

The Convergence Analysis of an Improved Newton-type Method for the Regularization of Nonlinear Ill-Posed Problems

Gui Zhang*, Xiqiang Liu, Yan Zhang and Bingyu Kou

Foundation Department, The Army Engineering University of PLA, Nanjing 211101, China

*Corresponding author

Abstract—An improved Newton-type iteration method for regularizing the abstract nonlinear ill-posed operator equation is presented and also certain stopping criterion to determine the iteration is proposed in this paper by using the Newton-Landber iteration and the linear Tikhonov regularization. Under the condition that the Fréchet-derivation operator is uniformly boundary and further assumptions on the closeness and smoothness of the exact solution, the local convergence of the approximate solution is obtained.

Keywords—nonlinear; ill-posed; operator equations; Newton-type method; convergence

I. INTRODUCTION

This paper concerns with an abstract nonlinear ill-posed operator equation

$$F(x) = y, \quad (1)$$

where $F : D(F) \subset X \rightarrow Y$ is a nonlinear operator between Hilbert spaces X and Y with inner products (\cdot, \cdot) and norms $\|\cdot\|$, respectively. Assume that (1) has a solution x_* and the available approximate data y^δ satisfying $F(x) = y^\delta$ and

$$\|y^\delta - y\| \leq \delta \quad (2)$$

with given noise level $\delta > 0$. We are mainly interested in problems of the form (1) for which the solution x_* does not depend continuously on the right-hand side data y . Such problems are called inverse problems, they are always ill-posed and mostly arise from practical problems in nature science, engineering and technology research fields^[1], such as determination of atmospheric temperature profiles from telemetry data (atmospheric exploration), remote sensing in the ocean, location of tumors by tomography, seismic prospecting, system identification, etc.

In order to obtain reasonable approximations to x_* , many methods especially the linear regularization Tikhonov method

had been presented to achieve this aim. Due to their excellent convergence properties in the well posed situation, Newton-type methods have also been applied to nonlinear inverse problems recently. Kaltenbacher mentions several methods in [2], for example the Levenberg Marquardt method, Gauss-Newton method and Newton-Landber iteration etc.(cf. [3-6])

By using the Newton iteration, Landweber iteration and the linear Tikhonov regularization method, a new iteration is presented in [7], under the condition that the Fréchet-derivation operator is uniformly boundary, the local convergence of the iteration is obtained under certain conditions, and the scheme is proved to be a regularization method under the stopping criterion. An improvement on the Newton-type iteration is presented in this paper, because of the improved restriction conditions of several constants, the application of this method to practice will have much more abroad.

II. THE IMPROVED NEWTON-TYPE METHOD

Assume that F is Fréchet differentiable with $F'(x)$ denoting the Fréchet derivative of F at $x \in D(F)$.

Impose the scaling condition

$$\|F'(x)\| \leq M, \quad x \in B_\rho(x_0), \quad (3)$$

where x_0 is an initial guess which may incorporate a priori knowledge of an exact solution x_* , $B_\rho(x_0)$ is an open ball of radius $\rho > 0$ and centric at x_0 .

Define

$$\begin{aligned} A_n &= F'(x_n) , \quad A_n^\delta = F'(x_n^\delta) , \\ \tilde{\alpha}_n &= (I + \alpha A_n^* A_n)^{-1} , \\ \hat{\alpha}_n &= (I + \alpha A_n^* A_n^\delta)^{-1} , \\ \tilde{\alpha}_n^\delta &= (I + \alpha A_n^{\delta*} A_n^\delta)^{-1} , \\ \hat{\alpha}_n^\delta &= (I + \alpha A_n^{\delta*} A_n^{\delta\delta})^{-1} . \end{aligned}$$

where $F'^*(\cdot)$ is the adjoint operator of F' , $0 \leq \alpha \leq 1$ is a given constant which satisfy:

$$\alpha = \begin{cases} 0 & , \quad 0 < M \leq 1 , \\ 1 - \frac{1}{M^2} & , \quad 1 < M < +\infty . \end{cases}$$

and $M = \sup_{B_\rho(x_0)} \|F'(x)\|$.

The improved Newton-type method considered in this paper is

$$\begin{cases} x_{0,0} = x_0 \\ x_{n,k+1} = x_{n,k} - \alpha \tilde{\alpha}_n A_n^* [F(x_n) - y^\delta + F'(x_n)(x_{n,k} - x_n)] \\ \quad 0 \leq k \leq k_n, n=1,2,3,\dots \\ x_{n+1} = x_{n+1,0} = x_{n,k_n} \end{cases} \quad (4)$$

where

$$\|F'(x)\| \leq (1 - \alpha)^{-1/2} , \quad x \in B_\rho(x_0) \quad (5)$$

Assume that F is Fréchet-differentiable with Lipschitz-continuous Fréchet derivative in $B_\rho(x_0)$; moreover, use the local property of F in $B_\rho(x_0)$:

$$\|F(x) - F(\bar{x}) - F'(x)(x - \bar{x})\| \leq \eta_\alpha \|F(x) - F(\bar{x})\| , \quad (6)$$

$$x , \bar{x} \in B_\rho(x_0) \subset D(F) , \quad 0 < \eta_\alpha < \frac{1}{2} ,$$

and

$$\|F'(x_n^\delta)(x_{n,k}^\delta - x_n^\delta)\| < \varsigma_\alpha \|F(x_n^\delta) - y^\delta\| , \quad (7)$$

the constants η_α and ς_α satisfy

$$(1 - \alpha)(1 - \varsigma_\alpha) - \eta_\alpha > 0 .$$

For given $\delta > 0$, the iteration is stopped at the index $n = N(\delta)$ such that

$$\|y^\delta - F(x_{N(\delta)}^\delta)\| \leq \tau_\alpha \delta < \|y^\delta - F(x_n^\delta)\| \quad (8)$$

for

$$0 \leq n < N(\delta)$$

where

$$\tau_\alpha > \frac{1 + \eta_\alpha}{(1 - \alpha)(1 - \varsigma_\alpha) - \eta_\alpha} > 2 . \quad (9)$$

III. CONVERGENCE ANALYSIS OF THE NEW ITERATION

Some Lemmas need to be proved before the convergence theorems.

Lemma 1. Let (6) and (7) hold, and x_* be a solution of (1) in $B_{\rho/2}(x_0)$, then for all $0 \leq n \leq N(\delta)$,

$$d_{n,k} \leq 2\alpha(1 + \varsigma_\alpha) \|F(x_n^\delta) - y^\delta\|$$

$$\{(1 + \eta_\alpha)\delta - [(1 - \alpha)(1 - \varsigma_\alpha) - \eta_\alpha] \|F(x_n^\delta) - y^\delta\|\} , \quad (10)$$

where $d_{n,k} = \|x_{n,k+1}^\delta - x_*\|^2 - \|x_{n,k}^\delta - x_*\|^2$ and $0 \leq k \leq k_n$.

Proof. For all $0 \leq n \leq N(\delta)$, let

$$\begin{aligned} \gamma_n &:= F(x_n^\delta) - F(x_*) - A_n(x_n^\delta - x_*) , \\ f_n &:= \hat{\alpha}_n^\delta [A_n^\delta(x_{n,k}^\delta - x_*) + \gamma_n + y - y^\delta] . \end{aligned}$$

From iteration (4):

$$\begin{aligned} &x_{n,k+1}^\delta - x_{n,k}^\delta \\ &= -\alpha \tilde{\alpha}_n^\delta A_n^{\delta*} [A_n^\delta(x_{n,k}^\delta - x_*) + \gamma_n + y - y^\delta] \end{aligned}$$

so,

$$d_{n,k} = 2(x_{n,k}^\delta - x_*^\delta, x_{n,k+1}^\delta - x_{n,k}^\delta) + \|x_{n,k+1}^\delta - x_{n,k}^\delta\|^2 \\ \leq 2\alpha \|f_n\| \cdot (-\|f_n\| + \|\gamma_n + y - y^\delta\|) .$$

Let $\left\{E_\lambda \mid \lambda \in \sigma(A_n^\delta A_n^{*\delta}) \subset \left[0, \frac{1}{1-\alpha}\right]\right\}$ be the spectral family generated by $A_n^\delta A_n^{*\delta}$, then for any $y \in Y$,

$$\|\hat{\alpha}_n^\delta y\|^2 = \int_0^{\frac{1}{4}} \frac{1}{(1+\alpha\lambda)^2} d\|E_\lambda y\|^2 ,$$

which gives

$$(1-\alpha)\|y\| \leq \|\hat{\alpha}_n^\delta y\| \leq 1 . \quad (11)$$

By using (6), (7) and (11) it follows

$$(1-\alpha)(1-\zeta_\alpha)\|F(x_n^\delta) - y^\delta\| \leq \|f_n\| \leq (1+\zeta_\alpha)\|F(x_n^\delta) - y^\delta\| . \quad (12)$$

And

$$d_{n,k} \leq 2\alpha(1+\zeta_\alpha)\|F(x_n^\delta) - y^\delta\| \\ \left[(1+\eta_\alpha)\delta - ((1-\alpha)(1-\zeta_\alpha) - \eta_\alpha) \|F(x_n^\delta) - y^\delta\| \right]$$

Combining the above estimates with (12) thus obtain (10). Lemma 2. Under the conditions in lemma 1, the integer $N(\delta)$ defined by (8) and (9) always exists and is finite, and for any $0 \leq n \leq N(\delta)$, there holds

$$x_{n+1}^\delta \in B_{\rho/2}(x_*) \subseteq B_\rho(x_0) , \quad (13)$$

when $\delta = 0$,

$$\sum_{n=0}^{+\infty} \|F(x_n) - y\|^2 < +\infty . \quad (14)$$

Proof. Let

$$\beta_\alpha = 2\alpha(1+\zeta_\alpha)[(1-\alpha)(1-\zeta_\alpha) - \eta_\alpha] > 0 ,$$

$$\beta_\alpha^* = 2\alpha(1+\zeta_\alpha) \left[((1-\alpha)(1-\zeta_\alpha) - \eta_\alpha) - \frac{\eta_\alpha + 1}{\tau_\alpha} \right] > 0 .$$

From (10) and (8), (9) obtain

$$d_{n,k} \leq 2\alpha(1+\zeta_\alpha)\|F(x_n^\delta) - y^\delta\| \cdot \\ \{(1+\eta_\alpha)\delta - [(1-\alpha)(1-\zeta_\alpha) - \eta_\alpha]\tau_\alpha\delta\} \\ \leq 2\alpha(1+\zeta_\alpha)\|F(x_n^\delta) - y^\delta\|\tau_\alpha\delta \\ \left[\frac{\eta_\alpha + 1}{\tau_\alpha} - ((1-\alpha)(1-\zeta_\alpha) - \eta_\alpha) \right] \\ = -\beta_\alpha^*\|F(x_n^\delta) - y^\delta\|^2 \leq 0 .$$

which shows that for all $0 \leq k \leq k_n$

$$\|x_{n,k+1}^\delta - x_*^\delta\| \leq \|x_{n,k}^\delta - x_*^\delta\| . \quad (15)$$

Using the definition of $\{x_n^\delta\}$

$$\|x_{n+1}^\delta - x_*^\delta\| = \|x_{n,k_n}^\delta - x_*^\delta\| \\ \leq \|x_{n,k_n-1}^\delta - x_*^\delta\| \leq \dots \leq \|x_{n,0}^\delta - x_*^\delta\| = \|x_{n-1,k_{n-1}}^\delta - x_*^\delta\| \\ \leq \|x_{n-1,k_{n-1}-1}^\delta - x_*^\delta\| \leq \dots \leq \|x_{n-1,0}^\delta - x_*^\delta\| \\ = \|x_{n-2,k_{n-2}}^\delta - x_*^\delta\| \leq \dots \leq \|x_0 - x_*^\delta\| \leq \frac{\rho}{2} ,$$

so $x_{n+1}^\delta \in B_{\rho/2}(x_*) \subseteq B_\rho(x_0)$, $0 \leq n < N(\delta)$.

For the case $\delta = 0$,

$$d_{n,k} \leq -2\alpha(1+\zeta_\alpha)[(1-\alpha)(1-\zeta_\alpha) - \eta_\alpha]\|F(x_n) - y\|^2 \\ = -\beta_\alpha\|F(x_n) - y\|^2 ,$$

that is $\|x_{n+1} - x_*\|^2 + \beta_\alpha\|F(x_n) - y\|^2 \leq \|x_n - x_*\|^2$.

By induction, it gets

$$\sum_{n=0}^{+\infty} \|F(x_n) - y\|^2 \leq \frac{\|x_0 - x_*\|^2}{\beta_\alpha} < +\infty .$$

So proved lemma 2.

Remark 1. If $\{x_n\}$ be the resultant from (4) by y^δ replaced

by y , then $\sum_{n=0}^{\infty} \|F(x_n) - y\|^2 \leq \frac{\|x_0 - x_*\|^2}{\beta_\alpha} < +\infty$.

Lemma 3. Let (6) hold and x_* be a solution of (1) in $B_{\rho/2}(x_0)$, then for any other solution \bar{x}_* of (1) in

$B_{\rho/2}(x_*)$ there holds $x_* - \bar{x}_* \in N(F'(x_*))$, where $N(F'(x_*))$ denotes the null spaces of $F'(x_*)$.

Proof: from (6), for $x, \bar{x} \in B_{\rho}(x_0)$

$$\frac{\|F'(x)(x - \bar{x})\|}{1 + \eta_\alpha} \leq \|F(x) - F(\bar{x})\| \leq \frac{\|F'(x)(x - \bar{x})\|}{1 - \eta_\alpha},$$

so it can easily to get Lemma 3.

The following Theorem gives the convergence of $\{x_n\}$.

Theorem 1. Let (3), (4), (5) and (6) hold and (1) is solvable in $B_{\rho/2}(x_0)$, then x_n converges to a solution x_* of (1) in $B_{\rho/2}(x_0)$. In addition, $N(F'(x^+)) \subset N(F'(x))$ for all $x \in B_{\rho}(x_0)$, then x_n converges to x^+ which is the unique solution of (1) with minimal distance to x_0 .

Proof. Let \tilde{x}_* be any solution of (1) in $B_{\rho/2}(x_0)$, and put $e_n := \tilde{x}_* - x_n$.

From lemma 2, it follows that $\{\|e_n\|\}$ is monotonically decreasing to some $\varepsilon \geq 0$. It will be showed next that $\{e_n\}$ is a Cauchy sequence. For any integers $p \geq q$, choose an integer l satisfying $p \geq l \geq q$ such that

$$\|y - F(x_l)\| \leq \|y - F(x_i)\|, \quad q \leq i \leq p, \quad (16)$$

because

$$\begin{aligned} \|e_p - e_q\| &\leq \|e_p - e_l\| + \|e_l - e_q\|, \\ \|e_p - e_l\|^2 &= 2(e_l - e_p, e_l) + \|e_p\|^2 - \|e_l\|^2, \\ \|e_l - e_q\|^2 &= 2(e_l - e_q, e_l) + \|e_q\|^2 - \|e_l\|^2. \end{aligned}$$

and

$$\|e_p\|^2 - \|e_l\|^2 \rightarrow 0, \quad \|e_q\|^2 - \|e_l\|^2 \rightarrow 0 \text{ as } q \rightarrow \infty,$$

From (4) :

$$x_{n,k+1} - x_{n,k} = \tilde{\alpha}_n (x_{n,k} - x_{n,k-1}),$$

For

$$x_{n-1,1} - x_{n-1,0} = -\alpha \tilde{\alpha}_{n-1} A_{n-1}^* (F(x_{n-1}) - y),$$

so

$$x_n - x_{n-1} = -\sum_{j=0}^{k_{n-1}-1} (\tilde{\alpha}_{n-1})^j \alpha \tilde{\alpha}_{n-1} A_{n-1}^* (F(x_{n-1}) - y)$$

And

$$\begin{aligned} & |(e_l - e_q, e_l)| \\ & \leq \alpha \sum_{i=q}^{l-1} \left(\sum_{j=1}^{k_i} \|(\hat{\alpha}_i)^j\| \cdot \|F(x_i) - y\| \right) \|A_i (\tilde{x}_* - x_i + x_i - x_l)\| \end{aligned}$$

from (4), (6) and (12):

$$\begin{aligned} & |(e_l - e_q, e_l)| \\ & \leq \alpha K \sum_{i=q}^{l-1} \|F(x_i) - y\| (1 + \eta_\alpha) (\|F(x_i) - y\| + \|F(x_i) - F(x_l)\|) \\ & = \alpha K (1 + \eta_\alpha). \end{aligned}$$

$$\left[\sum_{i=q}^{l-1} \|F(x_i) - y\|^2 + \sum_{i=q}^{l-1} \|F(x_i) - y\| (\|F(x_i) - y\| + \|F(x_i) - F(x_l)\|) \right],$$

from (16):

$$|(e_l - e_q, e_l)| \leq 3\alpha K (1 + \eta_\alpha) \sum_{i=q}^{l-1} \|F(x_i) - y\|^2.$$

Similarly, one can show that

$$|(e_l - e_p, e_l)| \leq 3\alpha K (1 + \eta_\alpha) \sum_{i=l}^{p-1} \|F(x_i) - y\|^2.$$

With these estimates it follows from (14) that $|(e_l - e_q, e_l)| \rightarrow 0$ and $|(e_l - e_p, e_l)| \rightarrow 0$ as $q \rightarrow \infty$.

Therefore,

$$\|e_p - e_l\| \rightarrow 0, \quad \|e_q - e_l\| \rightarrow 0, \quad \text{or} \quad \|e_p - e_q\| \rightarrow 0$$

which shows that $\{e_n\}$, and consequently $\{x_n\}$ is a Cauchy sequence. If assume $x_n \rightarrow x_*$ as $n \rightarrow \infty$, then by noting that $x_n \in B_{\rho/2}(x_0)$ and $\|y - F(x_n)\| \rightarrow 0$ as $n \rightarrow \infty$, know x_* is a solution of (1) in $B_{\rho/2}(x_0)$.

Because of Lemma 3, (1) has a unique solution x^+ of minimal distance to x_0 , which satisfies

$$x^+ - x_0 \in N(F'(x^+))^\perp.$$

If $N(F'(x^+)) \subseteq N(F'(x_n))$ for all $n=1,2,\dots$, then it is clear that

$$x_n - x_0 \in N(F'(x^+))^\perp, n=1,2,\dots,$$

hence,

$$x^+ - x_* = x^+ - x_0 + x_0 - x_* \in N(F'(x^+))^\perp.$$

This together, with lemma 3 implies that $x^+ = x_*$, i.e. $x_n \rightarrow x^+$ as $n \rightarrow +\infty$.

The next result shows that the discrepancy principle (7) and (8) renders (4) a regularization method.

Theorem 2. Under the assumptions of Theorem 1, if y^δ fulfills (2), and if the perturbed iteration is stopped with $N(\delta)$ according to the principle (8) and (9), then $x_{N(\delta)}^\delta \rightarrow x_*$ as $\delta \rightarrow 0$.

Proof. Let $\{\delta_n, n=1,2,\dots\}$ be a sequence converging to zero as $n \rightarrow +\infty$, and let $y_n := y^{\delta_n}$ be a corresponding sequence of perturbed data. For each pair (δ_n, y_n) , denote by $N_n := N(\delta_n)$ the corresponding stopping index determined from the discrepancy principle (8),(9). The proof will be split up in two separate cases.

1° Assume first that N' is a finite accumulation point of N_n . Without loss of generality can assume that $N_n = N'$ for all $n \in \mathbb{N}$. Thus from the definition of N_n it follows that

$$\|y_n - F(x_{N'}^\delta)\| \leq \tau_\alpha \delta_n.$$

Since $x_{N'}^\delta$ depends continuously on y^δ , have $x_{N'}^\delta \rightarrow x_{N'}$ as $n \rightarrow +\infty$.

And since F is Fréchet differentiable, F is Lipschitz continuous, also have

$$F(x_{N'}^\delta) \rightarrow F(x_{N'}) \text{ as } n \rightarrow +\infty.$$

Taking the limit yield $F(x_{N'}) = y$, thus $x_{N'} = x_*$ by Theorem 1, and obtain

$$x_{N'}^\delta \rightarrow x_* \text{ as } n \rightarrow +\infty.$$

2° It remains to consider the case where $N_n \rightarrow +\infty$ ($n \rightarrow +\infty$). Without loss of generality assume that N_n increases monotonically with n . Then for $n > m$ conclude from Lemma 1:

$$\begin{aligned} \|x_{N_n}^\delta - x_*\| &\leq \|x_{N_{n-1}}^\delta - x_*\| \leq \dots \leq \|x_{N_m}^\delta - x_*\| \\ &\leq \|x_{N_m}^\delta - x_{N_m}\| + \|x_{N_m} - x_*\|. \end{aligned}$$

Theorem 1 deduces that we can fix m so large that $\|x_{N_m} - x_*\|$ is sufficiently close to zero; now that N_m is fixed, apply 1° conclude that $\|x_{N_m}^\delta - x_{N_m}\| \rightarrow 0$ as $n \rightarrow +\infty$.

Hence $x_{N'}^\delta \rightarrow x_*$ as $n \rightarrow +\infty$, and the proof is complete.
Remark 2. The iteration Scheme and the proofs of theorem 1 and theorem 2 in this paper are similar to those in paper [8], the main difference lie in the improved restriction conditions $\|F'(x)\| \leq M$, which is $\|F'(x)\| \leq \frac{1}{2}$ in [8].

Based on it, several constants such as α, η_α and ζ_α are different in these two papers.

REFERENCES

- [1] ENGL H W. Regularization method for the stable solution of inverse problems[C]. Survey Math.Indust. 1993(3): 71-143.
- [2] BARBARA KALTENBACHER. Some Newton-type methods for the regularization of nonlinear ill-posed problems[J]. Inverse Problems, 1997(13): 729-753.
- [3] HANKE M. A Regularizaion Levenberg -Marquardt Scheme, with Applications to Inverse Groundwater Filtration Problems[J]. Inverse Problems, 1997(13): 79-95.
- [4] OTMAR SCHERZER. A Convergence Analysis of a method of steepest descent and a two-step algorithm for nonlinear ill-posed problems[J]. Numer.Funct. Anal. Optimiz, 1996(17): 197-214.
- [5] MARTIN HANKE, ANDREAS NEUBAUER, OTMAR SCHERZER. A convergence analysis of the Landweber iteration for nonlinear ill-posed problems[J]. Numer.Math. 1995(72): 21-37.
- [6] MOROZOV V. A. Regularisation Methods for non-linear ill-posed problems[M]. Boca Raton FL: Chemical Rubber Company, 1993.

- [7] Qi-Nian Jin and Zong-Yi Hou. On the choice of the regularization parameter for ordinary and iterated Tikhonov regularization of nonlinear ill-posed problems[J]. *Inverse Problems* 1997(13):815-827.
- [8] Zhang gui, Huang si-xun, On a newton-type method for the regularization of nonlinear ill-posed problems [J], *Annals of Mathematics*(Chinese Edition), 2003, 24(3): 321-330.