \), which we wanted to show.

Now suppose that \( G(x,y) = 1 \) with \( x+y \leq 1 \), \( x+y \in [0,1] \). Further, we consider the IFS given by \( \xi_i = x \) and \( \eta_i = y \) for every \( i = 1,2,\ldots,n \). This gives

\[
K_{IF}(A) = \frac{1}{n} \sum_{i=1}^{n} G(\xi_i, \eta_i) = 1.
\]

From the axiom \((K_{IFS1})\) it can happen iff either \( x=0 \), \( y=1 \) or \( x=1 \), \( y=0 \), which we wanted to show. In context of (3), suppose that there exist \( x,y \in [0,1] \) such that \( x+y < 1 \) but \( G(x,y) \neq G(y,x) \). Without loss of generality, we can assume that \( G(x,y) > G(y,x) \). Consider the IFS \( A \) defined by \( \xi_i = x \) and \( \eta_i = y \). Then we get

\[
K_{IF}(A) = \frac{1}{n} \sum_{i=1}^{n} G(\xi_i, \eta_i) > \frac{1}{n} \sum_{i=1}^{n} G(\eta_i, \xi_i) = K_{IF}(A^c).
\]

which contradicts axiom \((K_{IFS3})\). Finally (4) can be shown by a similar argument. The converse is just an easy calculation.

Theorem 5 Let \( M : [0,1]^2 \rightarrow [0,1] \) be a function such that \( M(x,x) : [0,1] \rightarrow [0,1] \) is strictly decreasing for every \( x \in [0,1] \) satisfying following conditions:

(a) Symmetry: \( M(x,y) = M(y,x) \),

(b) Idempotence: \( M(x,x) = x \), \( \forall x \in [0,1] \),

(c) Boundary condition: \( M(x,0) = x \).

Let \( f : [0,1] \rightarrow [0,1] \) be a mapping. Then the function \( G(x,y) = M(f(x),f(y)) \) satisfies properties 1-4 in theorem 4 if following properties holds:

(i) \( f(x) = 0 \) iff \( x = 0 \),
(ii) \( f(x) = 1 \) iff \( x = 1 \).
(iii) \(f\) is monotonic increasing in \([0,1]\).

**Proof** Given, \(G(x,y) = M(f(x), f(y))\) satisfying property 1-4 of theorem 4. We have from property (1) of theorem 4

\[
G(0, 0) = 0
\]

\[
\Rightarrow M(f(0), f(0)) = 0.
\]

Using idempotency of \(M\), we have

\[
f(0) = 0.
\]

Conversely, if possible suppose \(\exists x \neq 0\) such that \(f(x) = 0\).

Then

\[
G(x, 0) = M(f(x), f(0))
\]

\[= f(x) = 0.
\]

Therefore, \(G(x,0) = 0\) which is a contradiction to condition (1) of theorem 4. This proves (i). Now,

\[
G(1, 0) = M(f(1), f(0))
\]

\[= M(f(1), 0)
\]

\[= f(1).
\]

From condition (2) of theorem 4 we have,

\[
G(1, 0) = 1
\]

\[
\Rightarrow f(1) = 1.
\]

Let \(y_0(\neq 1)\) such that \(f(y_0) = 1\).

Then,

\[
G(y_0, 0) = M(f(y_0), f(0))
\]

\[= M(f(y_0), 0)
\]

\[= f(y_0)
\]

\[= 1.
\]

\[\Rightarrow G(y_0, 0) = 1\] which is a contradiction to condition (2) of theorem 4. Therefore, \(f(x) = 1 \iff x = 1\).

This proves (ii).

Next, let \(x, y \in [0,1]\).

If possible suppose that \(f(x)\) is monotonically decreasing function in \([0,1]\). Therefore, we assume that \(x \leq y\) such that \(f(x) > f(y)\).

Now, \(G(x, 1 - y) = M(f(x), f(1 - y))\) and \(G(y, 1 - y) = M(f(y), f(1 - y))\).

Then, \(M(f(x), f(1 - y)) < M(f(y), f(1 - y))\)

\[\Rightarrow G(x, 1 - y) < G(y, 1 - y).
\]

This contradicts condition (4) of theorem 4.

Therefore \(f(x)\) is monotonically increasing in \([0,1]\).

8. A measure of accuracy

The amount of knowledge in an IFS \(A\) can be considered as a amount of accuracy in \(A\). Intuitionistically, if \(A\) is crisp then it must be absolutely accurate and value of knowledge/accuracy may numerically be considered as 1. Now, if \(A\) is any IFS and \(B\) is another IFS and we want to calculate degree of accuracy in \(B\) relative to \(A\) i.e., when \(A\) is benchmark for accuracy of \(B\). The unorthodoxy in this situation is that \(A\) in itself may not be accurate (crisp).

Thus, intuitionistic argument says; the amount of accuracy in \(B\) can be maximum when \(A\) and \(B\) both are crisp sets and \(\xi_A(x_i) = \eta_A(x_i)\), \(\xi_B(x_i) = \eta_B(x_i)\). Otherwise, it can attain as much accuracy as that of \(A\) when \(A = B\).

The accuracy in \(B\) relative to \(A\) is zero if we have no knowledge about of \(A\) (i.e., \(\xi_A(x_i) = \eta_A(x_i) = 0\)).

This type of measure of accuracy of \(B\) relative to \(A\) is clearly different from measure of similarity \(S(A, B)\) between \(A\) and \(B\) due to the following facts:

- \(S(A; B)\) is symmetric but \(\mathcal{A}(A, B)\) is not symmetric.
Remark 2. There can be IFSs $A$ and $B$ where $S(A;B) = 1$ even if $A$ and $B$ both are Intuitionistic fuzzy sets but Accuracy $\mathcal{A}(A;B) \neq 1$ if $A$ and $B$ both are Intuitionistic fuzzy sets.

Therefore, this type of accuracy measure seems to be important for pattern recognition problems.

Inspired by these characteristics present in intuitionistic fuzzy sets, we introduce a notion of accuracy in an intuitionistic fuzzy set $B$ relative to a reference intuitionistic fuzzy set $A$. We also compute the degree of accuracy, the axiomatic definition of such an intuitive measure of accuracy is as follows:

**Definition 8** Let $A, B, C \in IFS(X)$. Let $\mathcal{A}$ be a mapping $\mathcal{A} : IFS(X) \times IFS(X) \to [0,1]$. $\mathcal{A}(A,B)$ is said to accuracy in $B$ relative to $A$ if it satisfies the following properties:

(A1) $0 \leq \mathcal{A}(A;B) \leq 1$;

(A2) $\mathcal{A}(A;B) = 0$ iff $\pi_A(x) = 1$;

(A3) $\mathcal{A}(A;B) = 1$ if both $A$ and $B$ are crisp sets and $A = B$. $(\xi_A(x) = \xi_B(x))$, $\eta_A(x) = \eta_B(x)$;

(A4) $\mathcal{A}(A;B) = K(A)$ if $A = B$.

We propose a measure of accuracy in $B$ relative to $A$ as follows. Let $A$ and $B$ be two fuzzy sets. The accuracy of the intuitionistic fuzzy set $B$ relative to intuitionistic fuzzy set $A$ is given by

$$\mathcal{A}(A;B) = \frac{1}{2} K(A) + \frac{1}{2} C(A,B),$$

where $K(A) = \frac{1}{n} \sum_{i=1}^{n} [\xi_A(x_i) + \eta_A(x_i)]$ and $C(A,B) = \frac{1}{n} \sum_{i=1}^{n} [\xi_A(x_i)\xi_B(x_i) + \eta_A(x_i)\eta_B(x_i)]$.

**Remark 1.** $A$ and $B$ may not be similar at all but $B$ may be accurate to some extent relative to $A$.

For example, if $A = (0, 1)$, $B = (1, 0)$ then $S(A,B) = 0$; $\mathcal{A}(A;B) = 0.5$.

Where we consider the similarity measure $S(A,B) = \frac{\sum_{i=1}^{n} \xi_A(x_i)\xi_B(x_i) + \eta_A(x_i)\eta_B(x_i)}{\sqrt{\sum_{i=1}^{n} \xi_A^2(x_i) + \eta_A^2(x_i)\sqrt{\sum_{i=1}^{n} \xi_B^2(x_i) + \eta_B^2(x_i)}}}$

**Remark 2.** There can be IFSs $A$ and $B$ with higher degree of similarity but very low degree of accuracy.

For example, if $A = (0.4, 0.2)$, $B = (0.5, 0.2)$ then $S(A,B) = 0.9965$; $\mathcal{A}(A;B) = 0.22$.

Where $S(A,B)$ is same as used in remark 1.

**Theorem 6** $\mathcal{A}$ is a valid measure of accuracy.

**Proof** We verify the axioms $A_1 - A_4$.

(A1) This is obvious from the definition.

(A2) Let $\pi_A(x_i) = 1$. This implies $\xi_A(x_i) = \eta_A(x_i) = 0 \forall x_i \in X$, thus

$$\mathcal{A}(A;B) = \frac{1}{2n} \sum_{i=1}^{n} [\xi_A^2(x_i) + \eta_A^2(x_i)] + \frac{1}{2n} \sum_{i=1}^{n} [\xi_A(x_i)\xi_B(x_i) + \eta_A(x_i)\eta_B(x_i)].$$

On the other hand, let

$$\mathcal{A}(A;B) = 0$$

$$\Rightarrow \frac{1}{2n} \sum_{i=1}^{n} [\xi_A^2(x_i) + \eta_A^2(x_i)] + \frac{1}{2n} \sum_{i=1}^{n} [\xi_A(x_i)\xi_B(x_i) + \eta_A(x_i)\eta_B(x_i)] = 0.$$ 

This is possible iff each term in sum on LHS is zero i.e. $\xi_A^2(x_i) = 0$, $\eta_A^2(x_i) = 0$ and $\xi_A(x_i)\xi_B(x_i) = 0$, $\eta_A(x_i)\eta_B(x_i) = 0$ which gives, $\xi_A(x_i) = \eta_A(x_i) = 0$ and $\pi_A(x_i) = 1$.

(A3) Let $A$ and $B$ be crisp sets. Now we consider $\xi_A(x_i) = \xi_B(x_i) = 1$, $\eta_A(x_i) = \eta_B(x_i) = 0$ and $\xi_A(x_i)\xi_B(x_i) = 0$, $\eta_A(x_i)\eta_B(x_i) = 0$ which gives, $\xi_A(x_i) = \eta_A(x_i) = 0$ and $\pi_A(x_i) = 1$.

(A4) We know that

$$\mathcal{A}(A;B) = \frac{1}{2n} \sum_{i=1}^{n} [\xi_A(x_i)\xi_B(x_i) + \eta_A(x_i)\eta_B(x_i)].$$

Now, when $A=B$

$$\mathcal{A}(A;B) = K(A).$$

Thus $\mathcal{A}(A;B)$ is a valid measure of accuracy.

**Theorem 7** Let $A$, $B$, $C$ be three intuitionistic fuzzy sets then $\mathcal{A}(A;B \cup C) + \mathcal{A}(A;B \cap C) = \mathcal{A}(A;B) + \mathcal{A}(A;C)$.
Proof Let

\[ Z_1 = \{ x | x \in X, \xi_B (x_i) \geq \xi_C (x_i) \text{ and } \eta_A (x_i) < \eta_C (x_i) \} \],

\[ Z_2 = \{ x | x \in X, \xi_B (x_i) < \xi_C (x_i) \text{ and } \eta_A (x_i) \geq \eta_C (x_i) \} \].

where \( \xi_A (x_i), \xi_B (x_i) \) and \( \xi_C (x_i) \) are the membership functions and \( \eta_A (x_i), \eta_B (x_i), \) and \( \eta_C (x_i) \) are the non-membership functions of A, B and C respectively. Now

\[
\mathcal{A} (A; B \cup C) = \frac{1}{2n} \sum_{i=1}^{n} [\xi_A^2 (x_i) + \eta_A^2 (x_i)] + \frac{1}{2n} \sum_{i=1}^{n} [\xi_A (x_i) + \eta_A (x_i)] \xi_B (x_i) + \eta_A (x_i) \eta_B (x_i)]
\]

\[
= \frac{1}{2n} \sum_{i=1}^{n} [\xi_A^2 (x_i) + \eta_A^2 (x_i)] + \frac{1}{2n} \sum_{i=1}^{n} [\xi_A (x_i) + \eta_A (x_i)] \xi_B (x_i) + \eta_A (x_i) \eta_B (x_i)]
\]

and

\[
\mathcal{A} (A; B \cap C) = \frac{1}{2n} \sum_{i=1}^{n} [\xi_A^2 (x_i) + \eta_A^2 (x_i)] + \frac{1}{2n} \sum_{i=1}^{n} [\xi_A (x_i) + \eta_A (x_i)] \xi_B (x_i) + \eta_A (x_i) \eta_B (x_i)]
\]

Therefore, we have

\[
\mathcal{A} (A; B \cup C) + \mathcal{A} (A; B \cap C) = \mathcal{A} (A; B) + \mathcal{A} (A; C).
\]

Theorem 8 Let A, B, C be three intuitionistic fuzzy sets then

\[
\mathcal{A} (A \cup B; C) + \mathcal{A} (A \cap B; C) = \mathcal{A} (A; C) + \mathcal{A} (B; C).
\]

Proof Let

\[ Z_1 = \{ x | x \in X, \xi_A (x_i) \geq \xi_B (x_i) \text{ and } \eta_A (x_i) < \eta_B (x_i) \} \],

\[ Z_2 = \{ x | x \in X, \xi_A (x_i) < \xi_B (x_i) \text{ and } \eta_A (x_i) \geq \eta_B (x_i) \} \].

where \( \xi_A (x_i), \xi_B (x_i) \) and \( \xi_C (x_i) \) are the membership functions and \( \eta_A (x_i), \eta_B (x_i), \) and \( \eta_C (x_i) \) are the non-membership functions of A, B and C respectively. We have

\[
\mathcal{A} (A \cup B; C) = \frac{1}{2n} \sum_{i=1}^{n} [\xi_A^2 (x_i) + \eta_A^2 (x_i)] + \frac{1}{2n} \sum_{i=1}^{n} [\xi_A (x_i) + \eta_A (x_i)] \xi_B (x_i) + \eta_A (x_i) \eta_B (x_i)]
\]

\[
= \frac{1}{2n} \sum_{i=1}^{n} [\xi_A^2 (x_i) + \eta_A^2 (x_i)] + \frac{1}{2n} \sum_{i=1}^{n} [\xi_A (x_i) + \eta_A (x_i)] \xi_B (x_i) + \eta_A (x_i) \eta_B (x_i)]
\]

and

\[
\mathcal{A} (A \cap B; C) = \frac{1}{2n} \sum_{i=1}^{n} [\xi_A^2 (x_i) + \eta_A^2 (x_i)] + \frac{1}{2n} \sum_{i=1}^{n} [\xi_A (x_i) + \eta_A (x_i)] \xi_B (x_i) + \eta_A (x_i) \eta_B (x_i)]
\]

Therefore,
\[ \mathcal{A}(A \cup B; C) + \mathcal{A}(A \cap B; C) = \mathcal{A}(A; C) + \mathcal{A}(B; C). \]

**Theorem 9** Let \( A, B, C \) be three intuitionistic fuzzy sets then
\[
\mathcal{A}(A \cup B; A \cap B) + \mathcal{A}(A \cap B; A \cup B) = \mathcal{A}(A; B) + \mathcal{A}(B; A).
\]

**Proof** Let
\[ Z_1 = \{x \in X, \xi_A(x) \geq \xi_B(x) \& \eta_A(x) < \eta_B(x)\}, \]
\[ Z_2 = \{x \in X, \xi_A(x) < \xi_B(x) \& \eta_A(x) \geq \eta_B(x)\}. \]

where \( \xi_A(x), \xi_B(x) \) and \( \xi_C(x) \) are the membership functions and \( \eta_A(x), \eta_B(x) \) and \( \eta_C(x) \) are the non-membership functions of \( A, B \) and \( C \) respectively. We have
\[
\mathcal{A}(A \cup B; A \cap B) = \frac{1}{2n} \sum_{x \in Z_1} \xi_A(x) + \xi_B(x) + \frac{1}{2n} \sum_{x \in Z_2} \xi_A(x) \xi_B(x) + \eta_A(x) \eta_B(x) + \frac{1}{2n} \sum_{x \in Z_1} \eta_A(x) \eta_B(x) + \frac{1}{2n} \sum_{x \in Z_2} \eta_A(x) \eta_B(x).
\]

and \( \mathcal{A}(A \cap B; A \cup B) \)
\[
\mathcal{A}(A \cap B; A \cup B) = \frac{1}{2n} \sum_{x \in Z_1} \xi_A(x) \xi_B(x) + \eta_A(x) \eta_B(x) + \frac{1}{2n} \sum_{x \in Z_2} \xi_A(x) \xi_B(x) + \eta_A(x) \eta_B(x).
\]

**Theorem 10** Let \( A, B \) be two intuitionistic fuzzy sets. Then
(a) \( \mathcal{A}(A; B) = \mathcal{A}(\overline{A}; \overline{B}) \),
(b) \( \mathcal{A}(A; \overline{A}) = \mathcal{A}(\overline{A}; A) \),
(c) \( \mathcal{A}(A; B) = \mathcal{A}(\overline{A}; B) \),
(d) \( \mathcal{A}(A; B) + \mathcal{A}(\overline{A}; B) = \mathcal{A}(\overline{A}; \overline{B}) + \mathcal{A}(A; B) \).

**Proof** The proof of this theorem is an easy calculation.

9. **Application of accuracy measure in pattern recognition**

**Problem formulation:** Let \( C_1, C_2, \ldots, C_n \) be some known patterns characterized by IFS in the universal set \( Y = \{z_1, z_2, \ldots, z_k\} \) as follows:
\[ C_i = \{ (z_j, \xi_C(z_j), \eta_C(z_j)) | z_j \in Y, \ j = 1, 2, \ldots, k \}. \]

Let
\[ B = \{ (z_j, \xi_B(z_j), \eta_B(z_j)) | z_j \in Y, \ j = 1, 2, \ldots, k \}. \]

be an unknown pattern. The problem is to classify pattern \( B \) into one of the known patterns \( C_i \).

The solution of the problem can be obtained as follows:

1. **Distance or Dissimilarity measure approach**
   Let \( d(C_i; B) = \text{Distance or Dissimilarity of pattern } B \text{ from } C_i \). Then \( B \) is assigned to \( C_i \)
   \[ i^* = \arg \min_{i=1, 2, \ldots, n} \{d(C_i; B)\}. \]
2. Similarity measure approach
Let \( S(C_i;B) \) = Similarity of pattern B from \( C_i \). Then B is assigned to \( C_i \).

\[
i^* = \arg \max_{i=1,2,\ldots,n} \{ S(C_i;B) \}.
\]

3. Accuracy measure approach
Let \( \alpha(C_i;B) \) = Accuracy of pattern B from \( C_i \). Then B is assigned to \( C_i \).

\[
i^* = \arg \max_{i=1,2,\ldots,n} \{ \alpha(C_i;B) \}.
\]

Xiao et al.\(^{41}\) conducted a comprehensive investigation of pattern recognition problem by distance/dissimilarity measures. Boran et al.\(^7\) gave comparative study of various existing similarity measures in pattern recognition problems. In the comparative studies regarding distance/dissimilarity measures (Xiao et al.\(^{41}\) and the references there in) and similarity measures (Boran et al.\(^7\) and the references there in), we observe that there is neither a distance/dissimilarity measure nor a similarity measure which suits to every problem of pattern recognition. This happens due to some counter intuitive situations. Thus, a new distance/dissimilarity, similarity measure or some alternative model is always desirable for pattern recognition problems. Our proposed accuracy measure is also an alternative and may be more effective than existing distance/dissimilarity and similarity measures in some pattern recognition problems. For the sake of comparative study and demonstration of effectiveness of the proposed accuracy measure, we consider the examples from Boran and Akay\(^7\) in pattern recognition problem for applying accuracy measure approach.

**Example 6**\(^7\) Let \( C_1, C_2 \) and \( C_3 \) be the IFSs which represents three known patterns in the universal set \( Y = \{z_1, z_2, z_3, z_4\} \) respectively which are given below

\[
C_1 = \{ \langle z_1, 0.5, 0.3 | z_1 \in Y \rangle, \langle z_2, 0.7, 0.0 | z_2 \in Y \rangle, \langle z_3, 0.4, 0.5 | z_3 \in Y \rangle, \langle z_4, 0.7, 0.3 | z_4 \in Y \rangle \},
\]

\[
C_2 = \{ \langle z_1, 0.5, 0.2 | z_1 \in Y \rangle, \langle z_2, 0.6, 0.1 | z_2 \in Y \rangle, \langle z_3, 0.2, 0.7 | z_3 \in Y \rangle, \langle z_4, 0.7, 0.3 | z_4 \in Y \rangle \},
\]

\[
C_3 = \{ \langle z_1, 0.5, 0.4 | z_1 \in Y \rangle, \langle z_2, 0.7, 0.1 | z_2 \in Y \rangle, \langle z_3, 0.4, 0.6 | z_3 \in Y \rangle, \langle z_4, 0.7, 0.2 | z_4 \in Y \rangle \},
\]

Now our aim is that the unknown pattern which is represented by the IFS B can be classified into one of the patterns \( C_1, C_2, \) or \( C_3 \). The unknown pattern B is given below

\[
B = \{ \langle z_1, 0.4, 0.3 | z_1 \in Y \rangle, \langle z_2, 0.7, 0.1 | z_2 \in Y \rangle, \langle z_3, 0.3, 0.6 | z_3 \in Y \rangle, \langle z_4, 0.7, 0.3 | z_4 \in Y \rangle \},
\]

For some existing distance measures, the degrees of dissimilarity/distances \( d(C_1, B), d(C_2, B) \) and \( d(C_3, B) \) are calculated \(^{41}\). The results obtained are shown in Table 7.

Table 7. The distance between known and unknown patterns in Example 6 (Patterns not discriminated are in bold italic). \( p = 1 \) in \( d^p_i \) and \( t = 2, p = 1 \) in \( d^{2p}_i \)

<table>
<thead>
<tr>
<th>Distances</th>
<th>( d(C_1, B) )</th>
<th>( d(C_2, B) )</th>
<th>( d(C_3, B) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d^4_L )</td>
<td>0.1083</td>
<td>0.1208</td>
<td>0.0917</td>
</tr>
<tr>
<td>( d^3_H )</td>
<td>0.0750</td>
<td>0.0750</td>
<td>0.1000</td>
</tr>
<tr>
<td>( d^3_E )</td>
<td>0.0866</td>
<td>0.0866</td>
<td>0.1118</td>
</tr>
<tr>
<td>( l^3_0 )</td>
<td>0.0750</td>
<td>0.0750</td>
<td>0.0750</td>
</tr>
<tr>
<td>( l^4_0 )</td>
<td>0.0750</td>
<td>0.0750</td>
<td>0.1000</td>
</tr>
<tr>
<td>( d^1_0 )</td>
<td>0.0625</td>
<td>0.0687</td>
<td>0.0625</td>
</tr>
<tr>
<td>( d^2_0 )</td>
<td>0.0500</td>
<td>0.0625</td>
<td>0.0500</td>
</tr>
<tr>
<td>( d^2_0 )</td>
<td>0.0500</td>
<td>0.0500</td>
<td>0.0500</td>
</tr>
<tr>
<td>( d^3_L )</td>
<td>NaN</td>
<td>NaN</td>
<td>NaN</td>
</tr>
<tr>
<td>( d^4_L )</td>
<td>0.0500</td>
<td>0.062</td>
<td>0.0333</td>
</tr>
</tbody>
</table>

From the results obtained in Table 7, we observe that the distance \( d_0 H, d_0 E, l_0, d_0^1, d_0^2, d_0^3, d_0^4 \) and \( d_0^2 \) are not able to classify the pattern B into one of the problem \( C_i (i = 1, 2, 3) \) as value of \( d(C_i; B) \) is same to the value of \( i \) (shown in bold). Only, the distance \( d_L \) can classify B in the pattern \( C_3 \).

Boran and Akay\(^7\) also considers the same example and uses similarity measure approach. The results are shown in Table 8 for various existing similarity measures.
Table 8. The similarity between known and unknown patterns in Example 6 (Patterns not discriminated are in bold italic) \((p = 1 \text{ in } S_{HB}, S_{C}^{p}, S_{L}^{p}, S_{s}^{p} \text{ and } p = 1, t = 2 \text{ in } S_{L}^{t})\)

<table>
<thead>
<tr>
<th>Similarity measures</th>
<th>(S(C_{1}, B))</th>
<th>(S(C_{2}, B))</th>
<th>(S(C_{3}, B))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_{C}^{30})</td>
<td>0.925</td>
<td>0.863</td>
<td>0.925</td>
</tr>
<tr>
<td>(S_{H}^{33})</td>
<td>0.975</td>
<td>0.963</td>
<td>0.975</td>
</tr>
<tr>
<td>(S_{L}^{31})</td>
<td>0.950</td>
<td>0.938</td>
<td>0.963</td>
</tr>
<tr>
<td>(S_{O}^{15})</td>
<td>0.929</td>
<td>0.921</td>
<td>0.929</td>
</tr>
<tr>
<td>(S_{DC}^{13})</td>
<td>0.950</td>
<td>0.938</td>
<td>0.950</td>
</tr>
<tr>
<td>(S_{HB}^{46})</td>
<td>0.950</td>
<td>0.938</td>
<td>0.950</td>
</tr>
<tr>
<td>(S_{C}^{16})</td>
<td>0.950</td>
<td>0.938</td>
<td>0.950</td>
</tr>
<tr>
<td>(S_{S}^{16})</td>
<td>0.950</td>
<td>0.938</td>
<td>0.950</td>
</tr>
<tr>
<td>(S_{L}^{12})</td>
<td>0.925</td>
<td>0.925</td>
<td>0.925</td>
</tr>
<tr>
<td>(S_{s}^{12})</td>
<td>0.860</td>
<td>0.860</td>
<td>0.860</td>
</tr>
<tr>
<td>(C_{IFS}^{36})</td>
<td>0.991</td>
<td>0.987</td>
<td>0.996</td>
</tr>
<tr>
<td>(S_{C}^{16})</td>
<td>0.950</td>
<td>0.938</td>
<td>0.967</td>
</tr>
</tbody>
</table>

We observe that the similarity measures \(S_{C}, S_{H}, S_{O}, S_{HB}, S_{C}^{p}, S_{L}^{p}, S_{s}^{p}\) and \(S_{s}^{p}\) are unable to classify pattern \(B\). But the similarity measures \(S_{L}, S_{DC}, S_{C}^{3}, S_{IFS}\) and \(S_{s}^{C}\) can classify pattern \(B\) into pattern \(C_{3}\).

Using both approaches distance/dissimilarity and similarity measure the pattern \(B\) is classified into the pattern \(C_{3}\).

Now, we apply accuracy measure approach. The value of accuracy of pattern \(B\) from \(C_{1}, C_{2}\), and \(C_{3}\) are as follows: \(\alpha/(C_{1}, B) = 0.445, \alpha/(C_{2}, B) = 0.4412\) and \(\alpha/(C_{3}, B) = 0.4538\).

Therefore, the accuracy measure approach also classify the pattern \(B\) into the pattern \(C_{3}\). Hence, our accuracy measure approach is effective in this pattern recognition problem.

Example 7 Let \(C_{1}, C_{2}\) and \(C_{3}\) be the IFSs which represents three known patterns in the universal set \(Y = \{z_{1}, z_{2}, z_{3}, z_{4}\}\) respectively which are given below:

\[
\begin{align*}
C_{1} & = \{(z_{1}, 0.8, 0.1|z_{1} \in Y), (z_{2}, 0.5, 0.3|z_{2} \in Y), \\
C_{2} & = \{(z_{3}, 0.5, 0.5|z_{3} \in Y), (z_{4}, 0.6, 0.1|z_{4} \in Y), \\
C_{3} & = \{(z_{3}, 0.6, 0.3|z_{3} \in Y), (z_{4}, 0.7, 0.2|z_{4} \in Y), \\
\end{align*}
\]

The unknown pattern \(B\) is given below:

\[
\begin{align*}
B & = \{(z_{1}, 0.7, 0.2|z_{1} \in Y), (z_{2}, 0.5, 0.2|z_{2} \in Y), \\
& \quad (z_{3}, 1, 0|z_{3} \in Y), (z_{4}, 0.4, 0.3|z_{4} \in Y)\}.
\end{align*}
\]

For some existing distance measures, the degrees of dissimilarity/distances \(d(C_{1}, B), d(C_{2}, B)\) and \(d(C_{3}, B)\) are calculated by Xiao et al. 41. The results obtained are shown in Table 9.

Table 9. The distance between known and unknown patterns in Example 7 (Patterns not discriminated are in bold italic) \((p = 1 \text{ in } d_{L}^{3}, d_{L}^{6}, p = 1, t = 2 \text{ in } d_{d}^{t})\)

<table>
<thead>
<tr>
<th>Distances</th>
<th>(d(C_{1}, B))</th>
<th>(d(C_{2}, B))</th>
<th>(d(C_{3}, B))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d_{L}^{3})</td>
<td>0.3625</td>
<td>0.5792</td>
<td>0.3042</td>
</tr>
<tr>
<td>(d_{E}^{34})</td>
<td>0.2250</td>
<td>0.3500</td>
<td>0.2250</td>
</tr>
<tr>
<td>(d_{E}^{35})</td>
<td>0.2784</td>
<td>0.5148</td>
<td>0.2345</td>
</tr>
<tr>
<td>(l_{b}^{32})</td>
<td>0.2250</td>
<td>0.3500</td>
<td>0.2250</td>
</tr>
<tr>
<td>(l_{ch}^{39})</td>
<td>0.2250</td>
<td>0.3500</td>
<td>0.2250</td>
</tr>
<tr>
<td>(d_{l}^{35})</td>
<td>0.2188</td>
<td>0.2813</td>
<td>0.1937</td>
</tr>
<tr>
<td>(d_{l}^{35})</td>
<td>0.2125</td>
<td>0.2125</td>
<td>0.1625</td>
</tr>
<tr>
<td>(d_{l}^{124})</td>
<td>0.2350</td>
<td>0.3250</td>
<td>0.1625</td>
</tr>
<tr>
<td>(d_{l}^{24})</td>
<td>0.2125</td>
<td>0.2125</td>
<td>0.1625</td>
</tr>
</tbody>
</table>

From the results obtained in Table 9, we observe that the distance \(d_{L}, l_{b}, l_{ch}, d_{l}^{3}, d_{l}^{3} \) are not able to classify the pattern \(B\) into one of the problem \(C_{i}\) \(i = 1, 2, 3\) as value of \(d(C_{i}, B)\) is same to the value of \(i\) (shown in bold). The distances \(d_{L}, d_{E}, d_{l}^{3}\) and \(d_{l}^{3}\) are able to classify pattern \(B\) in the pattern \(C_{3}\).

We also considers the same example and use similarity measure approach. The degree of similarity for various measures is shown in Table 10.

Table 10. The similarity between known and unknown patterns in Example 7 (Patterns not discriminated are in bold italic) \((p = 1 \text{ in } S_{HB}, S_{C}^{p}, \text{ and } p = 1, t = 2 \text{ in } S_{L}^{t})\)

<table>
<thead>
<tr>
<th>Similarity measures</th>
<th>(S(C_{1}, B))</th>
<th>(S(C_{2}, B))</th>
<th>(S(C_{3}, B))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_{C}^{30})</td>
<td>0.7875</td>
<td>0.7875</td>
<td>0.825</td>
</tr>
<tr>
<td>(S_{H}^{33})</td>
<td>0.8625</td>
<td>0.7875</td>
<td>0.825</td>
</tr>
<tr>
<td>(S_{L}^{31})</td>
<td>0.7875</td>
<td>0.7875</td>
<td>0.825</td>
</tr>
<tr>
<td>(S_{O}^{15})</td>
<td>0.8461</td>
<td>0.8095</td>
<td>0.8661</td>
</tr>
<tr>
<td>(S_{DC}^{13})</td>
<td>0.515</td>
<td>0.6125</td>
<td>0.5331</td>
</tr>
<tr>
<td>(S_{HB}^{46})</td>
<td>0.7875</td>
<td>0.7875</td>
<td>0.825</td>
</tr>
<tr>
<td>(S_{C}^{16})</td>
<td>0.9813</td>
<td>0.9969</td>
<td>0.9919</td>
</tr>
<tr>
<td>(S_{L}^{12})</td>
<td>0.975</td>
<td>0.95</td>
<td>0.975</td>
</tr>
<tr>
<td>(S_{S}^{12})</td>
<td>0.9609</td>
<td>0.9228</td>
<td>0.9609</td>
</tr>
<tr>
<td>(S_{O}^{12})</td>
<td>0.9512</td>
<td>0.9048</td>
<td>0.9512</td>
</tr>
<tr>
<td>(C_{IFS}^{36})</td>
<td>0.8926</td>
<td>NaN</td>
<td>0.9451</td>
</tr>
</tbody>
</table>

We observe that the similarity measures \(S_{C}, S_{L}, S_{HB}, S_{C}^{p}, S_{L}^{p}, S_{O}, S_{HY}, S_{HY}, C_{IFS}\) and \(S_{s}^{C}\) are unable to classify pattern \(B\). But the similarity measures \(S_{H}, S_{O}, S_{DC}\) and \(S_{s}^{C}\) are able to classify pattern \(B\) into pattern \(C_{3}\).
Using both approaches distance/dissimilarity and similarity measure the pattern B is classified into the pattern C3.

Now, we apply accuracy measure approach. The value of accuracy of pattern B from C1, C2, and C3 are as follows: \( \Delta(C_1, B) = 0.46625, \Delta(C_2, B) = 0.3725 \) and \( \Delta(C_3, B) = 0.47125 \).

Therefore, the accuracy approach also classify the pattern B into the pattern C3. Hence, the accuracy measure approach is effective in this pattern recognition problem.

10. Conclusion and future studies

In this study, we have introduced a measure of knowledge contained in an IFS and investigated the applicability and effectiveness of this knowledge measure in MADM problems. It has been observed that some existing knowledge measures are useful for a large degree of hesitancy in IFS while our proposed knowledge measure is useful in the problems in which the IFS have the small degree of hesitancy. We have also introduced an unorthodox measure of accuracy of an IFS relative to a given IFS. Further, we have shown the effectiveness and application of the proposed accuracy measure in pattern recognition problems through illustrative examples.

The proposed accuracy measure has been found to be an effective alternative to similarity and dissimilarity measures in the study of pattern recognition problems. The usefulness and some potential applications of intuitionistic fuzzy information measures presented in this work may be summarized as follows:

1). The optimization problem dealt with fuzzy entropy or intuitionistic fuzzy entropy alone may provide better insight to the experts if knowledge measure is also considered along with entropy measure.

2). The accuracy measure between IFSs (asymmetric similarity measure) may provide robust solutions to the problems where asymmetric similarity measures are desired.

3). In some counter-intuitive situations the proposed accuracy measure recognise the pattern but some similarity measures are unable to do so.

4). The proposed accuracy measure can be applied in the problems of binary image segmentation.

Our future studies includes:

1). Development of an aggregated or hybrid intuitionistic fuzzy information comprising intuitionistic fuzzy entropy and intuitionistic fuzzy knowledge measure.

2). To apply intuitionistic fuzzy accuracy measure in image segmentation problem.

3). Extension of the proposed knowledge measure and accuracy measure to interval-valued intuitionistic fuzzy sets/hesitant fuzzy sets.

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References

17. X.D. Liu, S.H. Zheng and F.L. Xiong. Entropy and sub-
sethood for general interval-valued intuitionistic fuzzy sets.
Goswami and G. Beliakov. Uncertainties with Atanassovs
intuitionistic fuzzy sets:Fuzziness and lack of knowledge.
19. E. Szmidt and J. Kacprzyk. Entropy for intuitionistic fuzzy
20. I. K. Vlachos. Subsethood, entropy and cardinality for in-
terval-valued fuzzy sets-an algebraic derivation. Fuzzy Sets
21. C. P. Wei, P. Wang and Y. Z. Zhang. Entropy, similarity mea-
sure of interval-valued intuitionistic fuzzy sets and their ap-
22. W. Y. Zeng and H. X. Li. Relationship between similarity
measure and entropy of interval-valued fuzzy sets. Fuzzy
measures for vague sets and its applications. Information
24. H. Zhang and L. Yu. New distance measures between intu-
itionistic fuzzy sets and interval-valued fuzzy sets. Informa-
25. H. Y. Zhang, W. X. Zhang and C. L. Mei. Entropy of in-
terval-valued fuzzy sets based on distance and its relation-
ship with similarity measure. Knowledge-Based Systems,
26. E. Szmidt and J. Kacprzyk. Some problems with entropy
measures for the Atanassov intuitionistic fuzzy sets. Lecture
27. E. Szmidt, J. Kacprzyk and P. Bujnowski. How to measure
amount of knowledge conveyed by Atanassovs intuitionis-
28. K. Guo. Knowledge measure for Atanassov’s intuitionis-
tic fuzzy sets. IEEE Transactions on Fuzzy Systems, 24:1072–
1078, 2016.
29. K. Guo and Q. Song. On the entropy for Atanassov’s intu-
initionistic fuzzy sets: An interpretation from the perspective
30. S. M. Chen. Measures of similarity between vague sets.
31. L. Fan and X. Zhangyan. Similarity measures between vague
sets. Journal of systems and software, 12: 922-927,
32. P. Grzegorzewski. Distneces between intuitionistic fuzzy
sets and/or intervalvalued fuzzy sets based on the Hausdorff
metric.
33. D. H. Hong and C. Kim. A note on similarity measures be-
tween vague sets and between elements. Information sci-
34. E. Szmidt and J. Kacprzyk. Distances between intuitionistic
35. W. Wang and X. Xin. Distance measure between intuitionis-
tic fuzzy sets. Pattern recognition Letters, 26:2063–2069,
2005.
36. J. Ye. Cosine similarity measures for intuitionistic fuzzy sets
and their applications. Mathematical and Computer Mod-
37. D. Wu, J. Lu and G. Zhang. Similarity measure models and
algorithms for hierarchical cases. Expert systems with ap-
personalized e-learning recommender system. IEEE trans-
39. Y. Zhang, L. Shang, L. Huang, A. L. Porter, J. Lu and D.
Zhu. A hybrid similarity measure method for pattern portfo-
Topic analysis and forecasting for science, technology and
innovation: Methodology and a case study focusing on big
data research. Technological forecasting and social change,
41. L. Xiao, L. Weimin and Z. Wei. Intuitive distance for intu-
initionistic fuzzy sets with applications in pattern recogni-
tion. Applied intelligence, DOI: 10.1007/s10489–017–1091–0,
2017.
42. I. Montes, N. R. Pal, V. Janis and S. Montes. Divergence
measures for intuitionistic fuzzy sets. IEEE Transactions on
43. J. Mao, D. Yao and C. Wang. A novel cross-entropy and en-
tropy measures of IFSs and their applications. Knowledge-
44. S. K. De, R. Biswas, and A. R. Roy, Some operations on in-
tuitionistic fuzzy sets. Fuzzy sets and systems, 105:179–191,
2000.
45. S. M. Chen and J. M. Tan. Handling multicriteria fuzzy deci-
sion making problem based on vague set theory. Fuzzy sets
and systems, 67: 163–172, 1994. Fuzzy sets and systems,
46. H. B. Mitchell. On the Dengfeng-Chuntian similarity mea-
sure and its application to pattern recognition. Pattern recog-
multi-criteria decision-making method based on intuitionis-
tic fuzzy entropy. Control and decision, 27:1694–1698,
2012.
48. Z. Xu. Intuitionistic fuzzy aggregation operators. IEEE
49. Y. Yang and F. Chiclana. Consistency of 2D and 3D dis-
tances of intuitionistic fuzzy sets. Expert systems with ap-