Abstract—The asymptotic expansion method is used to derive the solutions of a homogeneous isotropic strata subjected to uniform loading on the ground surface. The study develops a mathematical model for the excess pore fluid pressure and land deformation of the strata. Analytical solutions are derived through the application of Laplace transform with respect to the variable of time. The results can improve understanding of the asymptotic expansion theory on coupled consolidation.

Keywords—asymptotic expansion; coupled consolidation; Laplace transform; analytical solution

I. INTRODUCTION

Responses of strata due to ground surface loading is an important engineering issue. Regarding impact on engineering safety, many studies were concentrated on mechanical and hydraulic behavior due to coupled consolidation. Hydraulic disturbance usually results in a volumetric change of fluid and solid skeleton. The volumetric change can increase excess pore fluid pressure and lead to decrease in effective stress. The loss of shear resistance of solid skeleton may result in a hydraulic failure in the strata. The simulation and its validation are major concern for the safety improvement of the engineering construction.


The present investigation is focused on deriving analytical solutions of an isotropic pervious ground space due to uniform loading. The soil or rock mass is modelled as a linearly elastic medium with isotropic properties, and the mechanical properties and hydraulic fluid flow are treated as isotropic. Using Laplace integral transform, the analytical solutions of excess pore fluid pressure and displacements of the strata due to uniform loading are obtained.

II. MATHEMATICAL MODEL

A. Basic Equations

Figure 1 displays a surface uniform loading on the pervious porous elastic strata, and the isotropic soil or rock is modeled as a homogeneous elastic medium. The constitutive law for an elastic medium can thus be expressed by

$$
\tau_{ij}^* = \frac{2Gv}{1-2\nu} \epsilon_i^* \epsilon_j^* - \alpha \rho^* \delta_{ij}, \ i, j = 1, 2, 3, \quad (1a)
$$

$$
\delta^* = \alpha \epsilon^* + \frac{p^*}{Q}, \quad (1b)
$$

where $\tau_{ij}^*$, $\epsilon_i^*$, and $p^*$ are the total stress components, strain components, and excess pore fluid pressure of the strata, respectively. The increment of fluid volume within unit volume of porous medium is denoted by $\delta^*$. The volumetric strain of the porous strata $\epsilon^* = \epsilon_{11}^* + \epsilon_{22}^* + \epsilon_{33}^*$. The material constants $G$ and $\nu$ are shear modulus and Poisson’s ratio of the isotropic strata. The coupled material constants $\alpha$ and $Q$ are defined by
\[
\alpha = \frac{3(v_u - \nu)}{B(1-2\nu)(1+v_u)}, \quad (2a)
\]

\[
\frac{1}{Q} = \frac{9(v_u - \nu)(1-2v_u)}{2GB^2(1-2\nu)(1+v_u)^3}, \quad (2b)
\]
in which \(B\) and \(v_u\) are Skempton pore water pressure coefficient and undrained Poisson’s ratio defined by Rice and Cleary [7]. The strain components \(e_{ij}^*\) are related to displacement components \(u_{ij}^*\) of the strata as below:

\[
e_{ij}^* = \frac{1}{2} \left( \frac{\partial u_{ij}^*}{\partial x_j} + \frac{\partial u_{ij}^*}{\partial x_i} \right), \quad i, j = 1, 2, 3. \quad (3)
\]

In general, the total stresses \(\tau_{ij}^*\) satisfy the equilibrium equations:

\[
\tau_{ij}^* + b_i^* = 0, \quad i, j = 1, 2, 3, \quad (4)
\]
where \(b_i^*\) denote the body force components. With the effect of body forces neglected, the equilibrium equations can be expressed in terms of displacements and excess pore fluid pressure as follows:

\[
GV^2 u_{ij}^* + \frac{G}{1-2\nu} \frac{\partial e_{ij}^*}{\partial t} - \alpha \frac{\partial p^*}{\partial x_i} = 0, \quad (5a)
\]

\[
GV^2 u_{ij}^* + \frac{G}{1-2\nu} \frac{\partial e_{ij}^*}{\partial x_j} - \alpha \frac{\partial p^*}{\partial x_2} = 0, \quad (5b)
\]

\[
GV^2 u_{ij}^* + \frac{G}{1-2\nu} \frac{\partial e_{ij}^*}{\partial x_3} - \alpha \frac{\partial p^*}{\partial x_3} = 0, \quad (5c)
\]
in which the Laplacian \(V^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \). The fluid flow should obey Darcy’s law (6a) and continuity equation (6b) as below:

\[
v_i^* = \frac{k}{\gamma_u} \frac{\partial p^*}{\partial x_i}, \quad i = 1, 2, 3, \quad (6a)
\]

\[
\frac{\partial \theta^*}{\partial t} = \frac{\partial v_{1}^*}{\partial x_1} + \frac{\partial v_{2}^*}{\partial x_2} + \frac{\partial v_{3}^*}{\partial x_3}, \quad (6b)
\]
in which \(v_i^*\) denotes the fluid volume flow components through a unit area in a set time. The symbols \(k\) and \(\gamma_u\) are permeability and unit weight of pore water of the porous medium. Using Darcy’s law (6a) and continuity equation (6b), the governing equation for fluid flow can be derived as:

\[
\frac{k}{\gamma_u} \nabla^2 p^* = \alpha \frac{\partial e_{ij}^*}{\partial t} + \frac{1}{Q} \frac{\partial p^*}{\partial t}. \quad (7)
\]

Equations (5a), (5b), (5c) and (7) govern the responses of the porous medium subjected to ground surface loading. In one-dimensional consolidation, the governing equations are simplified as following:

\[
1 \frac{\partial^2 u_{ij}^*}{\partial x_i^2} - \alpha \frac{\partial p^*}{\partial x_i} = 0, \quad (8a)
\]

\[
k \frac{\partial^2 p^*}{\partial x_i^2} = \alpha \frac{\partial^2 u_{ij}^*}{\partial x_i \partial t} + \frac{1}{Q} \frac{\partial p^*}{\partial t}. \quad (8b)
\]

The final compressibility \(a\) is defined as

\[
a = \frac{1-2\nu}{2G(1-\nu)}. \quad (9)
\]

B. Boundary Conditions and Initial Conditions

The uniform load acting on the ground surface is \(p_0\) as shown in Figure 1, and the study can be treated as one-dimensional consolidation model. The soil layer of ground surface boundary \(x_3 = 0\) is regarded as pervious for all times, while the lower boundary at \(x_3 = H^*\) is assumed impervious with a rigid rock shown below:

\[
p^* \left(0,t^*\right) = 0, \quad (10a)
\]

\[
\frac{\partial u_{ij}^* \left(0,t^*\right)}{\partial x_i} = -ap_0, \quad (10b)
\]

\[
\frac{\partial p^* \left(H^*,t^*\right)}{\partial x_3} = 0, \quad (10c)
\]

\[
\frac{\partial u_{ij}^* \left(H^*,t^*\right)}{\partial t} = 0. \quad (10d)
\]
The initial conditions can be treated as [8]:

\[ p^* (x^*, 0^*) = \frac{a - a_i}{a a} p_0, \]  

(10e)

\[ \frac{\partial w^* (x^*, 0^*)}{\partial x^*} = -a_i p_0, \]  

(10f)

where the instantaneous compressibility \( a_i = a / (1 + a^2 a Q) \). If the compressibility of solid skeleton and pore water can be neglected, then \( a_i = 0 \) [8].

III. ASYMPTOTIC EXPANSION

The non-dimensional characteristic displacement \( u_0 = a \sigma_0 H^* \) and characteristic time \( t_0 = \gamma_0 H^* / k \sigma_0 \), where layered stratum thickness is \( H^* \), and critical excess pore water pressure \( \sigma_0 = 10^{-3} G \) [9]. The space \( x^* \), time \( t^* \), vertical displacement \( u^* \) and excess pore water pressure \( p^* \) can be expressed as

\[ z = \frac{x^*}{H^*}, \ t = \frac{t^*}{t_0}, \ w = \frac{u^*}{u_0}, \ p = \frac{p^*}{\sigma_0}, \]  

(11)

in which the symbols \( z \) and \( t \) are dimensionless vertical space variable and dimensionless time variable, respectively; \( w \) and \( p \) are dimensionless vertical displacement and dimensionless excess pore water pressure, respectively. Therefore, the governing equations of (8a) and (8b) can be transformed to:

\[ \frac{\partial^2 w}{\partial z^2} - \alpha \frac{\partial p}{\partial z} = 0, \]  

(12a)

\[ \frac{\partial^2 p}{\partial z^2} - a a \sigma_0 \frac{\partial^2 w}{\partial z \partial t} - \sigma_0 \frac{\partial p}{\partial t} = 0. \]  

(12b)

Besides, the boundary conditions and initial conditions can be expressed as:

\[ p(0, t) = 0, \]  

(13a)

\[ \frac{\partial w(0, t)}{\partial z} = -\frac{p_0}{\sigma_0}, \]  

(13b)

\[ \frac{\partial p(1, t)}{\partial z} = 0, \]  

(13c)

Using proper asymptotic expansion parameter \( \epsilon = \sigma_0 / G = 10^{-3} \) and asymptotic expansion theory, the onedimensional consolidation settlement and excess pore water pressure can be expressed as

\[ w = \epsilon^0 w^{(0)} + \epsilon^1 w^{(1)} + \epsilon^2 w^{(2)} + \epsilon^3 w^{(3)} + \cdots, \]  

(14a)

\[ p = \epsilon^0 p^{(0)} + \epsilon^1 p^{(1)} + \epsilon^2 p^{(2)} + \epsilon^3 p^{(3)} + \cdots, \]  

(14b)

where \( w \) and \( p \) are the dimensionless vertical displacement and excess pore water pressure, respectively. The symbols \( w^{(i)} \) and \( p^{(i)} \) are the \( i \)th-order dimensionless vertical displacement and excess pore water pressure with respect to the asymptotic expansion parameter \( \epsilon^i \).

A. Basic Equations of Order \( \epsilon^0 \)

- Governing equations

\[ \frac{\partial^2 w^{(0)}}{\partial z^2} - \alpha \frac{\partial p^{(0)}}{\partial z} = 0, \]  

(15a)

\[ \frac{\partial^2 p^{(0)}}{\partial z^2} - \frac{\sigma_0}{Q} \frac{\partial p^{(0)}}{\partial t} = 0. \]  

(15b)

- Boundary conditions and initial conditions

\[ p^{(0)} (0, t) = 0, \]  

(16a)

\[ \frac{\partial w^{(0)} (0, t)}{\partial z} = -\frac{p_0}{\sigma_0}, \]  

(16b)

\[ \frac{\partial p^{(0)} (1, t)}{\partial z} = 0, \]  

(16c)
$$w^{(0)}(1,t) = 0,$$  
(16d)  
$$\frac{\partial^2 w^{(i)}}{\partial z^2} - \alpha \frac{\partial p^{(i)}}{\partial z} = 0,$$  
(19a)  
$$p^{(0)}(z,0^+) = \frac{a - a_0}{a \sigma_0} p_0,$$  
(16e)  
$$\frac{\partial^2 p^{(i)}}{\partial z^2} - \frac{\sigma_0}{Q} \frac{\partial p^{(i)}}{\partial t} = \frac{\alpha}{2\eta} \frac{\partial^2 w^{(i)}(s-1)}{\partial z \partial t}.$$  
(19b)  
$$\frac{\partial w^{(0)}(z,0^+)}{\partial z} = -\frac{a \sigma_0}{a \sigma_0}.$$  
(16f)  

B. Basic Equations of Order $e^i$

- Governing equations

$$\frac{\partial^2 w^{(i)}}{\partial z^2} - \alpha \frac{\partial p^{(i)}}{\partial z} = 0,$$  
(17a)  
$$\frac{\partial^2 p^{(i)}}{\partial z^2} - \frac{\sigma_0}{Q} \frac{\partial p^{(i)}}{\partial t} = \frac{\alpha}{2\eta} \frac{\partial^2 w^{(i)}}{\partial z \partial t}.$$  
(17b)  

- Boundary conditions and initial conditions

$$p^{(i)}(0,t) = 0,$$  
(18a)  
$$\frac{\partial w^{(i)}(0,t)}{\partial z} = 0,$$  
(18b)  
$$\frac{\partial p^{(i)}(1,t)}{\partial z} = 0,$$  
(18c)  
$$w^{(i)}(1,t) = 0,$$  
(18d)  
$$p^{(i)}(z,0^+) = 0,$$  
(18e)  
$$\frac{\partial w^{(i)}(z,0^+)}{\partial z} = 0.$$  
(18f)  

C. Basic Equations of Order $e^n$

- Governing equations

$$w(z,t) \approx \frac{p_0}{\sigma_0} (1-z) + \frac{p_0}{\sigma_0} \sum_{n=1}^{\infty} \frac{-4 (a - a_j)}{a \pi^2} \left( \frac{2}{(2n-1)^2} \right)^{\frac{2}{4}} e^{-\frac{(2n-1)^2 \pi^2}{16}} e^{-\frac{(2n-1)^2 \pi^2}{16}} e^{-\frac{2n-1}{4}},$$  
(21a)  

$$p(z,t) \approx \frac{2 (a - a_j)}{\pi a} \sum_{n=1}^{\infty} \left[ \frac{2}{\alpha (2n-1)} + \frac{(2n-1) \pi^2 \alpha}{4\eta} e^{-\frac{(2n-1)^2 \pi^2}{16}} e^{-\frac{2n-1}{4}} \right]$$
where the dimensionless consolidation parameter \( c \) and \( \eta \) are defined as \( c = Q / \sigma_0 \) and \( \eta = (1 - \nu) / (1 - 2\nu) \), respectively. The total settlement can be derived by letting \( z = 0 \) in equation (21a):

\[
\frac{w(0, t)}{\sigma_0} = \frac{P_0}{\sigma_0} \sum_{n=1}^{\infty} \frac{4(a - a_1)}{a^2 \pi^2} \left\{ \frac{2}{(2n-1)^2} + \frac{\pi^2 a^2}{4 \eta} c^2 t \right\} \times \left\{ 1 + e^{-\frac{(2n-1)^2 \pi^2}{4 \eta}} \right\}.
\]

(22)

V. CONCLUSIONS

Based on asymptotic expansion theory of elasticity of porous media, the coupled mathematical model is decoupled by using proper asymptotic expansion parameter \( \varepsilon \) as shown in the mathematical equations. The transient analytical solutions of a homogeneous isotropic elastic layer for one-dimensional deformation and excess pore fluid pressure subjected to uniform loading are presented by equations (21a) and (21b). The results can improve understanding of the asymptotic expansion theory on coupled consolidation.

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REFERENCES


The $i$th-order dimensionless vertical displacement with respect to the asymptotic expansion parameter $\varepsilon^i$ (Dimensionless)

$x^*_i$ Space variable in $i$-direction (m)

$z$ Dimensionless vertical space variable defined in equation (11) (m)

$\alpha$ Coupled material constant defined in equation (2a) (Dimensionless)

$\gamma_w$ Unit weight of pore water (N/m$^3$)

$\delta_{ij}$ Kronecker delta function (Dimensionless)

$\varepsilon$ Asymptotic expansion parameter, $\varepsilon = \sigma_0 / G = 10^{-3}$ (Dimensionless)

$\eta$ Parameter, $\eta = (1 - \nu) / (1 - 2\nu)$ (Dimensionless)

$\phi'$ Increment of fluid volume within unit volume of the isotropic strata (Dimensionless)

$\nu$ Poisson’s ratio of the isotropic strata (Dimensionless)

$\nu_s$ Undrained Poisson’s ratio of the isotropic strata (Dimensionless)

$\sigma_0$ Critical excess pore water pressure, $\sigma_0 = 10^{-3} G$ (Pa)

$\tau^*$ Total stress components of the isotropic strata (Pa)

$\nabla^2$ Laplacian, $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ (m$^{-2}$)