Research on the Enumerating Equation of Rectilinear Embedding- Counting Rooted Spherical Near Quadrangulations

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Abstract—This paper provides quartic functional equations satisfied by the enumerating functions of rooted planar near-quadrangulations with the size, the valency of the root-face and the number of non-rooted vertices. Rooted two-edge-connected planar near-quadrangulations are also counted. The quartic and the cubic functional equations are proposed for the first time, furthermore, explicit formulae for such two types of maps with above parameters are derived respectively after employing Lagrangian inversion. For two particular cases, the numbers of rooted planar trees and outerplanar quadrangulations are deduced directly. The studying results are helpful for rectilinear embedding in VLSI (Very Large Scale Integration), for the Gaussian crossing problem in graph theory, for the knot problem in topology, and for the enumeration of some other kinds of maps.

Keywords—quadrangulation; rectilinear embedding; lagrangian inversion; enumerating function

I. INTRODUCTION

Maps considered here are graphs embedded in a surface without edge-crossing and each face homeomorphic to an open disc, if the surface is the plane or the sphere, the map is called a planar map. A map is rooted if an edge is distinguished with an end (called root-vertex) and a side of the edge (called root). The selected edge is called root-edge. The face to which the root belongs is called root-face. A map is said to be 2-boundary if the root-face is a 2-gon. Without loss of generality, the root-face is chosen as the infinite face. All other faces are called inner faces.

A rooted planar near-quadrangulation is a rooted planar map in which each inner face is a quadrangle. If the root-face of a rooted planar near-quadrangulation is also a quadrangle, then the map is called a quadrangulation. An outerplanar near-quadrangulation can be treated as quadrangulations on the disc such that all the vertices are on the boundary of the disc. For only one quadrangulation, the quadrangle itself is outerplanar.

Quadrangulations (or their duals: rooted 4-regular planar maps) have been investigated by many scholars. Such as Tutte [1-2], Brown[3], Mullin and Schellenberg[4], Li and Liu[5]. Quadrangulations and 4-regular maps (or quartic maps as some scholars call them) are very important, the usage can be seen for rectilinear embedding in VLSI, for the Gaussian crossing problem in graph theory, for the knot problem in topology, and for the enumeration of some other kinds of maps.

In this paper, the enumeration of rooted planar near-quadrangulations is discussed mainly with the size, the valency of the root-face and the number of non-rooted vertices as parameters. Based on this, two explicit expression of rooted planar near-quadrangulations with the size and with the root face valency are derived respectively. For one special case, rooted 2 edge-connected near-quadrangulations are considered. Moreover, we calculate rooted planar quadrangulations via rooted 2 boundary planar near-quadrangulations.

A map denoted by \( M = (X_{a,P}, P) \), \( X_{a,P}(X) = \sum_{x \in X} K_x \), \( X \) is a finite set and \( P \) is a basic permutation on \( X \). For convenience, the notations and terminologies not mentioned here can be seen in Liu [6-8]. Let \( Q_{i} \) be the set of all rooted planar near-quadrangulations without 2-boundary ones; \( Q_{\infty} \) be the subset of \( Q_{1} \) and for any \( M \in Q_{\infty} \), \( M \) is 2 edge-connected; \( Q_{m} \) be the set of all general rooted planar near-quadrangulations including 2-boundary ones. \( Q_{m} \) be the set of all general rooted planar near-quadrangulations with the rooted vertex is nonseparable. For a map \( M \), the root, the root-vertex, the root-edge and the root-face are denoted by \( r(M) \), \( v_r(M) \) and \( e_r(M) \), \( f_r(M) \) respectively. Since the valency of all faces of planar near quadrangulations is four probably except the root-face, the root-face valency of any planar near quadrangulation has to be even. We define generating functions for \( Q_i, i = 1, 2e, q, rvn \).

\[
f_f(x,y) = \sum_{M \in Q} x^{2m(M)} y^{n(M)} z^{r(M)}
\]

where \( 2m(M), n(M) \) and \( r(M) \) are the valency of the root-face, the size and the number of non-rooted vertices. Further, we write that
For $i = 1, 2$, $e_i$ as some special enunfunctions.

In this paper, we obtain main results as follows:

**Theorem I.A.** The enumerating function $f(x, y)$ satisfies the following equation

$$x^2y^2 - (1 - x^2^2)yf_1 + (1 - 3x^2^2)yf_2 + 3xyf_1 - x^2 = 0 \quad (1)$$

Furthermore,

$$G_i(x) = f(x, 1), F_i(y) = f(1, y), h_i(x, z) = f(x, 1, z)$$

**Theorem I.B.** The enumerating function $f_2(x, y)$ satisfies the following equation

$$(x^2 - y)f_2^3 + (x^2 + y - 3)yf_2^2 - 3xyf_2 + y = 0 \quad (2)$$

Moreover, for $i + j \leq m$, $0 \leq i \leq m + 2$ $0 \leq j \leq m - 2$

$$F_{2i}(y) = 1 + \sum_1 B_i y^n, G_{2i}(x) = 1 + \sum_1 B_i x^{2n}$$

*$\ast$

**Theorem I.C.** The enumerating function $h_{m_{ij}}(x, z)$ satisfies the following equation

$$x^{2z}h_{m_{ij}}^2 + (1 - x^2)h_{m_{ij}} + x^2 - x^2H_{m_{ij}} - 1 = 0 \quad (3)$$

where $H_{m_{ij}}$ is the coefficient of $x^2$ in $h_{m_{ij}}(x, z)$. And the explicit solution of (3) is

$$h_{m_{ij}}(x, z) = \sum_{n \geq 2m} C_{m,n} x^{2n} z^n = \sum_{n \geq 2m} \frac{3^{n-n}(2m)!}{m!(m-1)!(n-m)!(n+1)!}$$

II. **Establishment of The Equations**

For two maps $M_1$ and $M_2$ with their respective roots $r_1 = r(M_1)$ and $r_2 = r(M_2)$. The map $M = M_1 \cup M_2$, provided $M_1 \cap M_2 = \{v\}$ with $v = v_1 = v_2$ is defined to have its root, root-vertex and root-edge are as the same as those of $M_1$, but the root-face is the composition of $f_{r_1}(M_1)$ and $f_{r_2}(M_2)$, where $f_{r_i}(M_i)$ is the root-face of $M_i$ $(i = 1, 2)$. The operation for getting $M$ from $M_1$ and $M_2$ is called 1v-addition and is denoted $b(M = M_1 + M_2)$.

Further, for two sets of maps $M^{(1)}$ and $M^{(2)}$, the set of maps $M^{(1)} \oplus M^{(2)} = \{M_1 + M_2 | M_1 \in M^{(1)}, i = 1, 2\}$ is said to be the 1v-production of $M^{(1)}$ and $M^{(2)}$.

$\ast$

**FIGURE I.** QUADRANGLE AND $Q^{(0)}_{1}$

Here we consider $Q_1(x, y)$ first, which can be partitioned into three parts: Quadrangle and $1) Q^{(0)}_1 = \{v\}$, $v$ is the vertex map;

2) $Q^{(1)}_1 = \{M|e_r(M) is an isthmus \}$.

The link map $L = (Kr, vr(a \beta r))$ is included;

1) $Q^{(0)}_2 = \{M|e_r(M) belongs to a simple circuit \}$.

$Q_{m_{ij}}(x, y)$ follows the similar pattern of this partition.

**Lemma II.A.** Let $Q^{(1)}_{e_r(M)} = \{M \in Q_{e_r(M)} \}$ where $a = e_r(M)$ is the root-edge of $M$, then $Q^{(1)}_{<1>} = Q_1 \times Q_1$ (where $\times$ presents Descartes product between sets).

**Proof:** Because $\forall M \in Q^{(1)}_1$, the root-edge $a$ of $M$ is a cut edge, $M - a = M_1 + M_2, M_1, M_2 \in Q_1$. That implies $M - a \in Q_1 \times Q_1$. Hence $Q_{<1>} \subseteq Q_1 \times Q_1$. 

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Conversely, for \( \forall M \in Q_1 \times Q_1 \), we have \( M = M_1 + M_2 \). \( M_1, M_2 \in Q_1 \). The only map \( M' = (X', P') \) can be obtained by adding an edge \( K_r \), which connects \( v_r(M_1) \) and \( v_r(M_2) \) (See Figure I). Then \( M' \) is also outerplanar quadrangulation, and \( K_r \) is a cut edge, so \( M' \in Q_1^{(i)} \), \( M = M' - a \in Q_{\leq 1} \). This means that \( Q_1 \times Q_1 \subseteq Q_1^{(i)} \).

From Lemma II.A. we obtained the contribution of \( Q_1^{(i)} \) to \( Q_1 \) is

\[
 f_1^{(i)} = x^2 y f_1^2
\]

where \( x^2 \) presents the contribution of the root-edge to the rootface boundary of a map \( M \) in \( Q_1^{(i)} \).

**Lemma II.B.** \( Q_1 = \sum_{k \geq 0} Q_{r\nu n}^{\otimes k} \) Where \( Q_{r\nu n} = \{ M | \forall a \in Q_{\nu n}, v_r(M) \) is nonseparable \}.

**Proof:** \( \forall a \in Q_1 \), it is easily to see that \( M \) has \( k \) components \( M_i \) (\( 1 \leq i \leq k \)) and \( M_i \in Q_{\nu n} \), such that \( M = M_1 + M_2 + \ldots + M_k \), hence \( M \in Q_{r\nu n}^{\otimes k} \).

From Lemma II.B, \( f_1 = \sum_{k \geq 0} f_{r\nu n}^k = \frac{1}{1 - f_{r\nu n}} \), by simplification we obtained that \( f_{r\nu n} = \frac{f_1 - 1}{f_1} \).

For a map \( M \in Q_1^{(i)}, a = kr \) is the root-edge of \( M \), let \( M' - a \) be the map obtained by deleting the edge \( a \), such that the root of \( M' - a \) is \((Pa)(Pr)\), where \((Pr, Pa)(Pr, Pa)^2 Pr, a)\) is the quadrangle incident to \( a \).

**Lemma II.C.** Let \( Q_1^{(i)} = \{ M - a | \forall M \in Q_1^{(i)} \}. \)

Then \( Q_1^{(i)} = (Q_1 - Q_1^{(i)}) \times Q_{\nu n}^2 \)

**Proof:** \( \forall M' = (X', P') \in (Q_1 - Q_1^{(i)}) \times Q_{\nu n}^2 \), for the rootvertex \( v_r' \) is a cut vertex, we may obtained a map \( M = M' + a^* \) by join an edge \( a' = (v_{(Pa)}^{2m-1}r, v_{(Pa)}^{2m-1}r') \) where the root-face of \( M' \) is \((r', Pa^{2m}r', \ldots, (Pa)^{2m-1}r') \). Since \( a^* \) is an edge on a circuit, so \( M = M' + a^* \in Q_1^{(i)} \) (See Figure II), it is easily checked that \( M' = M \wedge a^* \).

By considering that the valency of the root-face of a map in \( Q_1^{(i)} \) is 2 less than that of the corresponding map in \((Q_1 - Q_1^{(i)}) \times Q_{\nu n}^2 \), we have \( Q_1^{(i)} \) to \( Q_1 \) is \( f_{r\nu n}^{(i)} = x^2 y f_{r\nu n}^2 \).

For \( f_{r\nu n} = \frac{f_1 - 1}{f_1} \), we get that

\[
 f_1^{(ii)} = x^{-2} y f_{r\nu n} (f_1)^2
\]

Since \( f_1 = f_1^{(o)} + f_1^{(i)} + f_1^{(ii)} \), from \( f_1^{(o)} = x^0 y^0 = 1 \) and (4), (5), we have

\[
 f_1 = 1 + x^2 y f_{r\nu n}^2 + x^2 y f_{r\nu n} (f_1)^2
\]

Multiplying by \( f_1^2 \) the two sides, Theorem I.A (1) follows from some rearrangement.

Here we consider \( Q_{\nu n}(x, y) \) in the same way as \( Q_1(x, y) \), which can be partitioned into three parts.

**Lemma II.D.** Let \( Q_{\nu n}^{(i)} = \{ M \wedge e_r(M) | M \in Q_{\nu n}^{(i)} \}. \)

Then \( Q_{\nu n}^{(i)} = Q_{\nu n} \oplus Q_{\nu n} \)

**Proof:** \( \forall M \in Q_{\nu n} \oplus Q_{\nu n} \), there exists \( M = M_1 + M_2 \), \( M_1 \), \( M_2 \in Q_{\nu n} \). After splitting the root-vertex of \( M' \), it will result in a map \( M' \) in \( Q_{\nu n}^{(i)} \) such that \( M = M' \wedge e_r(M') \), this procedure is reversible.
From **Lemma II.D**, we see that the contribution of \( Q_{m}^{(1)} \) to \( Q_{m}^{(1)} \) is
\[
\left( \frac{2}{n} \right) n_{q} n_{q} n_{q}^{2} = \frac{2}{n} n_{q} n_{q} n_{q}^{2}.
\]

**Lemma II.E.** Let
\[
Q_{m}^{(1)} = \left\{ M \mid e_{r}(M) \in Q_{m}^{(1)} \right\},
\]
then
\[
Q_{m}^{(1)} = \left\{ M \mid \forall M \in Q_{m}^{(1)} \subset Q_{m}^{(1)} \text{ the valency of } M \text{ is } 2 \right\}.
\]

**Proof:** \( \forall M \in Q_{m}^{(1)} \), \( M = M^* - e_{r}(M^*) \), \( M^* \in Q_{m}^{(1)} \). Since \( e_{r}(M^*) \) is on a circuit, so the valency of \( f_{j}(M) \) is not less than 4. Conversely, the only map \( M \) can be constructed by adding an edge \( (v(P_{B})^{n+1}, r_{B}) \) in \( M^* \).

Let \( H_{m} \) be the coefficient of \( x^2 \) in \( h_{m}(x, z) \).

From **Lemma II.E**, we have
\[
h_{m}^{(0)}(x) = x^2 (h_{m} - 1 - x^2 H_{m})
\]

Since \( h = h_{m} = h_{m}^{(0)} + h_{m}^{(1)} + h_{m}^{(2)} \), from \( h_{m}^{(1)} = x^2 z^{2} = 1 \), \( h_{m}^{(2)} = x^2 z h_{m}^{2} \) and (6), we get
\[
h_{m}^{(1)} = 1 + x^2 z h_{m}^{2} + x^2 (h_{m} - 1 - x^2 H_{m})
\]

By rearranging the terms, we soon find the equation (3) of **Theorem I.C.**

### III. PARAMETRIC EXPRESSIONS

Although from (1) it is allowed to find \( f \) directly by using Lagrangian inversion, because of the equation is quadruple, the result is rather complicated for usage. In order to find the enumerating explicit expressions, we have to do some transformation and find their parametric expressions firstly.

For the generating function \( f = f_{l} \) satisfies (1), then \( F = f_{l}(1, y) \) satisfies
\[
F^4 + (1 - \frac{1}{y}) F^3 + (\frac{1}{y} - 3) F^2 + 3 F = 1
\]

Because the left side of the equation (7) has the following decomposition, so we write
\[
F^3 + (2 - \frac{1}{y}) F^3 - 2 F + 2 = \frac{1}{1 - F} = 1 + F + F^2 + F^3 + \frac{F^4}{1 - F}
\]

which is in fact the form as
\[
\left( \frac{1 - F}{y} \right)^2 = \frac{1}{y} \left( \frac{F^2}{1 - F} \right)
\]

**Lemma III.A.** The generating function \( F = f_{l}(1, y) \) has the parametric expressions as follows:
\[
\begin{align*}
F &= \frac{1}{1 - t} \\
y &= \frac{t(1 - t)}{t(1 - t) + 1}
\end{align*}
\]

**Proof:** From equation (8), we introduce one parameter for \( F \), let \( t = \frac{1 - F}{F} \) then the parametric expression is derived.

By Ren [9], according to the duality, the (3) have the following parametric expressions:
\[
\begin{align*}
z &= \frac{(2 - 3 \theta) \theta}{4} \\
h &= \frac{x^2 - 1 + (1 + \theta x^2)(1 - \theta x^2) z^2}{2x^2 z}
\end{align*}
\]

where,
\[
\begin{align*}
-2z &= -v + 2u, 4z + 1 = -2uv + u^2 \\
-uv^2z &= -4y(1 + q) \theta = u + 1
\end{align*}
\]

### IV. EXPLICIT FORMULAE

In this section, we enumerate the explicit expressions of the enumerating functions \( F = f_{l}(1, y) \) by employing Lagrangian inversion based on the results as described above.

For the sake of brevity, let
\[
\delta_x^i = \begin{cases} 
\frac{1}{l!} \frac{d^i}{dx^i} & i \geq 0 \\
0 & i < 0 \\
\end{cases}
\]

which is called the coefficient operator [10], or the partial, here we use the notation

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In (9), we can see that
\[ t = y \varphi(t) \] where \( \varphi(t) = t^3 + \frac{1}{1-t} \). By applying Lagrangian inversion with one parameter,
\[
[y^n] F = \frac{1}{n} \left[ t^{n-1} \right] (\varphi^n(t)) \frac{dF}{dt}
\]
We get
\[
[y^n] F = \frac{1}{n} \left[ t^{n-1} \right] \left( \frac{1}{(1-t)^{n+2}} (1 + t^3 (1-t))^n \right)
\]
\[
= \frac{1}{n} \left[ t^{n-1} \right] \sum_{k=0}^{n} \binom{n}{k} t^{3k} (1-t)^{n-k-2}
\]
\[
= \frac{1}{n} \left[ t^{n-3k-1} \right] \sum_{k=0}^{n} \binom{n}{k} (1-t)^{n-k-2}
\]
\[
= \frac{1}{n} \sum_{k=0}^{n-1} \binom{n}{k} \left( \frac{2n-4k}{n-3k-1} \right) y^n
\]
So
\[
F(y) = \frac{1}{n} \sum_{k=0}^{n-1} \binom{n}{k} \left( \frac{2n-4k}{n-3k-1} \right) y^n
\]
which is equivalent to Theorem IA. According to this theorem, we compute the coefficients \( A_n \) of \( F(y) \), which are shown in TABLE I with respect to edges within 20 edges.

By Euler Formula, for any planar near quadrangulation, if the valency of root face is \( 2m \) and the number of non-root vertices is \( n \), we can derived the edge number is \( 2n - m \). Therefore, \( h_{mn} \) can be expanded as
\[
f_{mn} = \sum_{n=2m+1} C_{mn} y^n 2^{2m} (2n-2m-2)^n
\]

Corollary IV.A. (Reference [6])

Let \( T \) be the enumerating function of rooted planar trees, then
\[
T = \sum_{n=1}^{\infty} \frac{(2n)!}{n!(n+1)!} y^n
\]

Proof: Notice that if \( m = n \) in \( C_{mn} \), then the expression of \( T \) is obtained.

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Let \( F_{nq}(y) = \sum_{n=2} C_{nq} y^n \) denote the rooted general planar near-quadrangulation with edge as a parameter, from (10), we see that
\[
C_n = \sum_{n=2m+1} \frac{3^n (4m-2n)!}{(2m-n-1)! (2m-n)! (n-m)! (m+1)!}
\]

Further, let \( N_q(n) \) be the number of rooted general planar quadrangulation with \( n \) edges, \( N_{nq2}(n+1) \) be the number of \( Q_{nq2} \), rooted planar near quadrangulation with \( n + 1 \) edges and the valency of the root-face is 2.

Based on the discussion above, the corollary follows.

Corollary IV.B. \( F_{nq} = \sum_{n=2} (C_n - A_n) y^n \)

Corollary IV.C. \( N_{nq2}(n+1) = N_q(n) \)

Proof: For \( \forall M \in Q_{nq2} \) can be obtained by replacing the root-edge with an additional edge on maps in \( Q_q \), forming a multi-edge to the root-edge as the root-face boundary. On the other hand, \( \forall M \in Q_q \), \( M \) can be got by shrinking the multi-edge on the boundary of maps in \( Q_{nq2} \).
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