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Behavior of a Scale Factor for the Wiener Integral of a Stochastic Fourier Transform

Young Sik Kim
Department of Mathematics, Research Institute of Natural Sciences, Industry -University Cooperation Foundation, Hanyang University, 222 Wangsimni-ro, Seongdong-gu, Seoul 04763, Korea

***Abstract*—We investigate the behavior of a sacle factor for the Wiener integral of a stochastic Fourier transform of a measure on the abstract Wiener space.**

***Keywords*—abstract Wiener space; Wiener integral.; stochastic Fourier transform**

I. INTRODUCTION

In [1] and [2], R. H. Cameron and W. T. Martin developed Wiener integration theory about transformations of Wiener integrals on the Wiener space.

In [3] R. H. Cameron and W. T. Martin investigated the behavior of measure and measurability under change of scale in Wiener space.

L.Gross[6] and J.Kuelb[14] and H.H. Kuo[15] developed thr Wiener integration theory on the abstract Wiener space .

In [7] and [9], Y.S.Kim proved relationships among the Wiener integral and the analytic Feynman integral.

In this paper, we investigate the behavior of a scale factor for the Wiener integral about the stochastic Fourier transform $F(x) = \int_H \exp\{i\,(h,x)^{\sim}\}\,d\,\mu(h)$ of a measure $\mu \in M(H)$, where $M(H)$ is the class of complex valued countably additive measure $\mu \in M(H)$ defined on the Borel class of H on the abstract Wiener space. ■

II. DEFINITIONS AND PRELIMINARYS

Let H be a real separable infinite dimensional Hilbert space with inner product $<\cdot,\cdot>$ and norm $|\cdot| = \sqrt{<\cdot,\cdot>}$.

Let $||\cdot||_0$ be a measurable norm on H with respect to the Gauss measure $\mu \in M(H)$). Let B denote the completion of H with respect to $||\cdot||_0$. Let i denote the natural injection from H into B. The adjoint operator i^* of i is one-to-one and maps B^* continuously onto a dense subset of H^*, where H^* and B^* are topological duals of H and B, respectively. By identifying H with H^* and B^* with $i^* B^*$, we have a triplet (B^*, H, B) such that $B^* \subset H^* \equiv H \subset B$ and $<h, x> = (h, x)$ for all x in B^* and h in H, where $(\cdot,\cdot)$ denotes the natural dual pairing between B^* and B. By a well known result of Gross [6], $\mu \cdot i^{-1}$ has a unique countably additive extension m to the Borel σ-algebra $B(B)$ on B. Then (B,H,m) is called an abstract Wiener space and m is called a Wiener measure. We denote the Wiener integral of a functional F by $\int_B F(x)dm(x)$.

Let $\{e_j\}_{j=1}^{\infty}$ denote a complete orthonormal system in H such that e_j' s are in B^*. For each $h \in H$ and $x \in B$, we define a stochastic inner product $(\cdot,\cdot)^{\sim}$ between H and B as follows :

$$(h,x)^{\sim} = \begin{cases} \lim_{n\to\infty} \sum_{j=1}^{n} \langle h, e_j\rangle \, (e_j, x), & \textit{if the limit exists} \\ 0, & \textit{otherwise}. \end{cases} \quad (1)$$

It is well known that for every $h \in H$, $(h,x)^{\sim}$ exists for $\mu -$ a.e. x in B and it has a Gaussian distribution with mean zero and variance $|h|^2$. Furthermore, it is easy to show that $(h,x)^{\sim} = (h,x)$ for $\mu -$ a.e. x in B if $h \in B^*$, $(h,x)^{\sim}$ is essentially independent of the complete orthonormal set used in its definition, and finally that if $\{(h_1,x)^{\sim}, \cdots, (h_n,x)^{\sim}\}$ is an orthonormal set of elements in H, then $(h_1,x)^{\sim}, \cdots, (h_n,x)^{\sim}$ are independent Gaussian functionals with mean zero and variance one. Note that if both h and x are in H, then $(h,x)^{\sim} = \langle h, x\rangle$.

Throughout this paper, let R^n denote the n -dimensional Euclidean space and let $C, C_+, C_+^{\sim}$ denote the complex numbers, the complex numbers with positive real part, and the non-zero complex numbers with nonnegative real part, respectively.

Definition 2.1. Let (B,H,m) be an abstract Wiener space.

Let $C_+ = \{z | Re(z) > 0\}$ and $C_+^{\sim} = \{z | Re(z) \geq 0\}$. Let F be a complex-valued scale invariant measurable function on B such that the integral

$$J_F(r) = \int_B F\left(r^{-\frac{1}{2}}x\right) dm(x) \quad (2)$$

exists for all real $r > 0$. If there exists an analytic function $J_F^*(z)$ analytic on C_+ such that $J_F^*(r) = J_F(r)$ for all real $r > 0$, then we define $J_F^*(z)$ to be the analytic Wiener integral of F over B with parameter $z \in C_+$ and for each $z \in C_+$, we write

$$\int_B^{anw_z} F(x) dm(x) = J_F^*(z) \quad (3)$$

Let q be a non-zero real number and let F be a function whose analytic Wiener integral exists for each $z \in C_+$. If the limit exists, then we call it the **analytic Feynman integral** of F over B with parameter q, and we write

$$\int_B^{anf_q} F(x) dm(x) = \lim_{z\to -iq} \int_B^{anf_q} F(x) dm(x) \quad (4)$$

where z approaches $-iq$ through C_+ and $i^2 = -1$. ■

Now we introduce the Fresnel class of functions in the abstract Wiener space.

Definition 2.2. Let Let (B,H,m) be an abstract Wiener space. The Fresnel class F(B) is defined by

$$F(B) = \left\{[F]: F(x) = \int_H exp\{i\,(h,x)^\sim\}\,d\mu(h), x \in B\right\}. \quad (5)$$

where $\mu \in M(H)$ and M(H) is the space of complex valued countably additive measure μ defined on B(H), the Borel class of H. We will identify a function with its s-equivalencd class and think of F(B) as a collection of functions on B rather than as a class of equivalence classes. ■

The following is a well-known **Wiener integration formula** for the Wiener integral on the abstract Wiener space.

Theorem 2.3. Let Let (B,H,m) be an abstract Wiener space. and let F be a function on B of the form $F(x) = f(\,(h,x)^\sim\,)$, where $f: R \to \boldsymbol{C}$ is a Lebesgue measurable function. Then

$$\int_B f(\,(h,x)^\sim\,)dm(x) = \left(\frac{1}{2\pi|h|^2}\right)^{\frac{1}{2}} \int_R f(u)\cdot exp\left\{-\frac{1}{2|h|^2}u^2\right\}\text{du} \quad (6)$$

where " = " means that if either side exists, then both sides exists and they are equal. ■

Remark. In the next section, we will several times the following formula :

$$\int_R exp\{-au^2 + i\,bu\}\,du = \sqrt{\frac{\pi}{a}}\ exp\left\{-\frac{1}{2|h|^2}\,u^2\right\} \quad (7)$$

where a is a complex number with Re(a) > 0, and b is a real number and $i^2 = -1$. ■

III. THE MAIN RESULT

First, we obtain the Wiener integral of the stochastic Fourier transform of a measure $\mu \in M(H)$ in the Fresnel class F(B) on the abstract Wiener space.

Theorem 3.1 Let (B,H,m) be an abstract Wiener space. Let F be the stochastic Fourier transform of a measure $\mu \in M(H)$ in the Fresnel class F(B) of the form (2.5). Then for real $\rho > 0$, the Wiener integral of the function F exists and is of the form :

$$\int_B F(\rho\,x)\,dm(x) = \int_H exp\left\{-\frac{\rho^2}{2}|h|^2\right\}\text{du}\,, \quad (8)$$

Proof. By the Wiener integration formula in Theorem 2.3, we can easily have that for real $\rho > 0$,

$$\int_B F(\rho x)dm(x)$$

$$= \int_B \left[\int_H exp\{i\,(\,h,\rho x)^\sim\}d\mu(h)\right] dm(x)$$

$$= \int_H \left[\int_B exp\{i\,(\,h,\rho x)^\sim\}dm(x)\right] d\mu(h)$$

$$= \int_H exp\left\{-\frac{\rho^2}{2}|h|^2\right\} d\mu(h). \quad (9)$$

Note that for all real $\rho > 0$, $\int_B F(\rho x)dm(x) \leq \parallel \mu \parallel < \infty$.

Therefore, the Wiener integral exists for all real $\rho > 0$.■

By the above result, we can investigate a very interesting behavior of the scale factor for the Wiener integral which was first defined by the author in [13].

Definition 3.2. We define the scale factor for the Wiener integral by the real number $\rho > 0$ of the absolute value of the Wiener integral :

$$G(\rho) = \left|\int_B F(\rho x)dm(x)\right|$$

where G : R → R is a real valued function on R. ■

Remark. For x ∈ B, we shall interprete it as followings :

(1). For real $\rho > 1$, ρ x is a magnification of x ∈ B.

(2). For real $0 < \rho < 1$, ρ x is a minimization of x ∈ B. ■

(a). Behavior of a scale factor for the Wiener integral in the Fresnel class F(B) on the abstract Wiener space.

Whenever we magnify and minimize x ∈ B, the Wiener integral varies very interestingly according to the varying scale factor :

(1) $\int_B F\left(\frac{1}{100}x\right)dm(x) = \int_H exp\left\{-\frac{1}{2\times 10^4}\,|h|^2\right\}\,d\mu(h)$

(2) $\int_B F\left(\frac{1}{10}x\right)dm(x) = \int_H exp\left\{-\frac{1}{2\times 10^2}\,|h|^2\right\}\,d\mu(h)$

(3) $\int_B F(x)dm(x) = \int_H exp\{-\,|h|^2\}\,d\mu(h)$

(4) $\int_B F(10x)dm(x) = \int_H exp\left\{-\frac{10^2}{2}\,|h|^2\right\}d\mu(h)$

(5) $\int_B F(100x)dm(x) = \int_H exp\left\{-\frac{10^4}{2}\,|h|^2\right\}\,d\mu(h)$. ■

(b). Interpretation of a scale factor for the Wiener integral in the Fresnel class F(B) on the abstract Wiener space.

(1) Whenever the scale factor $\rho > 1$ is increasing, the Wiener integral decreases very rapidly.

(2) Whenever the scale factor $0 < \rho < 1$ is decreasing, the Wiener integral increases very rapidly.

(3) The scale factor $\rho > 0$ plays a very interesting behavior of the magnification and the minimization of the Wiener integral !

(4) The function $G(\rho) = \left|\int_B F(\rho x)dm(x)\right|$ is a decreasing function of a scale factor $\rho > 0$, whenever $\rho \to \infty$:

$$(a).\ 0 \leq \left| \int_B F(\rho x) dm(x) \right| \leq \| \mu \|$$

$$(b).\lim_{\rho \to 0} \left| \int_B F(\rho x) dm(x) \right| = \| \mu \|$$

$$(c).\lim_{\rho \to \infty} \left| \int_B F(\rho x) dm(x) \right| = 0$$

(5). Whenever the scale factor $\rho > 0$ increases, the Wiener integral decreases very rapidly. Whenever the scale factor $\rho > 0$ decreases, the Wiener integral increases very rapidly ! ■

Finally, we introduce the Motivation and the Application of the Wiener integral :

Remark.

(1) Motivation : The solution of the heat equation

$$\frac{\partial U}{\partial t} = -HU, \qquad U(0,\cdot) = \varphi(\cdot)$$

is

$$U(t, \varepsilon) = (e^{-tH}\varphi)(\varepsilon)$$

$$= E\left[e^{-\int_0^t V(x(s)+\varepsilon)ds} \cdot \varphi(x(t)+\varepsilon)\right],$$

where $\varphi \in L_2[R^d]$ and $\varepsilon \in R^d$ and $x(\cdot)$ is a $R^d -$ valued continuous function defined on $[0, t]$ such that $x(0) = 0$ and E denotes the expectation with respect to the Wiener path starting at time $t = 0$ and $H = -\Delta + V$ is the energy operator(or, Hamiltonian) and Δ is a Laplacian and $V: R^d \to R$ is a potential. This formula (4.12) is called the Feynman-Kac formula.

(2). **Application** of the Feynman-Kac formula(in various settings) have been given in the area : diffusion equation, the spectral theory of the $schr\ddot{o}dinger$ operator, quantum mechanics, statistical physics.(For more details, see the book [8]. ■

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