Research on the Enumerating Equation of Rectilinear Embedding- Counting Rooted Spherical Near Quadrangulations

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***Abstract*—This paper provides quartic functional equations satisfied by the enumerating functions of rooted planar near-quadrangulations with the size, the valency of the root-face and the number of non-rooted vertices. Rooted two edge-connected planar near-quadrangulations are also counted. The quartic and the cubic functional equations are proposed for the first time, furthermore, explicit formulae for such two types of maps with above parameters are derived respectively after employing Lagrangian inversion. For two particular cases, the numbers of rooted planar trees and outerplanar quadrangulations are deduced directly. The studying results are helpful for rectilinear embedding in VLSI(Very Large Scale Integration), for the Gaussian crossing problem in graph theory, for the knot problem in topology, and for the enumeration of some other kinds of maps.**

Keywords—quadrangulation; rectilinear embedding; lagrangian inversion; enumerating function

I. INTRODUCTION

Maps considered here are graphs embedded in a surface without edge-crossing and each face homeomorphic to an open disc, if the surface is the plane or the sphere, the map is called a *planar* map. A map is *rooted* if an edge is distinguished with an end (called *root-vertex*) and a side of the edge (called *root*). The selected edge is called *root-edge*. The face to which the root belongs is called *root-face*. A map is said to be *2-boundary* if the root-face is a 2-gon. Without loss of generality, the root-face is chosen as the infinite face. All other faces are called *inner faces*.

A rooted planar *near-quadrangulation* is a rooted planar map in which each inner face is a quadrangle. If the root-face of a rooted planar near-quadrangulation is also a quadrangle, then the map is called a *quadrangulation*. An outerplanar near-quadrangulation can be treated as quadrangulations on the disc such that all the vertices are on the boundary of the disc. For only one quadrangulation, the quadrangle itself is outerplanar.

Quadrangulations (or their duals: rooted 4-regular planar maps) have been investigated by many scholars. Such as Tutte [1-2], Brown[3], Mullin and Schellenberg[4], Li and Liu[5]. Quadrangulations and 4-regular maps (or quartic maps as some scholars called them) are very important, the usage can be seen for *rectilinear embedding* in *VLSI*, for the *Gaussian crossing problem* in graph theory, for the *knot problem* in topology, and for the enumeration of some other kinds of maps.

In this paper, the enumeration of rooted planar near-quadrangulations is discussed mainly with the size, the valency of the root-face and the number of non-rooted vertices as parameters. Based on this, two explicit expression of rooted planar near-quadrangulations with the size and with the root face valency are derived respectively. For one special case, rooted 2 edge-connected near-quadrangulations are considered. Moreover, we calculate rooted planar quadrangulations via rooted 2 boundary planar near-quadrangulations.

A map denoted by $M=(X_{\alpha,\beta},P)$, $X_{\alpha,\beta}(X)=\sum_{x\in X}Kx$, X is a finite set and P is a basic permutation on X . For convenience, the notations and terminologies not mentioned here can be seen in Liu [6-8]. Let Q_1 be the set of all rooted planar near-quadrangulations without 2-boundary ones; Q_{2e} be the subset of Q_1 and for any $M\in Q_{2e}$, M is 2 edge-connected; Q_{nq} be the set of all general rooted planar near-quadrangulations including 2-boundary ones. Q_{rvn} be the set of all general rooted planar near-quadrangulations with the rooted vertex is nonseparable. For a map M , the root, the root-vertex, the root-edge and the root-face are denoted by $r(M)$, $v_r(M)$ and $e_r(M)$, $f_r(M)$ respectively. Since the valency of all faces of planar near quadrangulations is four probably except the root-face, the root-face valency of any planar near quadrangulation has to be even. We define generating functions for Q_i , $i=1,2e$, nq, rvn.

$$f_i(x,y)=\sum_{M\in Q}x^{2m(M)}y^{n(M)}z^{t(M)}$$

where $2m(M)$, $n(M)$ and $t(M)$ are the valency of the root-face, the size and the number of non-rooted vertices. Further, we write that

$G_i(x) = f_i(x,1)$, $F_i(y) = f_i(1,y), h_i(x,z) = f_i(x,1,z)$

For $i = 1, 2e$, nq, rvn as some special enunfunctions.

In this paper, we obtain main results as follows:

Theorem I.A. The enumerating function $f_1(x,y)$ satisfies the following equation

$$x^2 y f_1^{\ 4} - (1 - x^{-2}y) f_1^3 + (1 - 3x^{-2}y) f_1^2 + 3x^{-2} y f_1 - x^2 y = 0 \quad (1)$$

Furthermore,

$$G_1(x) = 1 + \sum_{m \ge 1} A_m x^{2m}, A_m = \sum_{0 \le i \le m-1} \frac{(2m)!(3m-2i-3)!}{i!(2m-i)!(m-i-1)!(2m-i-1)!}$$

$$F_1(y) = 1 + \sum_{n \ge 1} A_n y^n, A_n = \sum_{k \ge 1}^{\left\lfloor \frac{n+2}{3} \right\rfloor} \frac{(n-1)!(2n-4k+4)!}{(k-1)!(n-k+1)!(n-k+2)!(n-3k+2)!};$$

Theorem I.B. The enumerating function $f_{2e}(x,y)$ satisfies the following equation

$$(x^2 - y) f_{2e}^{\ 3} - (x^2 + x^4 y - 3y) f_{2e}^{\ 2} - 3y f_{2e} + y = 0 \quad (2)$$

Moreover, for $i + j \le m$, $0 \le i \le m+2$ $\ 0 \le j \le m-2$

$$F_{2e}(y) = 1 + \sum_{n \ge 1} B_n y^n, G_{2e}(x) = 1 + \sum_{m \ge 1} B_m x^{2m};$$

$$B_n = \sum_{k \ge 1}^{\left\lfloor \frac{n-1}{3} \right\rfloor} \frac{(n-1)!(n-k-2)!}{k!(n-k)!(2k-1)!(n-3k-1)!};$$

$$B_m = \sum_{i,j} \frac{3^{m-i-j} 2^i (m+2)!(m-2)!(2m-i-j)!}{i!\, j!(2m-2i+4)!!(2m-2j-4)!!(m-i-j)!}$$

Theorem I.C. The enumerating function $h_{nq}(x,z)$ satisfies the following equation

$$x^4 z h_{nq}^{\ 2} + (1 - x^2) h_{nq} + x^2 - x^2 H_{nq} - 1 = 0 \quad (3)$$

where H_{nq} is the coefficient of x^2 in $h_{nq}(x,z)$. And the explicit solution of (3) is

$$h_{nq}(x,z) = \sum_{n \ge m \ge 1} C_{m,n} x^{2m} z^n \quad C_{m,n} = \frac{3^{n-m}(2m)!}{m!(m-1)!} \frac{(2n-m-1)!}{(n-m)!(n+1)!}$$

II. ESTABLISHMENT OF THE EQUATIONS

For two maps M_1 and M_2 with their respective roots $r_1 = r(M_1)$ and $r_2 = r(M_2)$. The map $M = M_1 \cup M_2$, provided $M_1 \cap M_2 = \{v\}$ with $v = v_{r_1} = v_{r_2}$ is defined to have its root, root-vertex and root-edge are as the same as those of M_1, but the root-face is the composition of $f_{r_1}(M_1)$ and $f_{r_2}(M_2)$, where $f_{r_i}(M_i)$ is the root-face of M_i ($i = 1, 2$). The operation for getting M from M_1 and M_2 is called *1v-addition* and is denoted b $M = M_1 \dot{+} M_2$.

Further, for two sets of maps $M^{(1)}$ and $M^{(2)}$, the set of maps $M^{(1)} \oplus M^{(2)} = \{M_1 \dot{+} M_2 \mid M_i \in M^{(i)}, i = 1, 2\}$ is said to be the *1v-production of* $M^{(1)}$ and $M^{(2)}$.

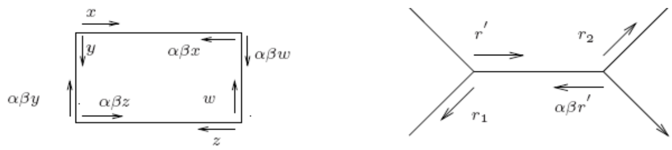

FIGURE I. QUADRANGLE AND $Q_1^{(I)}$

Here we consider $Q_1(x,y)$ first, which can be partitioned into three parts: Quadrangle and

1) $Q_1^{(0)} = \{v\}$, v is the vertex map;

2) $Q_1^{(I)} = \{M \mid e_r(M)$ is an isthmus$\}$.

The link map $L = (Kr, (r)(\alpha\beta r))$ is included;

1) $Q_1^{(II)} = \{M \mid e_r(M)$ belongs to a simple circuit$\}$.

$Q_{nq}(x,y)$ follows the similar pattern of this partition.

Lemma II.A. Let $Q_{\langle 1 \rangle}^{(I)} = \{M - a \mid \forall M \in Q_1^{(I)}\}$ where $a = e_r(M)$ is the root-edge of M, then $Q_{<1>}^{(I)} = Q_1 \times Q_1$ (where $\times$ presents Descartes product between sets).

Proof: Because $\forall M \in Q_1^{(I)}$, the root-edge a of M is a cut edge, $M - a = M_1 \dot{+} M_2, M_1, M_2 \in Q_1$. That implies $M - a \in Q_1 \times Q_1$. Hence $Q_{<1>}^{(I)} \subseteq Q_1 \times Q_1$.

Conversely, for $\forall M \in Q_1 \times Q_1$, we have $M = M_1 \dot{+} M_2$. $M_1, M_2 \in Q_1$. The only map $M^{'} = (X^{'}, P^{'})$ can be obtained by adding an edge $Kr^{'}$, which connects $v_r(M_1)$ and $v_r(M_2)$ (See **Figure I**). Then $M^{'}$ is also outerplanar quadrangulation, and $Kr^{'}$ is a cut edge, so $M^{'} \in Q_1^{(\mathrm{I})}$, $M = M^{'} - a \in Q_{<1>}^{(\mathrm{I})}$ .This means that $Q_1 \times Q_1 \subseteq Q_{\langle 1\rangle}^{(\mathrm{I})}$.

From **Lemma II.A**, we obtained the contribution of $Q_1^{(\mathrm{I})}$ to Q_1 is

$$f_1^{(\mathrm{I})} = x^2 y f_1^2 \tag{4}$$

where x^2 presents the contribution of the root-edge to the root-face boundary of a map M in $Q_1^{(\mathrm{I})}$.

Lemma II.B. $Q_1 = \sum_{k\geq 0} Q_{\mathrm{rvn}}^{\oplus k}$ Where $Q_{\mathrm{rvn}} = \{M \mid \forall M \in Q_{\mathrm{rvn}} \subset Q_1, v_r(M)$ is nonseparable$\}$.

Proof: $\forall M \in Q_1$, it is easily to see that M has k components M_i ($1 \leq i \leq k$) and $M_i \in Q_{\mathrm{rvn}}$, such that $M = M_1 \dot{+} M_2 \dot{+} \dot{+} M_3 \ldots + \dot{+} M_k$, hence $M \in Q_{\mathrm{rvn}}^{\oplus k}$.

From **Lemma II.B**, $f_1 = \sum_{k\geq 0} f_{\mathrm{rvn}}^k = \dfrac{1}{1 - f_{\mathrm{rvn}}}$, by simplification we obtained that $f_{\mathrm{rvn}} = \dfrac{f_1 - 1}{f_1}$.

For a map $M \in Q_1^{(\mathrm{II})}$, $a = kr$ is the root-edge of M, let $M \hat{-} a$ be the map obtained by deleting the edge a, such that the root of $M \hat{-} a$ is $(P\alpha\beta Pr)$, where $(Pr, P\alpha\beta Pr, (P\alpha\beta)^2 Pr, a)$ is the quadrangle incident to a.

Lemma II.C. Let $Q_{\langle 1\rangle}^{(\mathrm{II})} = \{M - a \mid \forall M \in Q_1^{(\mathrm{II})}\}$.

Then $Q_{\langle 1\rangle}^{(\mathrm{II})} = (Q_1 - Q_1^{(0)}) \times Q_{\mathrm{rvn}}^2$

Proof: $\forall M^* = (X^*, P^*) \in (Q_1 - Q_1^{(0)}) \times Q_{\mathrm{rvn}}^2$, for the root-vertex v_{r^*} is a cut vertex, we may obtained a map $M = M^* \hat{+} a^*$ by join an edge $a^* = (v_{(P\alpha\beta)^{2m-1}r^*}, v_{(P\alpha\beta)^2 r^*})$ where the root-face of M^* is $(r^*, P\alpha\beta r^*, \cdots\cdots, (P\alpha\beta)^{2m-1} r^*)$. Since a^* is an edge on a circuit, so $M = M^* \hat{+} a^* \in Q_1^{(\mathrm{II})}$ (See **Figure II**), it is easily checked that $M^* = M \hat{-} a^*$.

By considering that the valency of the root-face of a map in $Q_1^{(\mathrm{II})}$ is 2 less than that of the corresponding map in $(Q_1 - Q_1^{(0)}) \times Q_{\mathrm{rvn}}^2$, we have $Q_1^{(\mathrm{II})}$ to Q_1 is $f_1^{(\mathrm{II})} = x^{-2} y (f_1 - 1) f_{\mathrm{rvn}}^2$. For $f_{\mathrm{rvn}} = \dfrac{f_1 - 1}{f_1}$, we get that

$$f_1^{(\mathrm{II})} = x^{-2} y (f_1 - 1)\left(\frac{f_1 - 1}{f_1}\right)^2 \tag{5}$$

Since $f_1 = f_1^{(0)} + f_1^{(\mathrm{I})} + f_1^{(\mathrm{II})}$, from $f_1^{(0)} = x^0 y^0 = 1$ and (4), (5), we have

$$f_1 = 1 + x^2 y f_1^2 + x^{-2} y (f_1 - 1)\left(\frac{f_1 - 1}{f_1}\right)^2$$

Multiplying by f_1^2 the two sides, **Theorem I.A** (1) follows from some rearrangement.

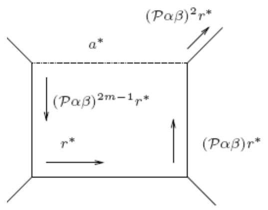

FIGURE II. $Q_1^{(\mathrm{II})}$

Here we consider $Q_{\mathrm{nq}}(x, y)$ in the same way as $Q_1(x, y)$, which can be partitioned into three parts.

Lemma II.D. Let $Q_{(\mathrm{nq})}^{(\mathrm{I})} = \{M \bullet e_r(M) \mid M \in Q_{\mathrm{nq}}^{(\mathrm{I})}\}$.

Then $Q_{(\mathrm{nq})}^{(\mathrm{I})} = Q_{\mathrm{nq}} \oplus Q_{\mathrm{nq}}$

Proof: $\forall M \in Q_{\mathrm{nq}} \oplus Q_{\mathrm{nq}}$, there exists $M = M_1 \dot{+} M_2$, M_1, $M_2 \in Q_{\mathrm{nq}}$. After splitting the root-vertex of M, it will result in a map $M^{'}$ in $Q_{\mathrm{nq}}^{(\mathrm{I})}$ such that $M = M^{'} \bullet e_r(M^{'})$, this procedure is reversible.

From **Lemma II.D**, we see that the contribution of $Q_{nq}^{(I)}$ to Q_{nq} is $h_{nq}^{(I)} = x^2 z h_{nq}{}^2$.

Lemma II.E. Let $Q_{\langle nq\rangle}^{(II)} = \{M - e_r(M) \mid M \in Q_{nq}^{(II)}\}$, then $Q_{\langle nq\rangle}^{(II)} = Q_{nq} - v - Q_{nq2}$, where

$$Q_{nq2} = \{M \mid \forall M \in Q_{nq2} \subset Q_{nq} \text{ the valency of } M \text{ is } 2\}.$$

Proof: $\forall M \in Q_{\langle nq\rangle}^{(II)}$, $M = M^* - e_{r^*}(M^*)$, $M^* \in Q_{nq}^{II}$. Since $e_{r^*}(M^*)$ is on a circuit, so the valency of $f_r(M)$ is not less than 4. Conversely, the only map M can be constructed by adding an edge $(v_{(P\alpha\beta)^{2m-1}r^*}, v_{(P\alpha\beta)^2 r^*})$ in M^*.

Let H_{nq} be the coefficient of x^2 in $h_{nq}(x,z)$.

From **Lemma II.E**, we have

$$h_{nq}^{(II)} = x^{-2}(h_{nq} - 1 - x^2 H_{nq}) \tag{6}$$

Since $h = h_{nq} = h_{nq}^{(0)} + h_{nq}^{(I)} + h_{nq}^{(II)}$, from $h_{nq}^{(0)} = x^0 z^0 = 1$, $h_{nq}^{(I)} = x^2 z h_{nq}{}^2$ and (6), we get

$$h_{nq} = 1 + x^2 z h_{nq}^2 + x^{-2}(h_{nq} - 1 - x^2 H_{nq})$$

By rearranging the terms, we soon find the equation (3) of **Theorem I.C**.

III. Parametric Expressions

Although from (1) it is allowed to find f_1 directly by using Lagrangian inversion, because of the equation is quadruple, the result is rather complicated for usage. In order to find the enumerating explicit expressions, we have to do some transformation and find their parametric expressions firstly.

For the generating function $f = f_1$ satisfies (1), then $F = f_1(1, y)$ satisfies

$$F^4 + (1 - \frac{1}{y})F^3 + (\frac{1}{y} - 3)F^2 + 3F = 1 \tag{7}$$

Because the left side of the equation (7) has the following decomposition, so we write

$$F^3 + (2 - \frac{1}{y})F^2 - F + 2 = \frac{1}{1-F} = 1 + F + F^2 + F^3 + \frac{F^4}{1-F}$$

which is in fact the form as

$$(\frac{1-F}{F})^2 = \frac{1}{y} + \frac{F^2}{1-F} \tag{8}$$

Lemma III.A. The generating function $F = f_1(1, y)$ has the parametric expressions as follows:

$$\begin{cases} F = \dfrac{1}{1-t} \\ y = \dfrac{t(1-t)}{t^3(1-t)+1} \end{cases} \tag{9}$$

Proof: From equation (8), we introduce one parameter for F, let $t = \frac{1-F}{F}$ then the parametric expression is derived.

By Ren [9], according to the duality, the (3) have the following parametric expressions:

$$\begin{cases} z = \theta \dfrac{(2-3\theta)}{4} \\ h = \dfrac{x^2 - 1 + (1+ux^2)(1-vx^2z)^{\frac{1}{2}}}{2x^4 z} \end{cases}$$

where, $\begin{cases} -2 = -vz + 2u, 4z + 1 = -2uvz + u^2 \\ -u^2 vz = -4y(1+q), \theta = u + 1 \end{cases}$.

IV. Explicit Formulae

In this section, we enumerate the explicit expressions of the enumerating functions $F = f_1(1, y)$ by employing Lagrangian inversion based on the results as described above.

For the sake of brevity, let

$$\partial_x^i = \begin{cases} \left.\dfrac{1}{i!}\dfrac{d^i}{dx^i}\right|_{x=0} & i \geq 0 \\ \left. x^{-i} \right|_{x=\infty} & i < 0 \end{cases}$$

which is called the coefficient operator [10], or the partial, here we use the notation

$$\partial_x^i f = [x^i] f$$

In (9), we can see that $t = y\varphi(t)$ where $\varphi(t) = t^3 + \frac{1}{1-t}$. By applying Lagrangian inversion with one parameter,

$$[y^n]F = \frac{1}{n}[t^{n-1}](\varphi^n(t) \bullet \frac{dF}{dt})$$

We get

$$\begin{aligned}
[y^n]F &= \frac{1}{n}[t^{n-1}](\frac{1}{(1-t)^{n+2}}(1+t^3(1-t))^n) \\
&= \frac{1}{n}[t^{n-1}]\sum_{k\geq 0}^{n}\binom{n}{k} t^{3k}(1-t)^{k-n-2} \\
&= \frac{1}{n}[t^{n-3k-1}]\sum_{k\geq 0}^{n}\binom{n}{k}(1-t)^{k-n-2} \\
&= \frac{1}{n}\sum_{k\geq 0}^{\left\lfloor \frac{n-1}{3}\right\rfloor}\binom{n}{k}\binom{2n-4k}{n-3k-1}
\end{aligned}$$

So

$$F(y) = \frac{1}{n}\sum_{k\geq 0}^{\left\lfloor \frac{n-1}{3}\right\rfloor}\binom{n}{k}\binom{2n-4k}{n-3k-1}y^n$$

which is equivalent to **Theorem I.A**. According to this theorem, we compute the coefficients A_n of $F(y)$, which are shown in **TABLE I** with respect to edges within 20 edges.

By Euler Formula, for any planar near quadrangulation, if the valency of root face is $2m$ and the number of non-root vertices is n, we can derived the edge number is $2n-m$. Therefore, h_{nq} can be expanded as

$$f_{\text{nq}} = \sum_{n\geq m\geq 1} C_{m,n} x^{2m} y^{2n-m} z^n \quad (10)$$

Corollary IV.A. (Reference [6])

Let T be the enumerating function of rooted planar trees, then $T = \sum_{n\geq 1}\frac{(2n)!}{n!(n+1)!}y^n$

Proof: Notice that if $m = n$ in $C_{m,n}$, then the expression of T is obtained.

TABLE I. NUMBERS OF OUTERPLANAR QUADRANGULATION

x	A_n
0	1
1	1
2	2
3	5
4	15
5	48
6	160
7	552
8	1953
9	7044
10	25806
11	95765
12	359216
13	1359767
14	5187754
15	19927572
16	77006278
17	299153776
18	1167638982
19	4576743657
20	18007689546

Let $F_{\text{nq}}(y) = \sum_{n\geq 0} C_n y^n$ denote the rooted general planar near-quadrangulation with edge as a parameter, from (10), we see that

$$C_n = \sum_{n\geq m\geq \left\lceil \frac{n+1}{2}\right\rceil} \frac{3^{n-m}(4m-2n)!}{(2m-n-1)!(2m-n)!}\frac{(n-1)!}{(n-m)!(m+1)!}$$

Further, let $N_{\text{q}}(n)$ be the number of rooted general planar quadrangulation with n edges, $N_{\text{nq2}}(n+1)$ be the number of Q_{nq2}, rooted planar near quadrangulation with $n+1$ edges and the valency of the root-face is 2.

Based on the discussion above, so the corollary follows.

Corollary IV.B. $F_{\text{nq2}} = \sum_{n\geq 1}(C_n - A_n)y^n$

Corollary IV.C. $N_{\text{nq2}}(n+1) = N_{\text{q}}(n)$

Proof: For $\forall M \in Q_{\text{nq2}}$ can be obtained by replacing the root-edge with an additional edge on maps in Q_{q}, forming a multi-edge to the root-edge as the root-face boundary. On the other hand, $\forall M \in Q_{\text{q}}$, M can be got by shrinking the multi-edge on the boundary of maps in Q_{nq2}.

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